COMPOSITE ITERATIVE METHODS FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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ABSTRACT. In this paper, we introduce two composite iterative methods (one implicit method and one explicit method) for finding a common element of the solution set of a general system of variational inequalities for continuous monotone mappings and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. First, this system of variational inequalities is proven to be equivalent to a fixed point problem of nonexpansive mapping. Then we establish strong convergence of the sequence generated by the proposed iterative methods to a common element of the solution set and the fixed point set, which is the unique solution of a certain variational inequality.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let $S : C \to C$ be a self-mapping on C. We denote by $Fix(S)$ the set of fixed points of S.

A mapping $F : C \to H$ is called *monotone* if

$$\langle x - y, Fx - Fy\rangle \geq 0, \quad \forall x,\ y \in C.$$

and F is called *α-inverse-strongly monotone* (see [5, 11]) if there exists a positive real number α such that

$$\langle x - y, Fx - Fy\rangle \geq \alpha\|Fx - Fy\|^2, \quad \forall x,\ y \in C.$$

The class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

A mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C.$$

and T is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings as a subclass.

Let F be a nonlinear mapping of C into H. The variational inequality problem (VIP) is to find a $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

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The results presented in this lecture are collected mainly from the work [8] by the author of this report.

We denote the set of solutions of VIP(1.1) by $VI(C, F)$. The variational inequality problem has been extensively studied in the literature; see [3, 5, 7, 10, 11, 14, 15, 17, 19] and the references therein.

In 2008, Ceng *et al.* [2] considered the following general system of variational inequalities:

$$\begin{cases} \langle \lambda F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \\ \langle \nu F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

where F_1 and F_2 are an α-inverse-strongly monotone mapping and a β-inverse-strongly monotone mapping, respectively; and $\lambda \in (0, 2\alpha)$ and $\nu \in (0, 2\beta)$ are two constants. For finding an element $Fix(S) \cap \Gamma$, where $S : C \to C$ is a nonexpansive mapping and Γ is the solution set of the problem (1.2), they introduced a relaxed extragradient method ([9]) and proved strong convergence to a common element of $Fix(S) \cap \Gamma$.

In 2016, Alofi *et al.* [1] also considered the problem (1.2) coupled with the fixed point problem, and introduced two composite iterative algorithms (one implicit algorithm and one explicit algorithm) based on Jung's composite iterative method [6] to find an element $Fix(T) \cap \Gamma$, where $T : C \to C$ is a k-strictly pseudocontractive mapping and Γ is the solution set of the problem (1.2), and showed strong convergence to a common element of $Fix(T) \cap \Gamma$. The following problems arise:

Question 1. Can we extend the class of inverse-strongly monotone mappings in [1, 2] to the more general class of continuous monotone mappings ?

Question 2. Can we extend the class of nonexpansive mappings in [2] or the class of strictly pseudocontractive mappings in [1] to the more general class of pseudocontractive mappings ?

In this paper, in order to give the affirmative answers to the above two questions, we consider a general system of variational inequalities slightly different from the problem (1.2). More precisely, we introduce the following general system of variational inequalities (GSVI) for two continuous monotone mappings F_1 and F_2 of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda F_1 x^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C \\ \langle \nu F_2 y^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

where $\lambda > 0$ and ν are two constants. The solution set of GSVI(1.3) is denoted by Ω. First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping. Second, by using Jung's composite iterative algorithms [6], we introduce a composite implicit iterative algorithm and a composite explicit iterative algorithm for finding a common element of $\Omega \cap Fix(T)$, where T is a continuous pseudocontractive mapping. Then we establish strong convergence of these two composite iterative algorithms to a common element of $\Omega \cap Fix(T)$, which is the unique solution of a certain variational inequality related to a minimization problem. As a direct consequence, we obtain strong convergence to a common element of $VI(C, F) \cap Fix(T)$, where F is a continuous monotone mapping.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that $P_C(x)$ is characterized by the property:

$$u = P_C(x) \Longleftrightarrow \langle x-u, u-y\rangle \geq 0, \quad \forall x \in H,\ y \in C. \tag{2.1}$$

In a Hilbert space H, we have

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x,\ y \in H. \tag{2.2}$$

We recall that

(1) an operator A is said to be *strongly positive* on H if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H;$$

(ii) a mapping $V : C \to H$ is said to be *l-Lipschitzian* if there exists a constant $l \geq 0$ such that

$$\|Vx - Vy\| \leq l\|x-y\|, \quad \forall x,\ y \in C;$$

(iii) a mapping $G : C \to H$ is said to be *ρ-strongly monotone* if there exists a constant $\rho > 0$ such that

$$\langle Gx - Gy, x-y\rangle \geq \rho\|x-y\|^2, \quad \forall x,\ y \in C.$$

The following lemma is an immediate consequence of an inner product.

