Attractive Point and Mean Convergence Theorems for Normally Generalized Hybrid Mappings in Hilbert Spaces

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Abstract

In this article, using Banach limits, we study the existence of attractive points of commutative normally 2-generalized hybrid mappings in Hilbert spaces. Then we prove a mean convergence theorem for the mappings in Hilbert spaces. Using these results, we obtain well-known attractive point and mean convergence theorems in Hilbert spaces.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $A(T)$ the set of *attractive points* [18] of T, i.e.,

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C\}.$$

We know from [18] that $A(T)$ is closed and convex. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [2] we know the first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$, where $F(T)$ is the set of fixed points of T. In 2010, Kocourek, Takahashi and Yao [6] defined a broad class of nonlinear mappings in a Hilbert space: Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [6] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called $(\alpha,\ \beta)$-*generalized hybrid*. We also know the following mapping: For $\lambda \in \mathbb{R}$, a mapping $U : C \to H$ is called λ-*hybrid* [1] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [8, 9] for $\alpha = 2$ and $\beta = 1$. It is also *hybrid* [17] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. In general, nonspreading and hybrid mappings are not continuous; see [5]. The nonlinear ergodic theorem by Baillon [2] for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [6]. Recently, Kohsaka [7] also proved the following theorem.

Theorem 1.1 ([7]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative λ and μ-hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

On the other hand, Takahashi and Takeuchi [18] proved the following attractive point and mean convergence theorem without convexity in a Hilbert space.

Theorem 1.2 ([18]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Assume that $\{T^n z\}$ for some $z \in C$ is bounded and define*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

for all $x \in C$ and $n \in \mathbb{N}$. Then $\{S_n x\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)} T^n x$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.

Maruyama, Takahashi and Yao [13] also defined a more broad class of nonlinear mappings called 2-generalized hybrid which contains generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H. A mapping $T : C \to C$ is 2-*generalized hybrid* [13] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1 \|T^2 x - Ty\|^2 + \alpha_2 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leq \beta_1 \|T^2 x - y\|^2 + \beta_2 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned} \tag{1.3}$$

for all $x, y \in C$. Such a mapping is called $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-2 generalized hybrid. Very recently, Kondo and Takahashi [10] introduced the following class of nonlinear mappings which covers 2-generalized hybrid mappings in Hilbert spaces. Let C be a nonempty subset of H. A mapping $T : C \to C$ is *normally* 2-*generalized hybrid* [10] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0, \quad \alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned} \alpha_2 \|T^2 x - Ty\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ + \beta_2 \|T^2 x - y\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0 \end{aligned} \tag{1.4}$$

for all $x, y \in C$.

In this article, motivated by Kohsaka' theorem (Theorem 1.1) and Takahashi and Takeuchi's therem (Theorem 1.2), we study the existence of attractive points of commutative normally 2-genralized hybrid mappings in Hilbert spaces. Then we prove a mean convergence theorem for the mappings in Hilbert spaces. Using these results, we obtain well-known attractive point and mean convergence theorems in Hilbert spaces.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [16]. Furthermore, in a Hilbert space, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$. Indeed, we have that

$$\begin{aligned}
2\langle x-y, z-w\rangle &= 2\langle x, z\rangle - 2\langle x, w\rangle - 2\langle y, z\rangle + 2\langle y, w\rangle \\
&= (-\|x\|^2 + 2\langle x, z\rangle - \|z\|^2) + (\|x\|^2 - 2\langle x, w\rangle + \|w\|^2) \\
&\quad + (\|y\|^2 - 2\langle y, z\rangle + \|z\|^2) + (-\|y\|^2 + 2\langle y, w\rangle - \|w\|^2) \\
&= \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2.
\end{aligned}$$

From (2.2), we have that

$$\langle (x-y) + (x-w), y-w\rangle = \|x-w\|^2 - \|x-y\|^2 \tag{2.3}$$

for all $x, y, w \in H$. Indeed, we have that

$$\begin{aligned}
&2\langle (x-y) + (x-w), y-w\rangle = 2\langle (x-w) - (y-x), (y-w) - 0\rangle \\
&= \|x-w-0\|^2 + \|y-x-(y-w)\|^2 - \|x-w-(y-w)\|^2 - \|y-x-0\|^2 \\
&= 2\|x-w\|^2 - 2\|y-x\|^2
\end{aligned}$$

and hence $\langle (x-y) + (x-w), y-w\rangle = \|x-w\|^2 - \|x-y\|^2$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (a_1, a_2, a_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(a_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(a_{n+1}) = \mu_n(a_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (a_1, a_2, a_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n.$$

In particular, if $f = (a_1, a_2, a_3, \dots) \in l^\infty$ and $a_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(a_n) = a$. See [15] for the proof of existence of a Banach limit and its other elementary properties.

Using a mean, we obtain the following result; see[12, 14]: Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H. \tag{2.4}$$

We call such a unique $z_0 \in H$ the *mean vector* of $\{x_n\}$ for μ.

