A Concise Approximation for the Early Exercise Boundary of American Options*

Toshikazu Kimura

Department of Civil, Environmental & Applied Systems Engineering
Kansai University

Abstract

This paper provides a concise approximation for the early exercise boundary (EEB) of an American option written on dividend-paying assets. Although a vast majority of traded options are of American-style optimally exercised before maturity, there are no explicit formulas for their prices as well as EEBs even in the standard model called *vanilla*. A closed-form EEB approximation is especially important in decision-making on optimal early exercise. Following a simple but indefinite idea of Carr et al. (1992) based on van Moerbeke (1976), we focus on a class of interpolation approximations with a square-root exponential weight. The unsettled problem there was how to determine the exponential decay rate. Applying the Laplace-Carson transform approach to this problem, we derive an explicit decay rate of the exponential weight to develop a pair of new EEB approximations for vanilla put/call options, both of which are consistent with the principal boundary features.

1 Introduction

European-style options, which can only be exercised at its maturity, have closed-form formulas for their values in the standard model pioneered by Black and Scholes (1973) and Merton (1973). Although a vast majority of traded options are of American-style optimally exercised before maturity, there are no closed-form formulas for their values even in the standard model called vanilla. The principal difficulty in analyzing American options may be the absence of an explicit expression for the early exercise boundary (EEB), which is an optimal level of critical asset value where early exercise occurs. Due to the lack of closed-form formulas for American option values, many approximate and/or numerical solutions have been developed so far.

The approximations previously established are summarized as follows:

- **interpolation approximations:**
  Johnson (1983); Blomeyer (1986); Broadie and Detemple (1996); Chen and Yeh (2002); Chung and Chang (2007); Li (2010b)
- **compound-options approximations:**
  Geske and Johnson (1984); Bunch and Johnson (1992); Ho et al. (1994)
- **quadratic approximations:**
  MacMillan (1986); Barone-Adesi and Whaley (1987); Barone-Adesi and Elliot (1991); Allegretto et al. (1995); Ju and Zhong (1999); Wong and Xu (2001); Andrikopoulos (2007); Li (2010a)

*This is an early draft of my paper Kimura (2019) in preparation. All of the proofs and computational results are omitted here.

- **approximations based on the integral representation:**
  Kim (1990); Ju (1998); Bunch and Johnson (2000); Little et al. (2000); AitSahlia and Lai (2001); Zhu and He (2007)
- **barrier-options approximations:**
  Bjerksund and Stensland (1993, 2002); Omberg (1987); Ingersoll (1998); Nunes (2007); Kimura (2018)
- **approximations based on the Laplace transform:**
  Carr (1998); Zhu (2006, 2011); Kimura (2012, 2014)

The first three class of these approximations are frequently called *analytical approximations*, where the word *analytical* has been locally used among researchers in finance. They have interpreted "analytical approximations" typically as solutions where a few standard numerical tools such as a root-finding algorithm (e.g., Newton-Raphson) or a simple one-dimensional numerical integration are required for just one or two times. However, solutions in which a Newton-Raphson algorithm is called repeatedly are excluded. There is no clear distinction between "analytical approximations" and "numerical methods", which means that the word "analytical" is insignificant.

From the viewpoint of option holders, our focus is on the EEB approximation, because a simple and accurate approximation is useful in their quick decision-making. The purpose of this paper is to provide new interpolation approximations for vanilla American put/call options written on a dividend-paying asset.

2 Preliminaries

2.1 Formulation

Assume that the capital market is well-defined and follows the efficient market hypothesis. Let $(S_t)_{t\geq 0}$ be the asset price process, $\delta \geq 0$ the continuous dividend rate, $\sigma > 0$ the volatility of the asset returns, and $r > 0$ the risk-free interest rate. Assume that the asset price $(S_t)_{t\geq 0}$ is a lognormal process

$$\frac{\mathrm{d}S_t}{S_t} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geq 0, \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$. $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to W and the probability measure $\mathbb{P}$ is chosen risk-neutrally so that the asset has mean rate of return r.

