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Approximate Minimality in Set Optimization and Application
（集合最適化における近似最適性とその応用）

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Abstract. This paper contains a weak optimality notion for set optimization. We consider a new concept of approximate efficiency for set optimization in terms of conical intervals and conical distance by using Tanaka's approximate minimality for vector optimization. Also, we show in the last part minimal element theorems and a variational principle by using our optimality.

1. Introduction

Weak optimality is a fundamental technique as approximation to be pripared for addressing optimization problems in which exact solutions does not exists. Loridan [1] proposed ε-efficiency in 1984, which is a well known weak efficiency on vector optimization.

In 1996, another weak optimality notion was given by Tanaka [2], characterized with ε-neighborhoods. It focuses on cases where Loridan's ε-efficiency does not make sense. In case systems prefer "too many" (unbounded) solutions to an ε-efficient solution, it may mean the Loridan's method does not choose appropriate solutions.

As a main part of this paper, we propose a weak set optimality in a similar way to Tanaka's approximate minimality. To realize this approach, we shall introduce set relations and conical intervals both taken to be like the pointwise ordering of vectors and neighborhoods of sets, respectively.

Finally, our definition is used to establish minimal element theomrems and a variational principle as application. The similar work has been done by researchers such as [5] directly inspiring the paper.

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2. Preliminaries

Throughout this paper, we let X be a topological vector space, C a convex solid (i.e., $\text{int}C \neq \emptyset$) cone in X. Also, $\leq_C$ is the pointwise ordering between two vectors in X ($x \leq_C y \;\Leftrightarrow\; y - x \in C$ for $x, y \in X$) and $\preceq_C$ is a binary relation between two subsets of X. Note that $\leq_C$ and $\preceq_C$ are usually denoted by $\leq$ and $\preceq$, respectively.

3. Motivation

Firstly, we begin with the definitions of Loridan's ε-efficiency and Tanaka's ε-approximately efficiency. Let S be a nonempty subset of $\mathbb{R}^n$, $\mathbb{R}^n_+$ the positive orthant of $\mathbb{R}^n$.

Definition 3.1 (ε-efficient point (Loridan [1], 1984))**.** $\bar{x} \in S$ is an ε-efficient point toward $d \in \mathbb{R}^n$ iff $(\bar{x} - \mathbb{R}^n_+) \cap (\varepsilon d + S \setminus \{\bar{x}\}) = \emptyset$ or equivalently, $\nexists x \in S$ such that $x + \varepsilon d \leq \bar{x}$ and $x \neq \bar{x}$.

Definition 3.2 (ε-approximately efficient point (Tanaka [2], 1996))**.** $\bar{x} \in S$ is an ε-approximately efficient point of S w.r.t. C iff $(\bar{x} - C) \cap (S \setminus B_\varepsilon(\bar{x})) = \emptyset$.

Definition 3.1 let us to give the translation by εd to the entire system but a considered point. This method has helped a lot of problems and their solutions. However, the essenciality of this weakness strongly depends on the shape of given sets. In this research, we look into pathological cases in which Definition 3.1 turns to be meaningless. Particularly, the following cases could distinguish the definitions.

Example. Let $S_1 := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ and $S_2 := -\mathbb{R}^2_{++} = -\text{int}\mathbb{R}^2_+$.
$(-2/3, -2/3)$ is a $(1/10)$-efficient point toward $(1, 1)$ of S_1 and so is a $(1/10)$-approximately efficient point with respect to $\mathbb{R}^2_+$. On the other hand, $(1/10, 1/10)$ is a $(1/10)$-efficient point toward $(1, 1)$ of S_2 while it fails to be an ε-approximately efficient point for any $\varepsilon > 0$. At this point, Definition 3.2 is in a sense, to complement Definition 3.1, regardless of a specified direction.

4. Main Research

We let X be a topological vector space, $\mathcal{A}$ a family of bounded subsets of X, $\preceq_C$ a set relation defined as $A \preceq_C B :\Leftrightarrow (A \subset B - C) \wedge (B \subset A + C)$. let us impose the order interval of a convex ordering cone. In this paper, we define a conical interval as a set of "ε-near points" of a set: $I_{C,k}(A; \varepsilon) := (A + \varepsilon - C) \cap (A - \varepsilon + C)$ for $k \in C \setminus -\text{cl}C$. The motivatin of this concept is shown in [3] as order interval: $[-x, x] := (-x + C) \cap (x - C)$ (e.g., $C \subset \mathbb{R}^n$ is a cone, $x \in \mathbb{R}^n$).

To begin with, we recall a Loridan-type basic efficiency.

