A new definition of resolvents for convex functions on complete geodesic spaces
完備測地距離空間上の凸関数に対するリゾルベントの新しい定義

東邦大学・理学部 梶村拓豊
Takuto Kajimura
Department of Information Science
Toho University
東邦大学・理学部 木村泰紀
Yasunori Kimura
Department of Information Science
Toho University

Abstract

In this paper, we propose a new resolvent in complete geodesic spaces and we show that it is well-defined as a single valued mapping. Moreover, we propose spherical nonspreadingness of sum-type and we show that the new resolvent satisfies this condition.

1 Introduction

Let X be a complete CAT(0) space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. A resolvent for f is defined by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + d(y, x)^2 \right\}$$

for all $x \in X$. In 1998, Mayer [7] proved its well-definedness; see also Jost [2]. It is known that J_f is nonspreading, that is,

$$2d(J_f x, J_f y)^2 \leqq d(J_f x, y)^2 + d(x, J_f y)^2$$

for all $x, y \in X$. See [6] for more details.

Let X be a complete CAT(1) space with $d(v, v') < \pi/2$ for all $v, v' \in X$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. In this case, a resolvent of f is defined by

$$Q_f x = \operatorname*{argmin}_{y \in X} \{f(y) + \tan d(y,x) \sin d(y,x)\}$$

for all $x \in X$. In 2016, Kimura and Kohsaka [5] proved its well-definedness. They also showed that the resolvent is spherically nonspreading of product-type, that is,

$$\cos^2 d(Q_f x, Q_f y) \geqq \cos d(Q_f x, y) \cos d(x, Q_f y)$$

for all $x, y \in X$.

In this paper, we propose a new resolvent in a complete CAT(1) space and we show that it is well-defined as a single-valued mapping. Moreover, we propose spherical nonspreadingness of sum-type and we show that the new resolvent satisfies this condition.

2 Preliminaries

Let X be a metric space with metric d. We denote by $\mathcal{F}(T)$ the set of all fixed points of a mapping from X into itself. A continuous mapping $c : [0,l] \to X$ is called geodesic if c satisfies $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s-t|$ for all $x, y \in X$ and $s, t \in [0,1]$. Its image, which is denoted by $[x,y]$, is called geodesic segment with endpoints x and X. X is said to be a geodesic space if there exists $[x,y]$ for all $x, y \in X$. In this paper, when X is a geodesic space, its geodesic is always assumed to be unique.

Let X be a geodesic space. There exists a unique point $z \in [x,y]$ such that $d(x,z) = (1-\alpha)d(x,y)$ and $d(z,y) = \alpha d(x,y)$ for all $x, y \in X$ and $\alpha \in [0,1]$. This point is called convex combination of x and X, which is denoted by $\alpha x \oplus (1-\alpha)y$. A subset $C \subset X$ is said to be convex if $[x,y] \subset X$ for all $x, y \in C$. A geodesic triangle of vertices $x, y, z \in X$ is defined by $[x,y] \cup [y,z] \cup [z,x]$, which is denoted by $\triangle(x,y,z)$.

Let M_κ^2 be a two dimensional model space for all $\kappa \in \mathbb{R}$. For example, $M_0^2 = \mathbb{R}^2$, $M_1^2 = \mathbb{S}^2$ and $M_{-1}^2 = \mathbb{H}^2$. A comparison triangle to $\triangle(x,y,z) \subset X$ of vertices $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ is defined by $[\bar{x},\bar{y}] \cup [\bar{y},\bar{z}] \cup [\bar{z},\bar{x}]$ with $d(x,y) = d(\bar{x},\bar{y}), d(y,x) = d(\bar{y},\bar{z})$ and $d(z,x) = d(\bar{z},\bar{x})$, which is denoted by $\bar{\triangle}(\bar{x},\bar{y},\bar{z})$. $\bar{z} \in [\bar{x},\bar{y}]$ is called comparison point of $z \in [x,y]$ if $d(x,z) = d(\bar{x},\bar{z})$ holds. For all $\kappa \in \mathbb{R}$, X is called a CAT(κ) space if $d(p,q) \leqq d(\bar{p},\bar{q})$ holds whenever $\bar{p}, \bar{q} \in \bar{\triangle}$ are comparison points for $p, q \in \triangle$. In general, if $\kappa < \kappa'$, then the CAT(κ) spaces are CAT(κ') spaces [1]. The following lemma is important to show the main theorem.

