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ATTRACTIVE POINT AND CONVERGENCE THEOREMS FOR HYBRID-TYPE SEQUENCES

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ABSTRACT. In this paper, we introduced a broad class of sequences which covers nonexpansive sequences, generalized hybrid sequences and 2-generalized hybrid sequences. Then, we get nonlinear ergodic theorems for the sequences by using the idea of attractive points. Furthermore, we get weak convergence theorems for weakly asymptotically regular sequences.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For a mapping $T : C \to H$, we denote by $F(T)$ the set of *fixed points* of T. In 1975, Baillon [3] proved the following first nonlinear ergodic theorem in a Hilbert space (see also [17]):

Theorem 1.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, $S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T.*

Kohsaka and Takahashi [8], and Takahashi [18] introduced the following nonlinear mappings. A mapping $T : C \to H$ is called nonspreading [8] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [18] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings (see also [5, 9, 21]). In general, nonspreading and hybrid mappings are not continuous mappings. Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced the class

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of λ-hybrid mappings in a Hilbert space. This class contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Kocourek, Takahashi and Yao [6] introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in Hilbert spaces. A mapping $T : C \to E$ is called generalized hybrid [6] if there are real numbers α, β such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. The nonlinear ergodic theorem by Baillon [3] for nonexpansive mapping has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [6]. Takahashi and Takeuchi [19] proved a nonlinear ergodic theorem of Baillon's type without convexity for generalized hybrid mappings by using the concept of attractive points.

Maruyama, Takahashi and Yao [13] defined a broad class of nonlinear mapping called 2-generalized hybrid which contains generalized hybrid mappings in Hilbert spaces. Let C be a nonempty subset of H. A mapping $T : C \to C$ is said to be 2-generalized hybrid [13] if there exist real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\begin{aligned} &\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ &\quad \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2 \end{aligned} \tag{1.1}$$

for all $x, y \in C$. Kondo and Takahashi [10] introduced the following class of nonlinear mapping which covers 2-generalized hybrid mappings in Hilbert spaces. A mapping $T : C \to C$ is said to be normally 2-generalized hybrid [10] if there exist real numbers $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ such that $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned} &\alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ &\quad + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$.

On the other hand, Rouhani [14] introduced the notion of generalized hybrid sequences in Hilbert spaces. A sequence $\{x_n\}$ in H is said to be generalized hybrid sequence if there exist real numbers α, β such that

$$\alpha\|x_{i+1} - x_{j+1}\|^2 + (1-\alpha)\|x_i - x_{j+1}\|^2 \leq \beta\|x_{i+1} - x_j\|^2 + (1-\beta)\|x_i - x_j\|^2$$

for all $i, j \in \mathbb{N}$. Further, Rouhani [15] also introduced the notion of 2-generalized hybrid sequences in Hilbert spaces. A sequence $\{x_n\}$ in H is said to be 2-generalized hybrid sequence if there exist real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$

such that

$$\alpha_1\|x_{i+2}-x_{j+1}\|^2+\alpha_2\|x_{i+1}-x_{j+1}\|^2+(1-\alpha_1-\alpha_2)\|x_i-x_{j+1}\|^2 \\ \leq \beta_1\|x_{i+2}-x_j\|^2+\beta_2\|x_{i+1}-x_j\|^2+(1-\beta_1-\beta_2)\|x_i-x_j\|^2 \qquad (1.3)$$

for all $i,j \geq 0$. Such a sequence $\{x_n\}$ is said to be an $(\alpha_1,\alpha_2,\beta_1,\beta_2)$-generalized hybrid sequence. We note that the class of $(0,\alpha,0,\beta)$-generalized hybrid sequences is the class of generalized hybrid sequences. Rouhani [15] proved a nonlinear ergodic theorem of Baillon's type for the sequences (see also [14]).

In this paper, motivated by Baillon [3], Hojo [4], Kondo and Takahashi [10] and Rouhani[14, 15], we introduced a broad class of sequences which covers nonexpansive sequences, generalized hybrid sequences [14] and 2-generalized hybrid sequences [15]. Then, we get nonlinear ergodic theorems for the sequences by using the idea of attractive points. Furthermore, we get weak convergence theorems for weakly asymptotically regular sequences.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T: C \to H$ is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x,y \in C$. For a mapping $T: C \to H$, we denote by $F(T)$ the set of *fixed points* of T.

Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x-P_C x\| \leq \|x-y\|$$

for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. It is characterized by

$$\langle P_C x-y, x-P_C x\rangle \geq 0$$

for all $y \in C$. See [17] for more details. The following result is well-known; see [17].

Lemma 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$)

to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H. We use the following notations:

$$F_1 = \{q \in H : \text{the sequence} \{\|x_n - q\|\} \text{ is nonincreasing}\};$$
$$F_\ell = \{q \in H : \lim_{n\to\infty} \|x_n - q\| \text{ exists}\}.$$

Lemma 2.2. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H. Then, F_1 and F_ℓ are closed convex subset of H.*

Using a mean, we obtain the following results (see [16]): Let H be a real Hilbert space, let $\{x_n\}$ be a bounded sequence in H and μ be a mean on ℓ^∞. Then, there exists a unique point $z_0 \in H$ $\overline{\text{co}}\{x_n : n \in \mathbb{N}\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A such that

$$(\mu)_n \langle x_n, z\rangle = \langle z_0, z\rangle \quad \forall z \in H.$$

We call such a unique point $z_0 \in H$ the mean values of $\{x_n\}$ for μ.

