数理解析研究所講究録 第2113巻 2019年 13-22

Numerical range and a conjugation on a Banach space

by

Muneo Chō and Haruna Motoyoshi

Abstract

We introduce a conjugation on a Banach space $\mathcal{X}$ and show properties of a conjugation. After that we show properties of numerical ranges of operators concerning with a conjugation C. Next we introduce (m, C)-symmetric and (m, C)-isometric operators on a Banach space and show spectral properties of such operators.

1 Conjugation on a Banach space

First we explain a conjugation on a complex Hilbert space.

Definition 1.1 *Let $\mathcal{H}$ be a complex Hilbert space. An operator C on $\mathcal{H}$ is antilinear if it holds that, for all x, $y \in \mathcal{H}$ and a, $b \in \mathbb{C}$,*

$$C(ax + by) = \bar{a}\, Cx + \bar{b}\, Cy.$$

Antilinear operator C is said to be a conjugation if it holds that, for all x, $y \in \mathcal{H}$ and a, $b \in \mathbb{C}$,

$$C^2 = I \text{ and } \langle Cx, Cy \rangle = \langle y, x \rangle,$$

where I is the identity operator on $\mathcal{H}$.

If C is a conjugation, then $\|Cx\| = \|x\|$ for all $x \in \mathcal{H}$. For a bounded linear operator T on a complex Hilbert space $\mathcal{H}$, let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_s(T)$, $\sigma_e(T)$ and $\sigma_w(T)$ denote the spectrum, the point spectrum, the approximate spectrum, the surjective spectrum, the essential spectrum and the Weyl spectrum of T, respectively. Then the following result is important.

Theorem 1.1 (S. Jung, E. Ko and J. E. Lee, [3]) *Let C be conjugation on $\mathcal{H}$. Then it holds the following statement hold:*

$$\sigma(CTC) = \overline{\sigma(T)},\ \sigma_p(CTC) = \overline{\sigma_p(T)},\ \sigma_a(CTC) = \overline{\sigma_a(T)},$$

$$\sigma_s(CTC) = \overline{\sigma_s(T)},\ \sigma_e(CTC) = \overline{\sigma_e(T)} \text{ and } \sigma_w(CTC) = \overline{\sigma_w(T)},$$

This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

where $\overline{E} = \{\overline{z} \,:\, z \in E\} \subset \mathbb{C}$.

S. Jung, E. Ko and J. E. Lee, *On complex symmetric operator matrices*, J. Math. Anal. Appl., **406**(2013), 373-385. This case doesn't need $CTC = T^*$. Only relation between T and CTC. Next we explain a conjugation on a Banach space.

Definition 1.2 *Let $\mathcal{X}$ be a complex Banach space with the norm $\|\cdot\|$ and C be an operator on $\mathcal{X}$. If C satisfies the following, then C is said to be a conjugation on a Banach space $\mathcal{X}$. For all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{C}$,*

$$(*) \qquad C^2 = I,\ C(\alpha x + \beta y) = \overline{\alpha} Cx + \overline{\beta} Cy \text{ and } \|C\| \leq 1,$$

where I is the identity operator on $\mathcal{X}$.

Of course, from the definition it holds $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$.

Theorem 1.2 *If C satisfies $(*)$ on a Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot\,,\,\cdot \rangle$, then $\langle Cx\,,\, Cy \rangle = \langle y\,,\, x \rangle$ for all $x, y \in \mathcal{H}$.*

Proof. Let $x,\, y \in \mathcal{H}$, $\alpha \in \mathbb{R}$ and let $Cy = z$. Since

$$\|Cx + \alpha z\| = \|C(x + \alpha Cz)\| \leq \|x + \alpha Cz\| = \|C(Cx + \alpha z)\| \ \leq \ \|Cx + \alpha z\|,$$

we have $\|Cx + \alpha z\| = \|x + \alpha Cz\|$. By taking square, we have $\mathrm{Re}\langle Cx, z\rangle = \mathrm{Re}\langle Cz, x\rangle$ and

$$\mathrm{Re}\ \langle Cx, Cy\rangle = \mathrm{Re}\ \langle Cx, z\rangle = \mathrm{Re}\ \langle Cz, x\rangle = \mathrm{Re}\ \langle C^2 y, x\rangle = \mathrm{Re}\ \langle y, x\rangle.$$