3 Attractive Point Theorems

Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T: C \to C$ is *normally* 2*-generalized hybrid* [10] if it satisfies (1.4), i.e., there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0, \quad \alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned}\alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0\end{aligned}$$

for all $x, y \in C$. We call such a mapping $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ *-normally* 2*-generalized hybrid*. We know that the class of the mappings above covers well-known mappings. For example, the class of $(1 - \alpha_1, -(1 - \beta_1), \alpha_1, -\beta_1, 0, 0)$-normally 2-generalized hybrid mappings is the class of generalized hybrid mappings in the sense of Kocourek, Takahashi and Yao [6]. If $x = Tx$ in (1.4), then for any $y \in C$,

$$\begin{aligned}\alpha_2\|x - Ty\|^2 + \alpha_1\|x - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ + \beta_2\|x - y\|^2 + \beta_1\|x - y\|^2 + \beta_0\|x - y\|^2 \leq 0\end{aligned}$$

and hence

$$(\alpha_2 + \alpha_1 + \alpha_0)\|x - Ty\|^2 \leq -(\beta_2 + \beta_1 + \beta_0)\|x - y\|^2.$$

From $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$, we have that

$$(\alpha_2 + \alpha_1 + \alpha_0)\|x - Ty\|^2 \leq -(\beta_2 + \beta_1 + \beta_0)\|x - y\|^2 \leq (\alpha_2 + \alpha_1 + \alpha_0)\|x - y\|^2.$$

Since $\alpha_2 + \alpha_1 + \alpha_0 > 0$, we have that

$$\|x - Ty\| \leq \|x - y\|, \quad \forall x \in F(T),\ y \in C. \tag{3.1}$$

So a normally 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Now, we prove an attractive point theorem for commutative normally 2-generalized hybrid mappings in a Hilbert space. Before proving the theorem, we have the following lemma from [4].

Lemma 3.1 ([4])**.** *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be mappings of C into itself. Suppose that there exist a mean μ on l^∞ and a sequence $\{x_n\} \subset H$ such that $\{x_n\}$ is bounded and*

$$\mu_n\|x_n - Sy\|^2 \leq \mu_n\|x_n - y\|^2 \ \ \textit{and} \ \ \mu_n\|x_n - Ty\|^2 \leq \mu_n\|x_n - y\|^2, \quad \forall y \in C.$$

Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex and $\{x_n\} \subset C$, then $F(S) \cap F(T)$ is nonempty.

By taking Banach limit and using Lemma 3.1, we obtain this theorem.

Theorem 3.2 ([3])**.** *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be commutative normally 2-generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

Using Theorem 3.2, we have the following theorem proved by Hojo, Takahashi and Takahashi [4] for commutative 2-generalized hybrid mappings in Hilbert spaces.

Theorem 3.3 ([4]). *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be commutative 2-generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

Using Theorem 3.2, we also have the attractive point theorem by Kondo and Takahashi [10] for normally 2-generalized hybrid mappings in Hilbert spaces.

Theorem 3.4 ([10]). *Let C be a nonempty subset of H and let $T : C \to C$ be a $(\alpha_n, \beta_n)_{n=0}^{2}$-normally 2-generalized hybrid mapping. Assume that there exists $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. Then, $A(T) \neq \emptyset$.*

4 Nonlinear Ergodic Theorems

In this section, we prove a mean convergence theorem for commutative normally 2-generalized hybrid mappings in Hilbert spaces.

Let $D = \{(k, l) : k, l \in \mathbb{N} \cup \{0\}\}$. Then D is a directed set by the binary relation:

$$(k, l) \leq (i, j) \quad \text{if} \quad k \leq i \;\text{ and }\; l \leq j.$$

Theorem 4.1 ([3]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let P be the metric projection of H onto $A(S) \cap A(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l)\in D} PS^kT^lx$. In particular, if C is closed and convex, $\{S_n x\}$ converges weakly to an element q of $F(S) \cap F(T)$.

Using Theorem 4.1, we can prove the following nonlinear ergodic theorem by Hojo, Takahashi and Takahashi [4] for commutative 2-generalized hybrid mappings in Hilbert spaces.

Theorem 4.2 ([4]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let P be the metric projection of H onto $A(S) \cap A(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l)\in D} PS^kT^lx$. In particular, if C is closed and convex, $\{S_n x\}$ converges weakly to an element q of $F(S) \cap F(T)$.

Using Theorem 4.1, we also have the following nonlinear ergodic theorem by Kondo and Takahashi [10].

Theorem 4.3 ([10]). *Let C be a nonempty subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$. Let $P_{A(T)}$ be the metric projection from H onto $A(T)$. Then, for any $x \in C$, the sequence $\left\{S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ converges weakly to $u \in A(T)$, where $u = \lim_{n\to\infty} P_{A(T)} T^n x$.*

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