We consider an American *put* option written on the asset price, which has a maturity $T > 0$ and strike price $K > 0$. Let

$$P \equiv P(t, S_t) = P(t, S_t; K, r, \delta), \quad 0 \leq t \leq T$$

denote the value of the American put option at time t. Similarly, let

$$C \equiv C(t, S_t) = C(t, S_t; K, r, \delta), \quad 0 \leq t \leq T$$

denote the value of the associated American call option with the same parameters as those in the put option. McDonald and Schroder (1998) proved that a *symmetric* relation holds between the American put and call values, i.e.,

$$C(t, S_t; K, r, \delta) = P(t, K; S_t, \delta, r). \tag{2}$$

See Carr and Chesney (1997) for another symmetric relation in more general settings.

From the theory of arbitrage pricing, the fair value of the American put option at time t is given by solving an *optimal stopping problem*

$$P(t, S_t) = \operatorname*{ess\,sup}_{\tau_e \in [t,T]} \mathbb{E}\left[\mathrm{e}^{-r(\tau_e - t)}(K - S_{\tau_e})^+ \,\middle|\, \mathcal{F}_t \right], \quad 0 \le t \le T, \tag{3}$$

where τ_e is a stopping time of the filtration $(\mathcal{F}_t)_{t\ge 0}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. Solving the optimal stopping problem (3) is equivalent to find the points (t, S_t) for which early exercise is optimal. Let $\mathcal{S}$ and $\mathcal{C}$ denote the *stopping region* and *continuation region*, respectively. The stopping region $\mathcal{S}$ is defined by

$$\mathcal{S} = \left\{ (t, S) \in [0, T] \times \mathbb{R}_+ \mid P(t, S) = (K - S)^+ \right\}.$$

Of course, the continuation region $\mathcal{C}$ is the complement of $\mathcal{S}$ in $[0, T] \times \mathbb{R}_+$. The boundary that separates $\mathcal{S}$ from $\mathcal{C}$ is the EEB, which is defined by

$$B_p(t) = \sup \left\{ S \in \mathbb{R}_+ \mid P(t, S) = (K - S)^+ \right\}, \quad 0 \le t \le T.$$

Similarly, define the EEB for the associated American call option by

$$B_c(t) = \inf \left\{ S \in \mathbb{R}_+ \mid C(t, S) = (S - K)^+ \right\}, \quad 0 \le t \le T.$$

Between these two boundaries $B_p(t) \equiv B_p(t; r, \delta)$ and $B_c(t) \equiv B_c(t; r, \delta)$, Carr and Chesney (1997) derived a simple symmetric relation such that

$$B_c(t; r, \delta) B_p(t; \delta, r) = K^2, \quad 0 \le t \le T. \tag{4}$$

McKean (1965) showed that the American put value and the early exercise boundary can be obtained by jointly solving a *free boundary problem*, which is specified by the so-called Black-Scholes-Merton partial differential equation (PDE)

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - \delta) S \frac{\partial P}{\partial S} - rP = 0, \quad S > B_p(t), \tag{5}$$

together with the boundary conditions

$$\left| \begin{aligned} & \lim_{S\uparrow\infty} P(t, S) = 0 \\ & \lim_{S\downarrow B_p(t)} P(t, S) = K - B_p(t) \\ & \lim_{S\downarrow B_p(t)} \frac{\partial P}{\partial S} = -1, \end{aligned} \right. \tag{6}$$

and the *terminal condition*

$$P(T, S) = (K - S)^+. \tag{7}$$

The second condition in (6) is often called the *value-matching condition*, while the third condition is called the *smooth-pasting* or *high-contact condition*.

2.2 Laplace-Carson transforms

It is sometimes convenient to work with the equations where the current time t is replaced by the time to expiry $\tau \equiv T - t$. For the sake of notational convenience, we write

$$\begin{aligned}\widetilde{S}_\tau &\equiv S_{T-\tau} = S_t\\ \widetilde{B}_p(\tau) &\equiv B_p(T-\tau) = B_p(t)\\ \widetilde{B}_c(\tau) &\equiv B_c(T-\tau) = B_c(t),\end{aligned}$$

and we refer to $(\widetilde{S}_\tau)_{\tau\geq 0}$ as the *backward running process* of $(S_t)_{t\geq 0}$.