By taking ix instead of x, we have $\mathrm{Im}\ \langle Cx, Cy\rangle = \mathrm{Im}\ \langle y, x\rangle$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$. □

Example 1.1 *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{X} = B(\mathcal{H})$. For conjugations C, J on $\mathcal{H}$, M_{CJ} on $\mathcal{X}$ is defined by*

$$M_{CJ}(T) := CTJ \quad (T \in B(\mathcal{H}) = \mathcal{X}).$$

Then M_{CJ} is a conjugation on a Banach space $\mathcal{X}$.

Definition 1.3 *Let C be a conjugation on a Banach space $\mathcal{X}$. The dual operator $C^* : \mathcal{X}^* \longrightarrow \mathcal{X}^*$ of C is defined by*

$$(C^*(f))(x) = \overline{f(Cx)} \quad (x \in \mathcal{X},\ f \in \mathcal{X}^*),$$

where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$ and $\overline{f(Cx)}$ is the complex conjugate of $f(Cx)$.

Theorem 1.3 *If C is a conjugation on $\mathcal{X}$, then C^* is a conjugation on $\mathcal{X}^*$.*

Proof. It is clear that $C^{*2} = I^*$ and

$$C^*(f + g) = C^*(f) + C^*(g) \text{ for all } f, g \in \mathcal{X}^*.$$

For $\lambda \in \mathbb{C}$ and $x \in \mathcal{X}$, it holds $(C^*(\lambda f))(x) = \overline{\lambda}\ \overline{f(Cx)} = \overline{\lambda}\ (C^*f)(x)$ and $C^*(\lambda f) = \overline{\lambda}\ C^*(f)$. Since, for all $f \in \mathcal{X}^*$, it holds

$$|(C^*f)(x)| = |\overline{f(Cx)}| \leq \|f\|\|Cx\| = \|f\|\|x\|,$$

we have $\|C^*f\| \leq \|f\|$ and $\|C^*\| \leq 1$. □

The same results hold for spectral properties of an operator on a Banach space concerning with a conjugation.

Theorem 1.4 *Let $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then it holds the following :*

$$\sigma(CTC) = \overline{\sigma(T)},\ \sigma_a(CTC) = \overline{\sigma_a(T)},\ \sigma_p(CTC) = \overline{\sigma_p(T)} \text{ and } \sigma_s(CTC) = \overline{\sigma_s(T)}.$$

2 Numerical range of Banach space operator

In this section, we explain definition of the numerical range $V(T)$ of T on a Banach space $\mathcal{X}$.

Definition 2.1 *Let Π be the set*

$$\Pi := \{(x, f) \in \mathcal{X} \times \mathcal{X}^* \ :\ \|f\| = f(x) = \|x\| = 1\}.$$

For an operator $T \in B(\mathcal{X})$, the numerical range $V(T)$ of T is given by

$$V(T) = \{f(Tx) \ :\ (x, f) \in \Pi\}.$$

Defitinion 2.2 *For $T \in B(\mathcal{X})$;*
- *T is Hermitian if $V(T) \subset \mathbb{R}$.*
- *T is positive if $V(T) \subset [0, \infty)$. In this case, we write $T \geq 0$.*
- *T is normal if there exist Hermitian operators H and K such that $T = H + iK$ and $HK = KH$.*
- *T is hyponormal if there exist Hermitian operators H and K such that $T = H + iK$ and $i(HK - KH) \geq 0$.*

Theorem 2.1 *If $(x, f) \in \Pi$, then $(Cx, C^*f) \in \Pi$.*

Proof. Let $(x, f) \in \Pi$. Then $\|f\| = f(x) = \|x\| = 1$.

$$(C^*f)(Cx) = \overline{f(C^2x)} = \overline{f(x)} = 1$$

Since $\|Cx\| = \|x\| = 1$, we have

$$\|C^*f\| = \sup_{\|x\|=1} |(C^*f)(x)| = \sup_{\|x\|=1} |f(Cx)| \leq \|f\|\|Cx\| = 1.$$

Therefore, we have $\|C^*f\| \le 1$ and $\|C^*f\| = 1$ and so $(Cx, C^*f) \in \Pi$. $\square$