Definition 4.1. Let C be a convex solid pointed ordering cone, $k \in C \setminus -\mathrm{cl}C$, $\varepsilon > 0$. $\bar{A} \in \mathcal{A}$ is an ε-minimal set toward k with respect to $\preceq$ iff $A \preceq \bar{A}$ for some $A \in \mathcal{A} \Rightarrow \bar{A} + \varepsilon k \preceq A$.

Next, our main generalizaion from [2] and an example contrasting difference between them are shown below.

Definition 4.2. Let C be a convex solid pointed ordering cone, $k \in C \setminus -\mathrm{cl}C$, $\varepsilon > 0$. $\bar{A} \in \mathcal{A}$ is an ε-approximately minimal set toward k with respect to $\preceq$ iff $A \preceq \bar{A}$ for some $A \in \mathcal{A} \Longrightarrow A \subset I_{C,k}(A;\varepsilon)$.

Example. Let $\mathcal{B} := \bigcup_{a>0, b\in\mathbb{R}} S(a,b)$ where $S(a,b) := \{(x,y) \mid (x-a)^2 + (y-b)^2 < a^2/4\}$. Then, for $(a,b) \in \mathbb{R}^2$, $S(a,b)$ is $(a/2)$-efficient toward $(1,1)$. On the other hand, any sets in $\mathcal{B}$ are not ε-approximately efficient.

5. APPLICATION

We show some application of the approximate minimality to Ekeland's variational principle. Here, we define X, Y are topological vector spaces, C_X, C_Y are convex solid cones in each space. $\mathcal{P}(\cdot)$ denotes the set of all subsets in a specified space.

First of all, we introduce topological structures named as "boundedness" for families of sets given by Hamel and Löhne.

Definition 5.1 (A. Hamel, A. Löhne (2006), [5])**.** A set $\mathcal{S} \subset \mathcal{P}(Y)$ is said to be $\precsim_{C_Y}$-bounded below if and only if there is a nonempty set $\bar{S} \subset \mathcal{P}(Y)$ such that $\bar{S} \precsim_{C_Y} S$ for all $S \in \mathcal{S}$. Similarly, $\mathcal{S} \subset \mathcal{P}(Y)$ is said to be $\preceq_{C_Y}$-bounded above if and only if $-\mathcal{S}$ is $\precsim_{C_Y}$-bounded below.

Similarly, the other type is defined by switching all the signs "$\precsim_{C_Y}$" or "$\preceq_{C_Y}$."

We set conical distance taken to be a quasi-metric function on $\mathcal{P}(Y)$, which is confirmed by the next proposition following it. The conical distance is defined with the conical interval between two specified sets.

Definition 5.2 (Conical distance)**.** Let Y be a topological vector space, $A, B \in \mathcal{P}(Y)$, $k \in C_Y \setminus -\mathrm{cl}C_Y$. The conical distance $D_{C_Y,k} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as $D_{C_Y,k}(A,B) := \max\{\inf\{t \geq 0 \mid A \subset I_{C_Y,k}(B;t)\}, \inf\{t \geq 0 \mid B \subset I_{C_Y,k}(A;t)\}\}$.

Proposition 5.1. Let Y be a topological vector space, $A, B \in \mathcal{P}(Y)$, $k \in C_Y \setminus -\mathrm{cl}C_Y$. Then, the following statements hold;

- $A_1 = A_2$ implies $D_{C,k}(A_1, A_2) = 0$;
- $D_{C_Y,k}(A_1, A_2) = D_{C_Y,k}(A_2, A_1)$;
- $D_{C_Y,k}(A_1, A_2) \leq D_{C_Y,k}(A_1, A_3) + D_{C_Y,k}(A_2, A_3)$.

Particularly, we denote $D_{C_Y,k}(A_1, A_2) = 0$ by $A_1 \approx A_2$.

This paper describes set relations as characterization of the position of two sets.

Definition 5.3 (Set relations in a product space)**.** For $V_1, V_2 \in \mathcal{P}(X)$ and $W_1, W_2 \in \mathcal{P}(Y)$ and $d \in C_X \setminus -\mathrm{cl}C_X$, $k \in C_Y \setminus -\mathrm{cl}C_Y$,

$$(V_1, W_1) \precsim^{d}_{C_X,C_Y} (V_2, W_2) \overset{\text{def}}{\Longleftrightarrow} W_1 + D_{C_X,d}(V_1, V_2)k \precsim_{C_Y} W_2;$$

$$(V_1, W_1) \preceq^{d}_{C_X,C_Y} (V_2, W_2) \overset{\text{def}}{\Longleftrightarrow} W_1 + D_{C_X,d}(V_1, V_2)k \preceq_{C_Y} W_2;$$

Unless otherwise specified, we let $\mathcal{A} \subset \mathcal{P}(X) \times \mathcal{P}(Y)$ and $\Psi(\mathcal{A}) := \{S \in \mathcal{P}(Y) \mid (V, S) \in \mathcal{A}$ for some $V \in \mathcal{P}(X)\}$.