Lemma 2.1 ([3]). *Let X be a complete* CAT(1) *space, $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2\pi$, and $\alpha \in [0,1]$. Then*

$$\begin{aligned}&\cos d(\alpha x \oplus (1-\alpha)y, z) \sin d(x,y) \\ &\geqq \cos d(x,z) \sin \alpha d(x,y) + \cos d(y,z) \sin(1-\alpha) d(x,y).\end{aligned}$$

Lemma 2.2 ([4]). *Let X, x, y, z, and α be the same as in Lemma 2.1. Then*

$$\cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right)\cos\frac{1}{2}d(x,y) \geqq \frac{1}{2}\cos d(x,z) + \frac{1}{2}\cos d(y,z).$$

Lemma 2.3 ([5]). *Let X, x, y, z, and α be the same as in Lemma 2.1. If $d(x,z) < \pi/2$ and $d(y,z) < \pi/2$, then*

$$\cos d(\alpha x \oplus (1-\alpha)y, z) \geqq \alpha \cos d(x,z) + (1-\alpha)\cos d(y,z).$$

Let X be a geodesic space and f a function from X into $]-\infty, \infty]$. The function f is said to be lower semicontinuous if the set $\{x \in X \mid f(x) \leqq a\}$ is closed for all $a \in \mathbb{R}$. If f is continuous, then it is lower semicontinuous. The domain of f is defined by $\{x \in X \mid f(x) \in \mathbb{R}\}$, which is denoted by $\mathrm{dom} f$. The function f is said to be proper if $\mathrm{dom} f$ is nonempty. The function is said to be convex if

$$f(\alpha x \oplus (1-\alpha)y) \leqq \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in X$ and $\alpha \in]0,1[$.

Lemma 2.4 ([5]). *Let X be a complete $\mathrm{CAT}(1)$ space with $d(v,v') < \pi/2$ for all $v, v' \in X$, f a proper lower semicontinuous convex function from X into $]-\infty,\infty]$ and p an element of X. Suppose that $f(x_n) \to \infty$ whenever $\{x_n\}$ is a sequence of X with $d(p, x_n) \to \pi/2$. Then $\mathrm{argmin}_X f$ is nonempty. Further, if*

$$x, y \in \mathrm{dom} f, x \neq y \Rightarrow f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

then $\mathrm{argmin}_X f$ consists of one point.

Lemma 2.5 ([5]). *Let X be a complete $\mathrm{CAT}(1)$ space with $d(v,v') < \pi/2$ for all $v, v' \in X$. Then every proper lower semicontinuous convex function from X into $]-\infty,\infty]$ is bounded below.*

3 Resolvents for convex functions in complete CAT(1) spaces

Let X be a metric space and T a mapping from X into itself. Then T is said to be spherically nonspreading of sum-type if

$$2\cos d(Tx, Ty) \geqq \cos d(Tx, y) + \cos d(x, Ty)$$

for all $x, y \in X$. It is obvious that if T is spherically nonspreading of sum-type, then T is spherically nonspreading of product-type.

In this section, we show that a new resolvent

$$R_f x := \operatorname*{argmin}_{y \in X}\{f(y) - \log(\cos d(y,x))\}$$

is well-defined, where f is a proper lower semicontinuous convex function. Moreover we show the fundamental properties of the new resolvent.

Throughout this section, we suppose that X is a complete CAT(1) space with $d(v, v') < \pi/2$ for all $v, v' \in X$.

Lemma 3.1. *Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. If*

$$g(\cdot) = f(\cdot) - \log(\cos d(\cdot, p))$$

for each $p \in X$, then g is a proper lower semicontinuous convex function from X into $]-\infty, \infty]$.