Theorem 2.2 *Let $\mathcal{X}$ be a complex Banach space, $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then $V(CTC) = \overline{V(T)}$.*

Proof. Let $z \in V(CTC)$. Then there exists $(x, f) \in \Pi$ such that $z = f(CTCx)$. We obtain $z = \overline{(C^*f)(TCx)}$. Since $(Cx, C^*f) \in \Pi$, we have $z \in \overline{V(T)}$ and $V(CTC) \subset \overline{V(T)}$ Therefore, we have $V(T) = V(C^2TC^2) \subset \overline{V(CTC)}$ and $V(CTC) = \overline{V(T)}$ $\square$

Theorem 2.3 *Let $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then following results hold.*
(1) *T is Hermitian if and only if CTC is Hermitian.*
(2) *T is positive if and only if CTC is positive.*
(3) *T is normal if and only if CTC is normal.*
(4) *T is hyponormal if and only if CTC is hyponormal.*
(5) *T is compact if and only if CTC is compact.*

Definition 2.3
• *Denote by $V_\omega(T)$ the set of all $z \in \mathbb{C}$ such that there exists a sequence $(x_n, f_n) \in \Pi$ which satisfies $w\text{-}\lim x_n = 0$ and $\lim f_n(Tx_n) = z$. The set $V_\omega(T)$ is said to be the sequential essential numerical range of T.*
• *For an operator $T \in B(\mathcal{X})$, the the essential numerical range $V_e(T)$ of T is given by*

$$V_e(T) := \{\mathcal{F}(T) : \mathcal{F} \in B(\mathcal{X})^*, \|\mathcal{F}\| = \mathcal{F}(I) = 1, \mathcal{F}(\mathcal{C}(\mathcal{X})) = \{0\}\},$$

where $\mathcal{C}(\mathcal{X})$ is the set of all compact operators on $\mathcal{X}$.
• *Denote by $W_e(T)$ the set of all $z \in \mathbb{C}$ with the property that there are nets $(x_\alpha) \subset \mathcal{X}, (f_\alpha) \subset \mathcal{X}^*$ suth that $\|f_\alpha\| = f_\alpha(x_\alpha) = 1$ $({}^\forall\alpha)$, $x_\alpha \to 0$ (weakly) and $f_\alpha(x_\alpha) \to z$. The set $W_e(T)$ is said to be the spatial essential numerical range of T.*

Theorem 2.4 *For any conjugation C, $w\text{-}\lim x_n = 0$ if and only if $w\text{-}\lim Cx_n = 0$.*

Proof. Assume $w\text{-}\lim_{n\to\infty} x_n = 0$. Then, for any $f \in \mathcal{X}^*$, since $C^*f \in \mathcal{X}^*$, we have $f(Cx_n) = \overline{(C^*f)(x_n)} \to 0$. Hence $w\text{-}\lim_{n\to\infty} Cx_n = 0$. Since $x_n = C^2x_n$, the converse is clear. $\square$

Theorem 2.5 *For any conjugation C, $V_\omega(CTC) = \overline{V_\omega(T)}$ and $W_e(CTC) = \overline{W_e(T)}$.*

Proof. Let $z \in V_\omega(CTC)$. There exists a sequence $\{(x_n, f_n)\}_{n=1}^\infty$ of Π such that $w\text{-}\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f_n(CTCx_n) = z$. We have

$$\lim_{n\to\infty}(C^*f_n)(TCx_n) = \overline{\lim_{n\to\infty} f_n(CTCx_n)} = \overline{z}.$$

Since $(Cx_n, C^*f_n) \in \Pi$ and $w\text{-}\lim_{n\to\infty} Cx_n = 0$ by Theorem2.4, we obtain $\overline{z} \in V_\omega(T)$ and $\overline{V_\omega(CTC)} \subset V_\omega(T)$. Hence we have $V_\omega(T) = V_\omega(C^2TC^2) \subset \overline{V_\omega(CTC)}$ and $V_\omega(CTC) = \overline{V_\omega(T)}$. The proof of $W_e(CTC) = \overline{W_e(T)}$ is almost the same. $\square$

