ON OPTIMALITY THEOREMS FOR ROBUST SEMI-INFINITE MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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ABSTRACT. We consider a semi-infinite multiobjective optimization problem with more than two differentiable objective functions and uncertain constraint functions, which is called a robust semi-infinite multiobjective optimization problem and give its robust counterpart (RSIMP) of the problem, which is regarded as the worst case of the uncertain semi-infinite multiobjective optimization problem. In this paper, we review necessary optimality theorems for weakly robust efficient solutions of (RSIMP), which were in the paper [16].

1. INTRODUCTION

Mathematical optimization problems in the face of data uncertainty have been treated by the worst case approach or the stochastic approach. The worst case approach for optimization problems, which has emerged as a powerful deterministic approach for studying optimization problems with data uncertainty, associates an uncertain optimization problem with its robust counterpart. Many researchers have investigated optimality and duality theories for linear or convex programming problems under uncertainty with the worst-case approach(the robust approach) ([1, 4, 5, 6, 7, 8, 9, 13, 15]). Moreover, many authors have studied optimality and duality theories for robust multiobjective optimization problems under different suitable constrained qualifications ([3, 10, 11, 12, 14, 16]). We consider a semi-infinite multiobjective optimization problem with more than two differentiable objective functions and uncertain constraint functions, which is called a robust semi-infinite multiobjective optimization problem and give its robust counterpart (RSIMP) of the problem,

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which is regarded as the worst case of the uncertain semi-infinite multiobjective optimization problem.

Consider the following semi-infinite multiobjective optimization problem in the absence of data uncertainty

$$\begin{aligned} \text{(SIMP)} \quad \min \quad & (f_1(x), \ldots, f_l(x)) \\ \text{s.t.} \quad & g_t(x) \leq 0, \ ^\forall t \in T, \end{aligned}$$

where $f_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, l$ and $g_t \colon \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are continuously differentiable and T is an index set with coordinately possible infinite.

The semi-infinite multiobjective optimization problem (SIMP) in the face of data uncertainty in the constraints can be captured by the problem

$$\begin{aligned} \text{(USIMP)} \quad \min \quad & (f_1(x), \ldots, f_l(x)) \\ \text{s.t.} \quad & g_t(x, v_t) \leq 0, \ ^\forall t \in T, \end{aligned}$$

where $f_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, l$ and $g_t \colon \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ are continuously differentiable and $v_t \in \mathbb{R}^q$ is an uncertain parameter which belongs to the convex compact set $\mathcal{V}_t \subset \mathbb{R}^q$, $t \in T$.

The uncertainty set-valued mapping $\mathcal{V} \colon T \rightrightarrows \mathbb{R}^q$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. So, $\text{gph}\mathcal{V} := \{(t, v_t) : v_t \in \mathcal{V}_t, \ t \in T\}$ and $v \in \mathcal{V}$ means that v is a *selection* of $\mathcal{V}$, i.e., $v \colon T \to \mathbb{R}^q$ and $v_t \in \mathcal{V}_t$ for all $t \in T$.

The robust counterpart of (USIMP):

$$\begin{aligned} \text{(RSIMP)} \quad \min \quad & (f_1(x), \ldots, f_l(x)) \\ \text{s.t.} \quad & g_t(x, v_t) \leq 0, \ ^\forall v_t \in \mathcal{V}_t, \ ^\forall t \in T. \end{aligned}$$

The robust feasible set F of (RSIMP) is defined by

$$F := \{x \in \mathbb{R}^n : g_t(x, v_t) \leq 0, \ ^\forall t \in T, \ ^\forall v_t \in \mathcal{V}_t\}.$$

Then $\bar{x} \in F$ is called a *weakly robust efficient solution* of (RSIMP) if there does not exist a robust feasible solution x of (RSIMP) such that

$$f_i(x) < f_i(\bar{x}), \ i = 1, \ldots, l.$$

In this paper, we review necessary optimality theorems for weakly robust efficient solutions of (RSIMP), which were in the paper [16].