Proof. Let $x, y \in X$ and $\alpha \in]0, 1[$. From Lemma 2.3, we know that

$$\cos d(\alpha x \oplus (1-\alpha)y, p) \geqq \alpha \cos d(x, p) + (1-\alpha) \cos d(y, p)$$

holds for all $p \in X$. Since $-\log t$ is decreasing and convex for all $t \geqq 0$, we get

$$\begin{aligned} -\log(\cos d(\alpha x \oplus (1-\alpha)y, p)) &\leqq -\log(\alpha \cos d(x, p) + (1-\alpha) \cos d(y, p)) \\ &\leqq -\alpha \log(\cos d(x, p)) - (1-\alpha) \log(\cos d(y, p)). \end{aligned}$$

Thus g is convex. On the other hand, it is obvious that g is proper and lower semicontinuous. □

Theorem 3.2. *Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. If*

$$g(\cdot) = f(\cdot) - \log(\cos d(\cdot, p))$$

for each $p \in X$, then $\operatorname{argmin}_X g$ *consists of one point.*

Proof. Let $\{x_n\}$ be a sequence of X with $\lim_{n\to\infty} d(x_n, p) = \pi/2$ for each $p \in X$. Then, it is obvious that $\lim_{n\to\infty}(-\log(\cos d(x_n, p))) \to \infty$. On the other hand, from Lemma 2.5, we know that there exists $K \in \mathbb{R}$ such that $f(x) \geqq K$ for all $x \in X$. So, we get

$$g(x_n) \geqq K + \log(\cos d(x_n, p)) \to \infty$$

and hence $g(x_n) \to \infty$. From Lemma 2.4 and 3.1, $\operatorname{argmin}_X g$ is nonempty.

We next show that $\operatorname{argmin}_X g$ consists of one point. Suppose that $x, y \in \operatorname{dom} f$ with $x \neq y$. Then, Lemma 2.2 implies that

$$\begin{aligned} \cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right) &> \cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right) \cos \frac{1}{2} d(x, y) \\ &\geqq \frac{1}{2} \cos d(x, p) + \frac{1}{2} \cos d(y, p) \end{aligned}$$

for all $p \in X$. Further, since $-\log t$ is decreasing and convex for all $t > 0$, we get

$$-\log\left(\cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right)\right) < -\log\left(\frac{1}{2}\cos d(x, p) + \frac{1}{2}\cos d(y, p)\right)$$

$$\leqq -\frac{1}{2}\log(\cos d(x,p)) - \frac{1}{2}\log(\cos d(y,p)).$$

Since f is convex, the inequality

$$g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}g(x) + \frac{1}{2}g(y)$$

holds. Thus $\operatorname{argmin}_X g$ consists of one point. □

Definition 3.3. Let f be a proper lower semicontinuous convex function from X into $]-\infty,\infty]$. Then we define a new resolvent $R_f : X \to X$ by

$$R_f x = \operatorname*{argmin}_{y\in X}\{f(y) - \log(\cos d(y,x))\}$$

for all $x \in X$. From Theorem 3.2, we know that R_f is well-defined.

Remark 3.4. Let C be a nonempty closed convex subset of X. If $f = i_C$, then $R_f = P_C$, where P_C is metric projection from X onto C. In fact,

$$\begin{aligned} R_f x &= \operatorname*{argmin}_{y\in C}\{f(y) - \log(\cos d(y,x))\} \\ &= \operatorname*{argmin}_{y\in C}\{-\log(\cos d(y,x))\} = \operatorname*{argmin}_{y\in C} d(y,x) = P_C x \end{aligned}$$

for all $x \in X$.

Theorem 3.5. *Let f be a proper lower semicontinuous convex function from X into $]-\infty,\infty]$ and R_f a resolvent of f. Then the following properties hold:*

(i) *R_f is spherically nonspreading of sum-type;*
(ii) *$\mathcal{F}(R_f) = \operatorname{argmin}_X f$.*

Proof. Put $T = R_f$. We first show (i). Let $x, y \in X$ with $Tx \neq Ty$ and put $z_t = tTx \oplus (1-t)Ty$ for all $t \in]0,1[$. Then, by the definition of T and convexity of f, we have

$$\begin{aligned} f(Ty) - \log(\cos d(Ty,y)) &\leqq f(z_t) - \log(\cos d(z_t,y)) \\ &\leqq tf(Tx) + (1-t)f(Ty) - \log(\cos d(z_t,y)) \end{aligned}$$

and hence

$$\begin{aligned} t(f(Tx) - f(Ty)) &\geqq \log(\cos d(z_t,y)) - \log(\cos d(Ty,y)) \\ &= \log\left(\frac{\cos d(z_t,y)}{\cos d(Ty,y)}\right). \end{aligned}$$