The put price for the backward running process $\widetilde{P}(\tau, \widetilde{S}_\tau)$ satisfies the PDE

$$-\frac{\partial \widetilde{P}}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widetilde{P}}{\partial S^2} + (r-\delta)S\frac{\partial \widetilde{P}}{\partial S} - r\widetilde{P} = 0, \quad S > \widetilde{B}_p(\tau), \tag{8}$$

with the boundary conditions

$$\left|\begin{array}{l}\lim\limits_{S\uparrow\infty} \widetilde{P}(\tau, S) = 0\\ \lim\limits_{S\downarrow \widetilde{B}_p(\tau)} \widetilde{P}(\tau, S) = K - \widetilde{B}_p(\tau)\\ \lim\limits_{S\downarrow \widetilde{B}_p(\tau)} \dfrac{\partial \widetilde{P}}{\partial S} = -1,\end{array}\right. \tag{9}$$

and the *initial condition*

$$\widetilde{P}(0, S) = (K - S)^+. \tag{10}$$

In order to value American vanilla options, Carr (1998) developed a fast and accurate method, which is called the *randomization* approach. The name "randomization" originates in its initial step of randomizing the maturity date T by an exponentially distributed random variable with mean $\lambda^{-1} = T$. Mathematically, the randomization approach is closely related to the Laplace-Carson transform (LCT): Let $f(\tau)$ be a function of exponential order, i.e., there exist some constants M and $\lambda_0 \geq 0$, for which $|f(\tau)| \leq M\mathrm{e}^{\lambda_0 \tau}$ for all $\tau \geq 0$. Then, the LCT $f^*(\lambda)$ of a function $f(\tau)$ is defined by

$$f^*(\lambda) \equiv \mathcal{LC}[f(\tau)](\lambda) = \int_0^\infty \lambda \mathrm{e}^{-\lambda\tau} f(\tau)\mathrm{d}\tau,$$

where λ is a complex number with $\mathrm{Re}(\lambda) > \lambda_0$.

Since the time-reversed quantities $\widetilde{P}(\tau, S)$ and $\widetilde{B}_p(\tau)$ are bounded functions of $\tau \in \mathbb{R}_+$, we can define the LCTs of these functions for $\mathrm{Re}(\lambda) > 0$. The randomization approach can be interpreted to mean that the LCT $P^*(\lambda, S) = \mathcal{LC}[\widetilde{P}(\tau, S)](\lambda)$ is an exponentially weighted sum (integral) of the time-reversed value $\widetilde{P}(\tau, S)$ for (infinitely many) different values of the maturity $T = \lambda^{-1} \in \mathbb{R}_+$, which makes $\widetilde{P}(\tau, S)$ and $P^*(\lambda, S)$ well defined for $\tau \geq 0$ and $\lambda > 0$, respectively.

From (8)–(10), the LCT $P^*(\lambda, S)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 P^*}{\mathrm{d}S^2} + (r-\delta)S\frac{\mathrm{d}P^*}{\mathrm{d}S} - (\lambda + r)P^* + \lambda(K-S)^+ = 0, \quad S > B_p^*, \tag{11}$$

together with the boundary conditions

$$\left|\begin{array}{l} \lim_{S\uparrow\infty} P^*(\lambda,S)=0 \\ \lim_{S\downarrow B_p^*} P^*(\lambda,S)=K-B_p^* \\ \lim_{S\downarrow B_p^*} \dfrac{\mathrm{d}P^*}{\mathrm{d}S}=-1, \end{array}\right. \tag{12}$$

where $B_p^* \equiv B_p^*(\lambda) = \mathcal{LC}[\widetilde{B}_p(\tau)](\lambda)$. Solving this boundary-value problem and the corresponding problem for the call case, we have