Theorem 5.1 (Minimal element theorem)**.** Let X, Y be topological vector spaces, C_X, C_Y convex cones in X, Y, $d \in C_Y \setminus -\mathrm{cl}C_Y$. Also let $\mathcal{A} \subset \mathcal{P}(X) \times \mathcal{P}(Y)$ satisfying for some $(V_0, W_0) \in \mathcal{A}$ and $\mathcal{A}_0 := \{(V, W) \in \mathcal{A} \mid (V, W) \precsim^{d}_{C_X,C_Y} (V_0, W_0)$ and the following holds:

- $\Psi(\mathcal{A}_0)$ is $\precsim_{C_Y}$–bounded above;
- $\Psi(\mathcal{A}_0)$ is $\precsim_{C_Y}$–bounded below;
- For any $\precsim^{d}_{C_X,C_Y}$–decreasing sequence $\{(V_n, W_n)\}_{n \in \mathbb{N}}$, there exists $(V, W) \in \mathcal{A}_0$ such that $(V, W) \precsim^{d}_{C_X,C_Y} (V_n, W_n)$ for all $n \in \mathbb{N}$.

Then, there exists $(\bar{V}, \bar{W}) \in \mathcal{A}$ such that

(i) $(\bar{V}, \bar{W}) \precsim^{d}_{C_X,C_Y} (V_0, W_0)$;
(ii) If $(\tilde{V}, \tilde{W}) \precsim^{d}_{C_X,C_Y} (\bar{V}, \bar{W})$ for some $(\tilde{V}, \tilde{W}) \in \mathcal{A}$, then $\tilde{V} \approx \bar{V}$.

This theorem directly follows from the Brézis–Browder principle ([4]) and it is obvious that the other case with the switched sign "$\preceq^{d}_{C_X,C_Y}$" also comes true.

To conclude this section, a variational principle for set-valued set functions is given by recasting Theorem 6.1 in [5] as application of our research with the following sircumstances:

- $\precsim_{C_Y}$–dom$F := \{V \in \mathcal{P}(X) \mid \exists$ a nonempty set $W \in \mathcal{P}(Y)$ s.t. $F(V) \precsim_{C_Y} W\}$;
- $\preceq_{C_Y}$–dom$F := \{V \in \mathcal{P}(X) \mid \exists$ a bounded set $W \in \mathcal{P}(Y)$ s.t. $F(V) \preceq_{C_Y} W\}$;
- graph$F := \{(V, W) \in \mathcal{P}(X) \times \mathcal{P}(Y) \mid F(V) = W\}$.

Theorem 5.2 (Variational principle)**.** Let X, Y be topological vector spaces, C_X, C_Y convex cones in X, Y, $k \in C_X \backslash -\mathrm{cl}C_X$, $d \in C_Y \backslash -\mathrm{cl}C_Y$, $\varepsilon > 0$. Also, let $F : \mathcal{P}(X) \to \mathcal{P}(Y)$, $V_0 \in \preceq_{C_Y}$–dom$F$, $S(V_0) := \{V \in \mathcal{P}(X) \mid (V, F(V)) \preceq^d_{C_X,C_Y} (V_0, F(V_0))\}$ and $\mathcal{A}_0 := \{(V, W) \in \mathrm{graph}F \mid V \in S(V_0)\}$ satisfying:

- $F(S(V_0))$ is $\preceq_{C_Y} d$–bounded below;
- $F(V_0)$ is an ε–approximate efficient point of $F(X)$;
- For all $\preceq^d_{C_X,C_Y}$–decreasing sequences $\{(V_n, W_n)\} \in \mathcal{A}_0$, there is $(V, W) \in \mathcal{A}_0$ such that $(V, W) \preceq^d_{C_X,C_Y} (V_n, W_n)$ for all $n \in \mathbb{N}$.

Then, there exists $\bar{V} \in \preceq_{C_X}$–dom$F$ such that

(i) $F(\bar{V}) + D_{C_X,k}(\bar{V}, V_0) \preceq_{C_Y} F(V_0)$;

(ii) $D_{C_Y,d}(\bar{V}, V_0) \leq \varepsilon$;

(iii) $F(V) + D_{C_X,k}(V, \bar{V}) \not\preceq_{C_Y} F(\bar{V})$ for all $V \not\approx \bar{V}$.

This theorem is established from the previous theorem by applying $\mathcal{A}$ to graph$F$. We remark that the other type with "$\precsim^d_{C_X,C_Y}$" cannot be given similarly due to the fact that $B \subset I_{C,d}(A; \varepsilon)$ is not equivalent to $A \subset I_{C,d}(B; \varepsilon)$ for some $A, B \in \mathcal{A}$ and $\varepsilon > 0$.

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