Lemma 2.1. *In a real Hilbert space H, there holds the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,\ y \in H.$$

We need the following lemmas for the proof of our main results.

Lemma 2.2 ([16])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\omega_n)s_n + \omega_n\delta_n + \nu_n, \quad \forall n \geq 1,$$

where $\{\omega_n\}$, $\{\delta_n\}$, and $\{\nu_n\}$ satisfy the following conditions:

(i) $\{\omega_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty}\omega_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1-\omega_n) = 0$*;*

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}\omega_n|\delta_n| < \infty$;

(iii) $\nu_n \geq 0$ $(n \geq 1)$, $\sum_{n=1}^{\infty}\nu_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([4])**.** (Demiclosedness principle) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I-S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x^*$ and $(I-S)x_n \to y$, then $(I-S)x^* = y$.*

Lemma 2.4 ([12])**.** *Let H be a real Hilbert space. Let $A : H \to H$ be a strongly positive bounded linear operator with a constant $\overline{\gamma} > 1$. Then*

$$\langle (A-I)x - (A-I)y, x-y\rangle \geq (\overline{\gamma}-1)\|x-y\|^2, \quad \forall x,\ y \in C.$$

That is, $A-I$ is strongly monotone with a constant $\overline{\gamma}-1$.

Lemma 2.5 ([12])**.** *Assume that A is a strongly positive bounded linear operator on H with a coefficient $\overline{\gamma} > 0$ and $0 < \zeta \leq \|A\|^{-1}$. Then $\|I-\zeta A\| \leq 1-\zeta\overline{\gamma}$.*

Lemma 2.6 ([17])**.** *Let H be a real Hilbert space. Let $G : H \to H$ be a ρ-Lipschitzian and η-strongly monotone mapping with constants ρ, $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 < t < \sigma \leq 1$. Then $S := \sigma I - t\mu G : H \to H$ is a contractive mapping with constant $\sigma - t\tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\rho^2)}$.*

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [18], respectively.

Lemma 2.7 ([18]). *Let C be a closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Fz\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $F_r : H \to C$ by

$$F_r x = \left\{ z \in C : \langle y - z, Fz\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *F_r is single-valued;*
(ii) *F_r is firmly nonexpansive, that is,*

$$\|F_r x - F_r y\|^2 \leq \langle x - y, F_r x - F_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(F_r) = \mathrm{VI}(C, F)$;*
(iv) *$\mathrm{VI}(C, F)$ is a closed convex subset of C.*

Lemma 2.8 ([18]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$;*
(iv) *$Fix(T)$ is a closed convex subset of C.*

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subspace subset of H;
- $A : C \to C$ is a strongly positive linear bounded self-adjoint operator with a constant $\overline{\gamma} \in (1, 2)$;
- $V : C \to C$ is l-Lipschitzian with constant $l \in [0, \infty)$;
- $G : C \to C$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- Constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- F_1 and $F_2 : C \to H$ are continuous monotone mapping;
- Ω is the solution set of GSVI (1.3) for F_1 and F_2;
- $F_{1\lambda} : H \to C$ is a mapping defined by

$$F_{1\lambda} x = \left\{ z \in C : \langle y - z, F_1 z\rangle + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$

  for $\lambda > 0$;

- $F_{2\nu} : H \to C$ is a mapping defined by
$$F_{2\nu}x = \left\{ z \in C : \langle y - z, F_2 z\rangle + \frac{1}{\nu}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$
for $\nu > 0$;
- $R : H \to C$ is a mapping defined by $Rx = F_{1\lambda}F_{2\nu}x$ for each $x \in H$;
- $T : C \to C$ is a continuous pseudocontractive mapping such that $Fix(T) \neq \emptyset$;
- $T_{r_t} : H \to C$ is a mapping defined by
$$T_{r_t}x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r_t}\langle y - z, (1 + r_t)z - x\rangle \leq 0, \quad \forall y \in C \right\}$$
for $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $T_{r_n} : H \to C$ is a mapping defined by
$$T_{r_n}x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0, \quad \forall y \in C \right\}$$
for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $\Omega \cap Fix(T) \neq \emptyset$.

By Lemma 2.7 and Lemma 2.8, we note that $F_{1\lambda}$, $F_{2\nu}$, T_{r_t}, and T_{r_n} are nonexpansive, and $Fix(T_{r_n}) = Fix(T) = Fix(T_{r_t})$.

First, we prove that the problem (1.3) is equivalent to a fixed point problem of nonexpansive mapping.

Proposition 3.1. *Let C be a closed convex subset of a real Hilbert space H. For given x^*, $y^* \in C$, (x^*, y^*) is a solution of* GSVI(1.3) *for continuous monotone mappings F_1 and F_2 if and only if x^* is a fixed point of the mapping $R : H \to C$ defined by*
$$Rx = F_{1\lambda}F_{2\nu}x, \quad \forall x \in H,$$
where $y^* = F_{2\nu}x^*$.