3. Nonlinear mean ergodic theorems

In this section, motivated by Baillon [3], Hojo [4], Kondo and Takahashi [10] and Rouhani [14, 15], we introduced a broad class of sequences which covers nonexpansive sequences, generalized hybrid sequences [14] and 2-generalized hybrid sequences [15]. Then, we get a strong convergence theorem and a nonlinear mean ergodic theorem for normally 2-generalized hybrid sequences in a Hilbert space H (see [2]). A sequence $\{x_n\}$ in H is said to be normally 2-generalized hybrid if there exist real number $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\begin{aligned} 0 \ge\ & \alpha_2\|x_{i+2} - x_{j+1}\|^2 + \alpha_1\|x_{i+1} - x_{j+1}\|^2 + \alpha_0\|x_i - x_{j+1}\|^2 \\ & + \beta_2\|x_{i+2} - x_j\|^2 + \beta_1\|x_{i+1} - x_j\|^2 + \beta_0\|x_i - x_j\|^2 \end{aligned} \tag{3.1}$$

for all $i, j \in \mathbb{Z}^+$ (see [2]). We call such a sequence an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-normally 2-generalized hybrid sequence. We note that the class of $(1-\alpha, -(1-\beta), \alpha, -\beta, 0, 0)$ -normally 2-generalized hybrid sequences is the class of generalized hybrid sequences (see [14]).

As in the proof of [12, Theorem 4], we have the following theorem (see also [11, 20]).

Theorem 3.1. *Let H be a Hilbert space and let $\{x_n\}$ be an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ -normally 2-generalized hybrid sequence in H. Assume that $\{x_n\}$ is bounded. Then, $\{Px_n\}$ converges strongly to some $v \in H$, where P is the metric projection from H onto F_1.*

Using the idea of attractive points and Theorem 3.1, we get a nonlinear ergodic theorem for normally 2-generalized hybrid sequences (see also [3, 4, 10, 14, 15]).

Theorem 3.2 ([2])**.** *Let H be a Hilbert space. Let $\{x_n\}$ be an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-normally 2-generalized hybrid sequence in H. Assume $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$ and $\alpha_2 + \alpha_1 + \alpha_0 > 0$. Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$*;*
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\left\{\frac{1}{n}\sum_{k=0}^{n-1} x_k\right\}$ *converges weakly to an element* $w \in H$*.*

Moreover, in this case $w = \lim_{n\to\infty} Px_n \in F_1$*.*

Remark 3.3. *In Theorem 3.2, we obtain that $q = w\text{-}\lim_{n\to\infty} S_n$ is the asymptotic center of $\{x_n\}$ (see, for instance* [17]*).*

By Theorem 3.2, we get the following nonlinear ergodic theorem by Rouhani [14] for generalized hybrid sequences (see also [2]).

Theorem 3.4 ([14])**.** *Let H be a Hilbert space and let $\{x_n\}$ be a generalized hybrid sequence. Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$*;*
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\left\{\frac{1}{n}\sum_{k=0}^{n-1} x_k\right\}$ *converges weakly to some* $w \in H$*.*

Moreover, in this case $w = \lim_{n\to\infty} Px_n \in F_1$*.*

By Theorem 3.2, we get the following nonlinear ergodic theorem by Rouhani [15] for 2-generalized hybrid sequences (see also [2]).

Theorem 3.5 ([15])**.** *Let H be a Hilbert space and let $\{x_n\}$ be a 2-generalized hybrid sequence in H. Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$*;*
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\left\{\frac{1}{n}\sum_{k=0}^{n-1} x_k\right\}$ *converges weakly to some* $w \in H$*.*

Moreover, in this case $w = \lim_{n\to\infty} Px_n \in F_1$.

4. Weak convergence theorems

In this section, we get weak convergence theorems for weakly asymptotically regular sequences (see also[14, 15]).

Theorem 4.1 ([2])**.** *Let* H *be a Hilbert space and let* $\{x_n\}$ *be a normally* 2*-generalized hybrid sequence in* H*. Assume* $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$ *and* $\alpha_2 + \alpha_1 + \alpha_0 > 0$.
And suppose that $\{x_n\}$ *is weakly asymptotically regular, i.e.,*

$$x_{n+1} - x_n \rightharpoonup 0.$$

Then, the following are equivalent.

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$*;*
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\{x_k\}$ *converges weakly to some* $u \in H$.

Moreover, in this case $u = \lim_{n\to\infty} Px_n \in F_1$.

Remark 4.2. *In Theorem 3.2, we have that* $u = w\text{-}\lim_{n\to\infty} x_n$ *is the asymptotic center of* $\{x_n\}$ *(see, for instance* [17]*).*

By Theorem 4.1, we also get the following weak convergence theorem by Rouhani [14] for generalized hybrid sequences (see also [2]).

Theorem 4.3 ([14])**.** *Let* H *be a Hilbert space and let* $\{x_n\}$ *be a generalized hybrid sequence in* H*. Suppose that* $\{x_n\}$ *is weakly asymptotically regular, i.e.,*

$$x_{n+1} - x_n \rightharpoonup 0.$$

Then, the following are equivalent.

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$*;*
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\{x_k\}$ *converges weakly to some* $u \in H$.

By Theorem 4.1, we also get the following weak convergence theorem by Rouhani [15] for 2-generalized hybrid sequences (see also [2]).

Theorem 4.4 ([15]). *Let H be a Hilbert space and let $\{x_n\}$ be a 2-generalized hybrid sequence in H. Suppose that $\{x_n\}$ is weakly asymptotically regular, i.e.,*

$$x_{n+1} - x_n \rightharpoonup 0$$

Then, the following are equivalent.

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$;
(iii) $\{x_n\}$ *is bounded in* H;
(iv) $\{x_k\}$ *converges weakly to some* $u \in H$.

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