Theorem 2.6 *For any conjugation C, $V_e(CTC) = \overline{V_e(T)}$.*

Proof. Let $\mathcal{F}(CTC) \in V_e(CTC)$. Then there exists $\mathcal{F} \in B(\mathcal{X})^*$ such that $\|\mathcal{F}\| = \mathcal{F}(I) = 1$, $\mathcal{F}(\mathcal{C}(\mathcal{X})) = \{0\}$. Since

$$|C^*\mathcal{F}(T)| = |\overline{\mathcal{F}(CTC)}| \leq \|\mathcal{F}\| \cdot \|CTC\| \leq \|T\|$$

and

$$C^*\mathcal{F}(I) = \overline{\mathcal{F}(CIC)} = \overline{\mathcal{F}(I)} = \overline{1} = 1,$$

we have $\|C^*\mathcal{F}\| = 1$. Moreover, by Theorem 2.3 (5), $C^*\mathcal{F}(\mathcal{C}(\mathcal{X})) = \{0\}$. Therefore, we obtain $\overline{\mathcal{F}(CTC)} \in V_e(T)$ and so $V_e(CTC) \subset \overline{V_e(T)}$. Hence we have $V_e(T) = V_e(C^2TC^2) \subset \overline{V_e(CTC)}$ and $V_e(CTC) = \overline{V_e(T)}$. $\square$

Theorem 2.7 *Let $T \in B(\mathcal{X})$ and C be a conjugation on $\mathcal{X}$. Then following results hold.*
(1) $x \in \ker(T)$ *if and only if* $Cx \in \ker(CTC)$.
(2) $x \in R(T)$ *if and only if* $Cx \in R(CTC)$.
(3) $R(T)$ *is closed if and only if* $R(CTC)$ *is closed.*

Proof. (1) If $x \in \ker(T)$, then we have $(CTC)Cx = CTx = 0$ and hence $Cx \in \ker(CTC)$. Conversely, if $Cx \in \ker(CTC)$, then we obtain $x = C^2x \in \ker(C^2TC^2) = \ker(T)$.
(2) Let $x \in R(T)$. Since $^\exists y \in \mathcal{X}$; $x = Ty$, it follows that $Cx = CTy = CTC(Cy)$ and hence $Cx \in R(CTC)$. Conversely, if $Cx \in R(CTC)$, then $x = C^2x \in R(C^2TC^2) = R(T)$.
(3)Let $R(T)$ be closed and $\{x_n\} \subset R(CTC)$ be a Cauchy sequence. By Theorem 2.7 (2), it follows $Cx_n \in R(C^2TC^2) = R(T)$. Since

$$\|Cx_m - Cx_n\| \leq \|C\|\|x_m - x_n\| \to 0 \text{ as } m, n \to \infty,$$

$\{Cx_n\} \subset R(T)$ is a Cauchy sequence. Since $R(T)$ is closed, $^\exists x_0 \in R(T)$; $x_0 = \lim_{n\to\infty} Cx_n$. Then $x_n = C^2x_n \to Cx_0$ and by Theorem 2.7 (2), we have $Cx_0 \in R(CTC)$. Therefore, $R(CTC)$ is closed. Conversely if $R(CTC)$ is closed, then $R(T) = R(C^2TC^2)$ is closed. $\square$

Definition 2.4 *Let $\sigma_{eap}(T)$ denote the set of all $z \in \mathbb{C}$ such that there exists a sequence $\{x_n\}$ of unit vectors which satisfies $x_n \to 0$ (weakly) and $(T - z)x_n \to 0$. The set $\sigma_{eap}(T)$ is said to be the essential approximate point spectrum of T.*

Theorem 2.8 *For $T \in B(\mathcal{X})$ and any conjugation C, $\sigma_{eap}(CTC) = \overline{\sigma_{eap}(T)}$.*

Proof. Let $z \in \sigma_{eap}(CTC)$. Take a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ (weakly) and $(CTC - z)x_n \to 0$ as $n \to \infty$. We have

$$C(T - \overline{z})Cx_n = (CTC - z)x_n \to 0 \text{ as } n \to \infty.$$

Thus we obtain $(T - \overline{z})Cx_n \to 0$ as $n \to \infty$. Since $\|Cx_n\| = \|x_n\| = 1$ and $Cx_n \to 0$ (weakly), we have $\overline{z} \in \sigma_{eap}(T)$ and hence $\sigma_{eap}(CTC) \subset \overline{\sigma_{eap}(T)}$. Therefore, we obtain $\sigma_{eap}(T) = \sigma_{eap}(C^2TC^2) \subset \overline{\sigma_{eap}(CTC)}$ and $\sigma_{eap}(CTC) = \overline{\sigma_{eap}(T)}$. $\square$