2. Necessary Optimality Theorems

Let $\mathcal{V} \colon T \rightrightarrows \mathbb{R}^q$ be an uncertainty set-valued mapping defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$ and $g_t \colon \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be a given continuously differentiable function. Now, we will assume that the following assumptions hold:

(A1) T is a compact metric space.
(A2) $\mathcal{V}$ is compact-valued and upper semi-continuous on T

(A3) $g_{t_n}(x_n, v_{t_n}) \to g_t(x, v_t)$, whenever $t_n \in T \to t \in T$, $v_{t_n} \in \mathcal{V}_{t_n} \to v_t \in \mathcal{V}_t$, and $x_n \in \mathbb{R}^n \to x \in \mathbb{R}^n$ as $n \to \infty$.

(A4) $\nabla g_{t_n}(x_n, v_{t_n}) \to \nabla g_t(x, v_t)$, whenever $t_n \in T \to t \in T$, $v_{t_n} \in \mathcal{V}_{t_n} \to v_t \in \mathcal{V}_t$, and $x_n \in \mathbb{R}^n \to x \in \mathbb{R}^n$ as $n \to \infty$.

Let $\bar{x} \in F$. Let us decompose T into two index sets $T = T_1(\bar{x}) \cup T_2(\bar{x})$, where $T_1(\bar{x}) := \{t \in T : \exists v_t \in \mathcal{V}_t \text{ s.t. } g_t(\bar{x}, v_t) = 0\}$ and $T_2(\bar{x}) := T \backslash T_1(\bar{x})$. Let $\mathcal{V}_t(\bar{x}) := \{v_t \in \mathcal{V}_t : g_t(\bar{x}, v_t) = 0\}$.

We define an *extended nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ)* at $\bar{x} \in F$ as follows:

$$\exists d \in \mathbb{R}^n \text{ s.t. } \nabla_x g_t(\bar{x}, v_t)^T d < 0, \ \forall t \in T_1(\bar{x}), \ \forall v_t \in \mathcal{V}_t(\bar{x}).$$

Now we give a robust necessary optimality theorem for a weakly robust efficient solution of (RSIMP), which was in [16].

Theorem 2.1. [16] *Assume that the conditions* (A1)-(A4) *hold. Let $\bar{x}$ be a weakly robust efficient solution of* (RSIMP). *Suppose that $g_t(x, \cdot)$ is concave on $\mathcal{V}_t$, for each $x \in \mathbb{R}^n$ and for each $t \in T$. Then there exist $\bar{\mu}_i \geq 0$, $i = 1, \ldots, l$, $(\bar{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, and $\bar{v}_t \in \mathcal{V}_t$, $t \in T$ such that $\sum_{i=1}^{l} \bar{\mu}_i + \sum_{t \in T} \bar{\lambda}_t = 1$,*

$$\sum_{i=1}^{l} \bar{\mu}_i \nabla f_i(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \nabla_x g_t(\bar{x}, \bar{v}_t) = 0$$
$$\text{and} \quad \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0, \ t \in T.$$

Moreover, if we further assume that the extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds, then there exist $\hat{\mu}_i \geq 0$, $i = 1, \ldots, l$, not all zero, $(\hat{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, and $\bar{v}_t \in \mathcal{V}_t$, $t \in T$ such that $\sum_{i=1}^{l} \hat{\mu}_i = 1$,

$$\sum_{i=1}^{l} \hat{\mu}_i \nabla f_i(\bar{x}) + \sum_{t \in T} \hat{\lambda}_t \nabla_x g_t(\bar{x}, \bar{v}_t) = 0$$
$$\text{and} \quad \hat{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0, \ t \in T.$$

We may apply Theorem 2.1 to robust linear semi-infinite multiobjective programming problems under uncertainty, which was studied by Goberna et al. [2, 3, 4].

Consider the following linear semi-infinite multiobjective optimization problem in the absence of data uncertainty:

$$\begin{array}{lll} \textbf{(LSIMP)} & \min & (c_1^T x, \ldots, c_l^T x) \\ & \text{s.t.} & a_t^T x \geq b_t, \ ^\forall t \in T, \end{array}$$

where c_i, $i = 1, \ldots, l$, $a_t \in \mathbb{R}^n$, and $b_t \in \mathbb{R}$, $t \in T$. The semi-infinite optimization problem in the face of data uncertainty in the linear constraints can be captured by the problem

$$\begin{array}{lll} \text{(ULSIMP)} & \min & (c_1^T x, \ldots, c_l^T x) \\ & \text{s.t.} & a_t^T x \geq b_t, \ ^\forall t \in T, \end{array}$$

where a_t and b_t are uncertain parameters, and (a_t, b_t) belongs to the set $\mathcal{V}_t \subset \mathbb{R}^{n+1}$ for all $t \in T$.