So, using Lemma 2.1 and putting $D = d(Tx,Ty)$, we get

$$e^{t(f(Tx)-f(Ty))}\sin D$$

$$
\begin{aligned}
&\geqq \frac{\cos d(z_t, y)\sin D}{\cos d(Ty, y)}\\
&\geqq \frac{\cos d(Tx, y)\sin tD + \cos d(Ty, y)\sin(1-t)D}{\cos d(Ty, y)}\\
&= \frac{\cos d(Tx, y)\sin tD + \cos d(Ty, y)\sin D\cos tD - \cos d(Ty, y)\cos D\sin tD}{\cos d(Ty, y)}\\
&= \sin tD\frac{\cos d(Tx, y) - \cos d(Ty, y)\cos D}{\cos d(Ty, y)} + \sin D\cos tD\\
&= \sin tD\frac{\cos d(Tx, y) - \cos d(Ty, y)\cos D}{\cos d(Ty, y)} + \sin D - 2\sin D\sin^2\frac{t}{2}D
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\sin D\left(\frac{e^{t(f(Tx)-f(Ty))}-1}{t}\right)\\
&\geqq \frac{\sin tD}{t}\frac{\cos d(Tx, y) - \cos d(Ty, y)\cos D}{\cos d(Ty, y)} - \frac{2}{t}\sin D\sin^2\frac{t}{2}D.
\end{aligned}
$$

Letting $t \downarrow 0$, we obtain

$$
\sin D(f(Tx) - f(Ty)) \geqq \frac{D}{\cos d(Ty, y)}(\cos d(Tx, y) - \cos d(Ty, y)\cos D).
$$

From this inequality, we also know that

$$
\sin D(f(Ty) - f(Tx)) \geqq \frac{D}{\cos d(Tx, x)}(\cos d(x, Ty) - \cos d(Tx, x)\cos D)
$$

holds. Adding these inequalities, we get

$$
\begin{aligned}
0 \geqq\ &\frac{1}{\cos d(Ty, y)}(\cos d(Tx, y) - \cos d(Ty, y)\cos D)\\
&+ \frac{1}{\cos d(Tx, x)}(\cos d(x, Ty) - \cos d(Tx, x)\cos D).
\end{aligned}
$$

So we have

$$
\begin{aligned}
2\cos D &\geqq \frac{1}{\cos d(Ty, y)}\cos d(Tx, y) + \frac{1}{\cos d(Tx, x)}\cos d(x, Ty)\\
&\geqq \cos d(Tx, y) + \cos d(x, Ty).
\end{aligned}
$$

Thus we get the conclusion.

We next show (ii). Let $u \in \operatorname{argmin}_X f$. Then we have

$$
f(u) - \log(\cos d(u, u)) = f(u) \leqq f(y) \leqq f(y) - \log(\cos d(y, u))
$$

for all $y \in X$. Hence we get

$$f(u) - \log(\cos d(u,u)) = \inf_{y \in X}\{f(y) - \log(\cos d(y,u))\}.$$

This implies that $u \in \mathcal{F}(T)$. Inversely, let $u \in \mathcal{F}(T)$ and $t \in]0,1[$. Then, by the definition of T and the convexity of f, we have

$$\begin{aligned} f(u) &= f(u) - \log(\cos d(u,u)) \\ &\leqq f(ty \oplus (1-t)u) - \log(\cos d(ty \oplus (1-t)u, u)) \\ &= f(ty \oplus (1-t)u) - \log(\cos td(y,u)) \\ &\leqq tf(y) + (1-t)f(u) - \log(\cos td(y,u)) \end{aligned}$$

and hence

$$f(u) \leqq f(y) - \frac{\log(\cos td(y,u))}{t}$$

for all $y \in X$. Letting $t \downarrow 0$, we obtain $f(u) \leqq f(y)$. This implies that $u \in \operatorname{argmin}_X f$. □

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