Definition 2.5 *An operator $T \in B(\mathcal{X})$ is Fredholm if and only if there exists operators $S_1, S_2 \in B(\mathcal{X})$ such that $TS_1 - I$ and $S_2T - I$ are compact operators. The essential spectrum σ_e of T is the set of all $z \in \mathbb{C}$ such that $T - z$ is not Fredholm.*

We have the following results.

Theorem 2.9 *For $T \in B(\mathcal{X})$ and a conjugation C on $\mathcal{X}$, T is Fredholm if and only if CTC is Fredholm.*

Theorem 2.10 *For $T \in B(\mathcal{X})$, $\sigma_e(CTC) = \overline{\sigma_e(T)}$.*

These definitions (numerical range, Fredholm, essential spectrum and others) are from the following paper: Barraa and Müller; On the essential numerical range, Acta Sci. Math. (Szeged) **71** (2005), 285-298.

3 (m, C)-Symmetric Operators on a Banach space

We introduce and show some properties of (m, C)-symmetric operators on a Banach space.

Definition 3.1 *For an operator $T \in B(\mathcal{X})$ and a conjugation C on $\mathcal{X}$, we define an operator $\alpha_m(T; C)$ by*

$$\alpha_m(T; C) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}CT^j.$$

An operator T is said to be (m, C)-symmetry if $\alpha_m(T; C) = 0$.

It hold that

$$CTC\,\alpha_m(T; C) - \alpha_m(T; C)\,T = \alpha_{m+1}(T; C).$$

Hence if T is (m, C)-symmetry, then T is (n, C)-symmetry for every $n \geq m$.

Example 3.1 *If Q is an n-nilpotent operator on $\mathcal{X}$, then Q is $(2n-1, C)$-symmetry for any conjugation C.*

Proof. By the definition, we have

$$\alpha_{2n-1}(Q; C) := \sum_{j=0}^{2n-1} (-1)^j \binom{2n-1}{j} CQ^{2n-1-j}CQ^j.$$

When $0 \leq j \leq n-1$, we have $Q^{2n-1-j} = 0$. When $j \geq n$, we obtain $Q^j = 0$. Therefore, we conclude $\alpha_{2n-1}(Q; C) = 0$. $\square$

Example 3.2 *Let $T \in B(\mathcal{H})$ satisfy $\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}CT^j = 0$ for some conjugation C on a Hilbert space $\mathcal{H}$. We define a conjugation $M_C(S)$ on $\mathcal{H}$ by $M_C(S) := CSC$ ($S \in B(\mathcal{H})$). Let an operator $L_T(S)$ be $L_T(S) := TS$ ($S \in B(\mathcal{H})$). Then L_T is an (m, M_C)-symmetric operator on a Banach space $B(\mathcal{H})$.*

Definition 3.2 *A pair (T, S) of operators T, $S \in B(\mathcal{H})$ is said to be C-doubly commuting if $TS = ST$ and $S \cdot CTC = CTC \cdot S$.*

• If (T, S) is C-doubly commuting, then it holds that

$$\alpha_n(T + S; C) = \sum_{j=0}^{n} \binom{n}{j} \alpha_{n-j}(T; C)\, \alpha_j(S; C)$$

and

$$\alpha_n(TS; C) = \sum_{j=0}^{n} \binom{n}{j} CT^jC \cdot \alpha_{n-j}(T; C)\, \alpha_j(S; C) \cdot S^{n-j}.$$

Theorem 3.1 *Let T be (m, C)-symmetry and S be (n, C)-symmetry. If (T, S) is C-doubly commuting, then $T + S$ is $(m + n - 1, C)$-symmetry.*

Proof. We have

$$\alpha_{m+n-1}(T + S; C) = \sum_{j=0}^{m+n-1} (-1)^j \binom{m+n-1}{j} \alpha_{m+n-1-j}(T; C) \cdot \alpha_j(S; C).$$

When $j \geq n$, we have $\alpha_j(S; C) = 0$. When $j \leq n-1$, we obtain $\alpha_{m+n-1-j}(T; C) = 0$ since $m+n-1-j \geq m+n-1-(n-1) = m$. Therefore, we conclude $\alpha_{m+n-1}(T+S; C) = 0$.
□

By Example 3.1 and Theorem 3.1, we have the following Theorem 3.2.