Let $(a_t, b_t) \in \mathcal{V}_t$, for $t \in T$. The set-valued mapping $\mathcal{V}\colon T \rightrightarrows \mathbb{R}^{n+1}$, is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$.

The robust counterpart of (ULSIMP) is

$$\begin{array}{lll} \text{(RLSIMP)} & \min & (c_1^T x, \ldots, c_l^T x) \\ & \text{s.t.} & a_t^T x \geq b_t, \ ^\forall (a_t, b_t) \in \mathcal{V}_t, \ ^\forall t \in T. \end{array}$$

Clearly, $F^L := \{x \in \mathbb{R}^n : a_t^T x \geq b_t, \ ^\forall (a_t, b_t) \in \mathcal{V}_t, \ ^\forall t \in T\}$ is the feasible set of (RLSIP).

We define an *extended linear Mangasarian-Fromovitz constraint qualification (ELMFCQ)* at $\bar{x} \in F^L$ as follows:

$$\exists d \in \mathbb{R}^n \text{ such that } \forall t \in T_1(\bar{x}), \ \forall (a_t, b_t) \in \mathcal{V}_t(\bar{x}), \ a_t^T d > 0.$$

Remark 2.2. [16] Goberna et al. [3] have established characterizations of robust solutions of (ULSIMP) under the *local Farkas-Minkowski* constraint qualification (LFMCQ) at $\bar{x} \in F^L$, that is, $D(F^L; \bar{x})^+ = A(\bar{x})$, where

$$A(\bar{x}) := \text{cone}\{a : (a, b) \in \bigcup_{t \in T} \mathcal{V}_t \text{ and } a^T \bar{x} = b\} \subset \mathbb{R}^n,$$

$$D(F^L; \bar{x}) := \{d \in \mathbb{R}^n : \exists \eta > 0 \text{ s.t. } \bar{x} + \eta d \in F^L\},$$

and $D(F^L; \bar{x})^+$ is the positive polar cone of $D(F^L; \bar{x})$. In the linear programming with finite uncertain linear constraints, generally, even if the extended linear Mangasarian-Fromovitz constraint qualification (ELMFCQ) does not hold, (LFMCQ) always holds.

We can get the following necessary optimality theorem for (ULSIP) from Theorem 2.1, which was in [16].

Theorem 2.3. [16] *Assume that the conditions* (A1) *and* (A2) *hold. Let* $\bar{x}$ *be a weakly robust efficient solution of* (ULSIP). *Then there exist* $\bar{\mu}_i \geq 0$,

$i = 1, \ldots, l$, $(\bar{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, *and* $\bar{v}_t = (\bar{a}_t, \bar{b}_t) \in \mathcal{V}_t$, $t \in T$ *such that*

$$\sum_{i=1}^{l} \bar{\mu}_i c_i - \sum_{t \in T} \bar{\lambda}_t \bar{a}_t = 0 \text{ and } \bar{\lambda}_t(\bar{a}_t^T \bar{x} - \bar{b}_t) = 0, \ t \in T.$$

Moreover, if we further assume that (ELMFCQ) *holds, then there exist* $\hat{\mu}_i \geq 0$, $i = 1, \ldots, l$, *not all zero,* $(\hat{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ *and* $\bar{v}_t = (\bar{a}_t, \bar{b}_t) \in \mathcal{V}_t$, $t \in T$ *such that* $\sum_{i=1}^{l} \hat{\mu}_i = 1$, $\sum_{i=1}^{l} \hat{\mu}_i + \sum_{t \in T} \hat{\lambda}_t \geq 1$,

$$\sum_{i=1}^{l} \hat{\mu}_i c_i - \sum_{t \in T} \hat{\lambda}_t \bar{a}_t = 0 \text{ and } \hat{\lambda}_t(\bar{a}_t^T \bar{x} - \bar{b}_t) = 0, \ t \in T.$$

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