Proposition 1 (Kimura (2010, 2014))**.** *The LCTs $B_p^*(\lambda)$ and $B_c^*(\lambda)$ are given by unique positive solutions of the functional equations*

$$\lambda\left(\frac{B_p^*}{K}\right)^{\theta_1} + \delta\theta_1 \frac{B_p^*}{K} + r(1-\theta_1) = 0,$$

and

$$\lambda\left(\frac{B_c^*}{K}\right)^{\theta_2} + \delta\theta_2 \frac{B_c^*}{K} + r(1-\theta_2) = 0,$$

respectively, where the parameters $\theta_i \equiv \theta_i(\lambda)$ $(i=1,2,\ \theta_1>1,\ \theta_2<0)$ are two roots of the quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + (r-\delta-\tfrac{1}{2}\sigma^2)\theta - (\lambda+r) = 0,$$

i.e., for $i=1,2$,

$$\theta_i(\lambda) = \frac{1}{\sigma^2}\left\{-(r-\delta-\tfrac{1}{2}\sigma^2) - (-1)^i\sqrt{(r-\delta-\tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+r)}\right\}. \tag{13}$$

Proposition 1 enables us to prove some known asymptotic properties of the time-reverse EEB as $\tau\to 0$ (Kim, 1990) and $\tau\to\infty$ (McKean, 1965; Merton, 1973):

Proposition 2.

$$\widetilde{B}_p(0) \equiv \lim_{\tau\to 0}\widetilde{B}_p(\tau) = \min\left(1,\frac{r}{\delta}\right)K,$$

and

$$\widetilde{B}_c(0) \equiv \lim_{\tau\to 0}\widetilde{B}_c(\tau) = \max\left(1,\frac{r}{\delta}\right)K.$$

Proposition 3.

$$\widetilde{B}_p(\infty) \equiv \lim_{\tau\to\infty}\widetilde{B}_p(\tau) = \frac{r}{\delta}\frac{\theta_1^\circ - 1}{\theta_1^\circ}K = \frac{\theta_2^\circ}{\theta_2^\circ - 1}K,$$

and

$$\widetilde{B}_c(\infty) \equiv \lim_{\tau\to\infty}\widetilde{B}_c(\tau) = \frac{r}{\delta}\frac{\theta_2^\circ - 1}{\theta_2^\circ}K = \frac{\theta_1^\circ}{\theta_1^\circ - 1}K,$$

where $\theta_i^\circ \equiv \theta_i(0)$ $(i=1,2)$.

Proposition 4 (Kimura (2012)). *For sufficiently large* $\tau > 0$,

$$\frac{\widetilde{B}_p(\tau)}{\widetilde{B}_p(\infty)} \sim \begin{cases} 1 + \dfrac{1}{\theta_1^\circ - 1} \exp\Big\{-\frac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)\tau\Big\}, & r < \delta \\[2ex] 1 - \dfrac{1}{\theta_2^\circ} \exp\Big\{-\frac{1}{2}\sigma^2(1-\theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)\tau\Big\}, & r \geq \delta, \end{cases}$$

and

$$\frac{\widetilde{B}_c(\tau)}{\widetilde{B}_c(\infty)} \sim \begin{cases} 1 + \dfrac{1}{\theta_2^\circ - 1} \exp\Big\{-\frac{1}{2}\sigma^2\theta_2^\circ(\theta_2^\circ - \theta_1^\circ)\tau\Big\}, & r > \delta \\[2ex] 1 - \dfrac{1}{\theta_1^\circ} \exp\Big\{-\frac{1}{2}\sigma^2(1-\theta_1^\circ)(\theta_2^\circ - \theta_1^\circ)\tau\Big\}, & r \leq \delta. \end{cases}$$

3 A Square-root Exponential Approximation

Carr et al. (1992) proposed an approximation form of $\widetilde{B}(\tau)$ for $\delta = 0$ that is an exponentially weighted average of the strike price K and the perpetual boundary, i.e., for $\alpha > 0$

$$\widetilde{B}_p(\tau) \approx K\mathrm{e}^{-\alpha\sqrt{\tau}} + \widetilde{B}_p(\infty)\left(1 - \mathrm{e}^{-\alpha\sqrt{\tau}}\right), \quad \tau > 0.$$