Theorem 3.2 *Let T be (m, C)-symmetry and Q be n-nilpotent. If (T, Q) is C-doubly commuting, then $T + Q$ is $(m + 2n - 2, C)$-symmetry.*

Theorem 3.3 *Let T be (m, C)-symmetry. Then*
(1) T^n is (m, C)-symmetry for any $n \in \mathbb{N}$.
(2) If T is invertible, then T^{-1} is (m, C)-symmetry.

Proof. (1) Since $\alpha_m(T; C) = 0$ and

$$\begin{aligned} (a^n - b^n)^m &= (a-b)^m(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})^m \\ &= (a-b)^m(\xi_0 a^{m(n-1)} + \xi_1 a^{m(n-1)-1}b + \cdots + \xi_{m(n-1)}b^{m(n-1)}) \end{aligned}$$

where ξ_i are coefficients ($i = 0, \dots, m(n-1)$), it follows that

$$\alpha_m(T^n; C) = \sum_{j=0}^{m(n-1)} \xi_j \, CT^{m(n-1)-j} C \cdot \alpha_m(T; C) \cdot T^j = 0.$$

Hence the operator T^n is (m, C)-symmetry.
(2) Suppose that T is invertible and (m, C)-symmetry. Since $\alpha_m(T; C) = 0$, we have

$$\begin{aligned} 0 &= CT^{-m}C \left(\alpha_m(T; C) \right) T^{-m} \\ &= CT^{-m}C \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j} C\, T^j \right) T^{-m} \\ &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} C(T^{-1})^j C \cdot (T^{-1})^{m-j}. \end{aligned}$$

Therefore, the operator T^{-1} is (m, C)-symmetry. $\square$

Next we show spectral properties of (m, C)-symmetric operators. It needs the following result.

Theorem 3.4 (C. Schmoeger, [5]) *Let $T \in B(\mathcal{X})$ and f be a polynomial. Then*

(1) $\sigma_a(f(T)) = f(\sigma_a(T))$ *and* (2) $\sigma_{eap}(f(T)) \subset f(\sigma_{eap}(T))$.

Theorem 3.5 *Let $T \in B(\mathcal{X})$ be (m, C)-symmetry.*
(1) *If $z \in \sigma_a(T)$ $(\sigma_p(T))$, then $\overline{z} \in \sigma_a(T)$ $(\sigma_p(T))$.*
(2) *If $z \in \sigma_{eap}(T)$, then $\overline{z} \in \sigma_{eap}(T)$.*

Proof. (1) Let $z \in \sigma_a(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T - z)x_n \to 0$ as $n \to \infty$. Since

$$\alpha_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CTC - z)^{m-j} (T - z)^j,$$

we have

$$0 = \lim_{n\to\infty} \alpha_m(T; C)x_n = \lim_{n\to\infty} (CTC - z)^m x_n.$$

By Theorem 3.5 for a polynomial $f(x) = z^m$, we obtain $0 \in \sigma_a(CTC - z)$ and hence $z \in \sigma_a(CTC)$. By Theorem 1.7, it holds $\overline{z} \in \sigma_a(T)$. $\square$

Theorem 3.6 *If T is (m, C)-symmetry, then* $\ker(T) \subset C(\ker(T^m))$.

Proof. If $x \in \ker(T)$, then we obtain

$$CT^m Cx = \sum_{j=1}^{m} (-1)^{j+1} \binom{m}{j} CT^{m-j} CT^j x = 0$$

and $T^m Cx = 0$. Hence we have $Cx \in \ker(T^m)$ and $x \in C(\ker(T^m))$. $\square$

4 (m,C)-Isometric Operators on a Banach space

We introduce and show some properties of an (m,C)-isometric operators on a Banach space.