First, we introduce the following composite algorithm that generates a net $\{x_t\}$ in an implicit way:

$$x_t = (I - \theta_t A)T_{r_t}Rx_t + \theta_t[t\gamma V x_t + (I - t\mu G)T_{r_t}Rx_t], \tag{3.1}$$

where $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ and $\theta_t \in (0, \|A\|^{-1}]$.

We summarize the basic properties of $\{x_t\}$, which can be proved by the same method as in [6].

Proposition 3.2. *Let $\{x_t\}$ be defined via* (3.1). *Then*

(i) $\{x_t\}$ *is bounded for* $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$;
(ii) $\lim_{t\to 0} \|x_t - T_{r_t}Rx_t\| = 0$ *provided* $\lim_{t\to 0} \theta_t = 0$;
(iii) $\lim_{t\to 0} \|x_t - y_t\| = 0$, *where* $y_t = t\gamma V x_t + (I - t\mu G)T_{r_t}Rx_t$;
(iv) $\lim_{t\to 0} \|x_t - Rx_t\| = 0$;
(v) x_t *defines a continuous path from* $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ *into* H *provided* $\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \|A\|^{-1}]$ *is continuous, and* $r_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \infty)$ *is continuous.*

We obtain the following theorem for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2) below.

Theorem 3.3. *Let the net $\{x_t\}$ be defined via* (3.1). *If $\lim_{t\to 0}\theta_t = 0$, then x_t converges strongly to $\widetilde{x}$ in $\Omega \cap Fix(T)$ as $t \to 0$, which solves the variational inequality*

$$\langle (A-I)\widetilde{x}, \widetilde{x}-p\rangle \le 0, \quad \forall p \in \Omega \cap Fix(T), \tag{3.2}$$

Equivalently, we have

$$P_{\Omega\cap Fix(T)}(2I-A)\widetilde{x} = \widetilde{x}.$$

Now, we propose the following composite algorithm which generates a sequence in an explicit way:

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) T_{r_n} R x_n, \\ x_{n+1} = (I - \beta_n A) T_{r_n} R x_n + \beta_n y_n, \quad \forall n \ge 0, \end{cases} \tag{3.3}$$

where $\{\alpha_n\} \in [0,1]$; $\{\beta_n\} \subset (0,1]$; $\{r_n\} \subset (0,\infty)$; and $x_0 \in C$ is an arbitrary initial guess, and establish strong convergence of this sequence to $\widetilde{x} \in \Omega \cap Fix(T)$, which is the unique solution of the variational inequality (3.2).

Theorem 3.4. *Let $\{x_n\}$ be the sequence generated by the explicit algorithm* (3.3). *Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following conditions:*

(C1) *$\{\alpha_n\} \subset [0,1]$ and $\{\beta_n\} \subset (0,1]$, $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$;*
(C2) *$\sum_{n=0}^{\infty} \beta_n = \infty$;*
(C3) *$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $|\beta_{n+1} - \beta_n| \le o(\beta_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition);*
(C4) *$\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$, and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.*

Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega \cap Fix(T)$, which is the unique solution of the variational inequality (3.2).

Taking $G \equiv I$, $\mu = 1$, and $\gamma = 1$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.5. *Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} y_n = \alpha_n V x_n + (1 - \alpha_n) T_{r_n} R x_n, \\ x_{n+1} = (I - \beta_n A) T_{r_n} R x_n + \beta_n y_n, \quad \forall n \ge 0. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the conditions (C1) – (C4) *in Theorem 3.5. Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega \cap Fix(T)$, which is the unique solution of the variational inequality* (3.2).

Remark 3.6. 1) The $\widetilde{x} \in \Omega \cap Fix(T)$ in our results is the unique solution of minimization problem

$$\min_{x\in D} \frac{1}{2}\langle (A-I)x, x\rangle, \tag{3.4}$$

where the constraint set D is $\Omega \cap Fix(T)$. In fact, the variational inequality (3.2) is the optimality condition for the minimization problem (3.4). Thus, for finding an element of $\Omega \cap Fix(T)$, where T is a continuous pseudocontractive mapping, and F_1 and F_2 are continuous monotone mappings, Theorem 3.4, Theorem 3.5 and Corollary 3.6 are new ones different from previous those introduced by some authors (for example, see [1, 2]).

2) Taking $F_1 = F_2 = F$, $\lambda = \nu$ and $x^* = y^*$ in GSVI(1.3) and replacing F_λ by F_{r_n} along with the condition (C4) on $\{r_n\}$, we can obtain a new result, which improves, supplements and develops the corresponding results of [3, 5, 7, 14, 15, 19].

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