However, they have not mentioned any specific definition of the decay rate α. Also, for $\delta > 0$, it should have the form

$$\begin{aligned} \widetilde{B}_p(\tau) &\approx \widetilde{B}_p(0)\mathrm{e}^{-\alpha\sqrt{\tau}} + \widetilde{B}_p(\infty)\left(1 - \mathrm{e}^{-\alpha\sqrt{\tau}}\right) \\ &= \widetilde{B}_p(\infty) + \left(\widetilde{B}_p(0) - \widetilde{B}_p(\infty)\right)\mathrm{e}^{-\alpha\sqrt{\tau}}. \end{aligned} \tag{14}$$

In order to determine α definitely, we need one more extra condition on $\widetilde{B}_p(\tau)$ other than its asymptotic properties.

For such a condition on $\widetilde{B}_p(\tau)$, we choose the value $\widetilde{B}_p(T) = B_p(0)$ at $\tau = T$ ($t = 0$), which can be approximated by

$$\widetilde{B}_p(T) \approx B_p^*(T^{-1}),$$

due to the randomization approach principle. If we set $\tau = T$ in the square-root exponential approximation (14), then we have

$$\widetilde{B}_p(T) \approx \widetilde{B}_p(\infty) + \left(\widetilde{B}_p(0) - \widetilde{B}_p(\infty)\right)\mathrm{e}^{-\alpha\sqrt{T}},$$

from which we obtain

$$-\alpha \approx \frac{1}{\sqrt{T}} \log\left(\frac{\widetilde{B}_p(T) - \widetilde{B}_p(\infty)}{\widetilde{B}_p(0) - \widetilde{B}_p(\infty)}\right).$$

Hence,

$$\mathrm{e}^{-\alpha\sqrt{\tau}} \approx \left(\frac{\widetilde{B}_p(T) - \widetilde{B}_p(\infty)}{\widetilde{B}_p(0) - \widetilde{B}_p(\infty)}\right)^{\sqrt{\frac{\tau}{T}}} \approx \left(\frac{B_p^*(T^{-1}) - \widetilde{B}_p(\infty)}{\widetilde{B}_p(0) - \widetilde{B}_p(\infty)}\right)^{\sqrt{\frac{\tau}{T}}},$$

so that we obtain for $\tau > 0$

$$\widetilde{B}_p(\tau) \approx \widetilde{B}_p(\infty) + \left(\widetilde{B}_p(0) - \widetilde{B}_p(\infty)\right) \left(\frac{B_p^*(T^{-1}) - \widetilde{B}_p(\infty)}{\widetilde{B}_p(0) - \widetilde{B}_p(\infty)}\right)^{\sqrt{\frac{\tau}{T}}}. \tag{15}$$

Applying the same argument as above to the call case, we have the main theorem:

Theorem 1.

$$\frac{\widetilde{B}_p(\tau)}{\widetilde{B}_p(\infty)} \approx \begin{cases} 1 + \dfrac{1}{\theta_1^\circ - 1}\left[(\theta_1^\circ - 1)\left\{\dfrac{\beta_1}{\widetilde{B}_p(\infty)} - 1\right\}\right]^{\sqrt{\frac{\tau}{T}}}, & r < \delta \\ 1 - \dfrac{1}{\theta_2^\circ}\left[-\theta_2^\circ\left\{\dfrac{\beta_1}{\widetilde{B}_p(\infty)} - 1\right\}\right]^{\sqrt{\frac{\tau}{T}}}, & r \geq \delta, \end{cases} \tag{16}$$

and

$$\frac{\widetilde{B}_c(\tau)}{\widetilde{B}_c(\infty)} \approx \begin{cases} 1 + \dfrac{1}{\theta_2^\circ - 1}\left[(\theta_2^\circ - 1)\left\{\dfrac{\beta_2}{\widetilde{B}_c(\infty)} - 1\right\}\right]^{\sqrt{\frac{\tau}{T}}}, & r > \delta \\ 1 - \dfrac{1}{\theta_1^\circ}\left[-\theta_1^\circ\left\{\dfrac{\beta_2}{\widetilde{B}_c(\infty)} - 1\right\}\right]^{\sqrt{\frac{\tau}{T}}}, & r \leq \delta, \end{cases} \tag{17}$$