Definition 4.1 *For an operator $T \in B(\mathcal{X})$ and a conjugation C on $\mathcal{X}$, we define an operator $\beta_m(T;C)$ by*

$$\beta_m(T;C) := \sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C\,T^{m-j}$$

An operator T is said to be (m,C)-isometry if $\beta_m(T;C) = 0$.

It hold that

$$CTC\,\beta_m(T;C)\,T - \beta_m(T;C) = \beta_{m+1}(T;C).$$

Hence if T is a (m,C)-isometry, then T is a (n,C)-isometry for every $n \geq m$. It holds similar results.

Example 4.1 *Let $T \in B(\mathcal{H})$ satisfy $\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C\,T^{m-j} = 0$ for some conjugation C on a Hilbert space $\mathcal{H}$. We define a conjugation M_C on $\mathcal{H}$ by $M_C(S) := CSC$ $(S \in B(\mathcal{H}))$. Let an operator L_T be $L_T(S) := TS$ $(S \in B(\mathcal{H}))$. Then L_T is an (m, M_C)-isometric operator on a Banach space $B(\mathcal{H})$.*

Theorem 4.1 *Let T is (m,C)-isometry. Then*
(1) $0 \notin \sigma_a(T)$.
(2) If $z \in \sigma_a(T)$, then $\overline{z}^{-1} \in \sigma_a(T)$.

The statement (2) holds for $\sigma_p(T)$ and $\sigma_{eap}(T)$. Therefore, if T is (m,C)-isometry, then $\|T\| \geq 1$.

Proof. (1) Assume that there exists a sequence $\{x_n\}$ of unit vectors such that $Tx_n \to 0$ as $n \to \infty$. Since it holds

$$0 = \beta_m(T;C) = \sum_{j=0}^{m-1}(-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j} + (-1)^m I,$$

we have $\lim_{n\to\infty} Ix_n = 0$. Hence, it's a contradiction and $0 \notin \sigma_a(T)$.
(2) Let $z \in \sigma_a(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T-z)x_n \to 0$. Since it holds

$$(z\,CTC - 1)^m x_n = \left(\sum_{j=0}^{m}(-1)^{j+1} \binom{m}{j} CT^{m-j}C(T^{m-j} - z^{m-j})\right) x_n \to 0,$$

we have $0 \in \sigma_a((zCTC-1)^m)$. By Theorem 3.5 for a polynomial $f(x) = z^m$, we obtain $0 \in \sigma_a(zCTC-1)$. By (1), since $z \neq 0$, we have $z^{-1} \in \sigma_a(CTC)$ and hence, by Theorem 1.7, it holds $\overline{z}^{-1} \in \sigma_a(T)$. $\square$

We have the following results.

Theorem 4.2 *Let T be (m, C)-isometry and Q be n-nilpotent. If (T, Q) is C-doubly commuting, then $T + Q$ is $(m + 2n - 2, C)$-isometry.*

Theorem 4.3 *Let T be (m, C)-isometry. Then*
(1) *T^n is (m, C)-isometry for any $n \in \mathbb{N}$.*
(2) *If T is invertible, then T^{-1} is (m, C)-isometry.*

Please see following references for details.

References

[1] M. Barraa and V. Müller, On the essential numerical range, Acta Sci. Math. (Szeged) **71** (2005), 285-298.

[2] M. Chō and K. Tanahashi, On conjugations for Banach spaces, Sci. Math. Jpn. **81** (2018), 37-45.

[3] S. Jung, E. Ko and J. E. Lee, On complex symmetric operators, J. Math. Anal. Appl. **406** (2013) 373-385.

[4] H. Motoyoshi, Linear operators and conjugations on a Banach space, to appear in Acta Sci. Math. (Szged).

[5] C. Schmoeger, The spectral mapping theorem for the essential approximate point spectrum, Colloquium Mathematicum, **74**(1997), 167-176.

Muneo Chō
Department of Mathematics, Kanagawa University, Hiratsuka 259-1293, Japan
e-mail: chiyom01@kanagawa-u.ac.jp

Haruna Motoyoshi
Department of Mathematics, Kanagawa University, Hiratsuka 259-1293, Japan
e-mail: r201770062cv@jindai.jp