where the perpetual values $\widetilde{B}_p(\infty)$ and $\widetilde{B}_c(\infty)$ are given in Proposition 3, and β_i $(i = 1, 2)$ is a unique positive solution of the functional equation

$$\frac{1}{T}\left(\frac{\beta_i}{K}\right)^{\theta_i^\star} + \delta\theta_i^\star \frac{\beta_i}{K} + r(1 - \theta_i^\star) = 0, \tag{18}$$

with $\theta_i^\star \equiv \theta_i(T^{-1})$ $(i = 1, 2)$.

Acknowledgments

I am grateful to seminar participants for their discussion in the RIMS workshop QMF2018 at Kyoto University and the MMDS Nakanoshima Workshop 2018 at Osaka University. This research was supported in part by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University, and by JSPS KAKENHI Grant Number 16K00037.

References

AitSahlia, F. and Lai, T. (2001). Exercise boundaries and efficient approximations to American option prices and hedge parameters. *Journal of Computational Finance*, **4**, 85–103.

Allegretto, W., Barone-Adesi, G., and Elliott, R.J. (1995). Numerical evaluation of the critical price and American options. *European Journal of Finance*, **1**, 69–78.

Andrikopoulos, A. (2007). On the quadratic approximation to the value of American options: a note. *Applied Financial Economics Letters*, **3**, 313–317.

Barone-Adesi, G. and Whaley, R.E. (1987). Efficient analytic approximation of American option values. *Journal of Finance*, **42**, 301–320.

Barone-Adesi, G. and Elliot, R.J. (1991). Approximations for the values of American options. *Stochastic Analysis and Applications*, **9**, 115-131.

Bjerksund, P. and Stensland, G. (1993). Closed-form approximation of American options. *Scandinavian Journal of Management*, **9**, Supplement, S88–S99.

Bjerksund, P. and Stensland, G. (2002). Closed form valuation of American options. working paper, NHH.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, **81**, 637–654.

Blomeyer, E.C. (1986). An analytic approximation for the American put price for options on stocks with dividends. *Journal of Financial and Quantitative Analysis*, **21**, 229–233.

Broadie, M. and Detemple, J. (1996). American option valuation: new bounds, approximations, and a comparison of existing methods. *Review of Financial Studies*, **9**, 1211–1250.

Bunch, D.S. and Johnson, H. (1992). A simple and numerically efficient valuation method for American puts using a modified Geske-Johnson approach. *Journal of Finance*, **47**, 809–816.

Bunch, D.S. and Johnson, H. (2000). The American put option and its critical stock price. *Journal of Finance*, **55**, 2333–2356.

Carr, P. (1998). Randomization and the American put. *Review of Financial Studies*, **11**, 597–626.

Carr, P., Jarrow, R. and Myneni, R. (1992). Alternative characterizations of American puts. *Mathematical Finance*, **2**, 87–106.

Carr, P. and Chesney, M. (1997). American put call symmetry. working paper, Morgan Stanley.

Chen, R.-R. and Yeh, S.-K. (2002). Analytical upper bounds for American option prices. *Journal of Financial and Quantitative Analysis*, **37**, 117–135.

Chung, S.-L. and Chang, H.-C. (2007). Generalized analytical upper bounds for American option prices. *Journal of Financial and Quantitative Analysis*, **42**, 209–228.

Chung, S.-L., Hung, M.-W. and Wang, J.-Y. (2010). Tight bounds on American option prices. *Journal of Banking & Finance*, **34**, 77–89.

Ekström, E. (2004). Convexity of the optimal stopping boundary for the American put option. *Journal of Mathematical Analysis and Applications*, **299**, 147–156.

Evans, J.D., Kuske, R. and Keller, J.B. (2002). American options on assets with dividends near expiry. *Mathematical Finance*, **12**, 219–237.

Geske, R. and Johnson, H.E. (1984). The American put option valued analytically. *Journal of Finance*, **39**, 1511–1524.

Ho, T.S., Stapleton, R.C. and Subrahmanyam, M.G. (1994). A simple technique for the valuation and hedging of American options. *Journal of Derivatives*, **2**, 52–66.

Ingersoll, J.E., Jr. (1998). Approximating American options and other financial contracts using barrier derivatives. *Journal of Computational Finance*, **2**, 85–112.

Jacka, S.D. (1991). Optimal stopping and the American put. *Mathematical Finance*, **1**, 1–14.

Johnson, H. (1983). An analytical approximation for the American put price. *Journal of Financial and Quantitative Analysis*, **18**, 141–148.

Ju, N. (1998). Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Review of Financial Studies*, **11**, 627–646.

Ju, N. and Zhong, R. (1999). An approximate formula for pricing American options. *Journal of Derivatives*, **7**, 31–40.

Kim, I.J. (1990). The analytical valuation of American options. *Review of Financial Studies*, **3**, 547–572.

Kimura, T. (2010). Alternative randomization for valuing American options. *Asia-Pacific Journal of Operational Research*, **27**, 167–187.

Kimura, T. (2012). Approximating the early exercise boundary for American-style options. *RIMS Kôkyûroku*, Research Institute for Mathematical Sciences, Kyoto University, No. 1818, 1–16.

Kimura, T. (2014). Approximating the early exercise boundary for American-style options. *RIMS Kôkyûroku*, Research Institute for Mathematical Sciences, Kyoto University, No. 1886, 1–15.

Kimura, T. (2018). An approximate barrier option model for valuing executive stock options. *Journal of the Operations Research Society of Japan*, **61**, 110–131.

Kimura, T. (2019). A square-root exponential approximation for the early exercise boundary of American options. In preparation.

Li, M. (2010a). Analytical approximations for the critical stock prices of American options: a performance comparison. *Review of Derivatives Research*, **13**, 75–99.

Li, M. (2010b). A quasi-analytical interpolation method for pricing American options under general multi-dimensional diffusion processes. *Review of Derivatives Research*, **13**, 177–217.

Little, T., Pant, V. and Hou, C. (2000). A new integral representation of the early exercise boundary for American put options. *Journal of Computational Finance*, **3**, 73–96.

MacMillan, L.W. (1986). Analytic approximation for the American put prices. *Advances in Futures and Options Research*, **1**, 119–139.

McDonald, R. and Schroder, M. (1998). A parity result for American options. *Journal of Computational Finance*, **1**, 5–13.

McKean, H.P. (1965). Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review*, **6**, 32–39.

Merton, R.C. (1973). The theory of rational option pricing. *Bell Journal of Economics and Management Science*, **4**, 141–183.

Nunes, J.P.V. (2007). A general characterization of the early exercise premium. EFMA Conference, Vienna.

Omberg, E. (1987). The valuation of American put options with exponential exercise policies. *Advances in Futures and Options Research*, **2**, 117–142.

van Moerbeke, P. (1976). On optimal stopping and free boundary problems. *Archive for Rational*

Mechanics and Analysis, **60**, 101–148.

Wong, W.K. and Xu, K. (2001). Refining the quadratic approximation formula for an American option. *International Journal of Theoretical and Applied Finance*, **4**, 773–781.

Zhu, S.-P. (2006). A new analytical approximation formula for the optimal exercise boundary of American put options. *International Journal of Theoretical and Applied Finance*, **9**, 1141–1177.

Zhu, S.-P. (2011). A simple approximation formula for calculating the optimal exercise boundary of American puts. *Journal of Applied Mathematics and Computing*, **37**, 611–623.

Zhu, S.-P. and He, Z.-W. (2007). Calculating the early exercise boundary of American put options with an approximation formula. *International Journal of Theoretical and Applied Finance*, **10**, 1203–1227.

Department of Civil, Environmental & Applied Systems Engineering
Faculty of Environmental & Urban Engineering
Kansai University
Suita 564-8680
JAPAN
E-mail address: t.kimura@kansai-u.ac.jp

関西大学・環境都市工学部　木村 俊一