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Power monotonicity for a class of operator means

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1 Introduction.

The theory of operator means is established by Kubo and Ando [4]: An *operator mean* $A \,\mathrm{m}\, B$ for positive invertible operators A, B is defined by a positive normalized operator monotone function f on $(0, \infty)$ by

$$A \,\mathrm{m}\, B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Here the *normalization* is $f(1) = 1$. One of the result of the Kubo-Ando theory is to give the bijection between the operator means and the positive normalized operator monotone functions on $(0, \infty)$ as above. In this bijection, f is often called the *representing function* of an operator mean $f(x) = 1 \,\mathrm{m}\, x$.

Related to this, we gave a path of positive function on $(0, \infty)$ in [2];

$$F_r(x) = \frac{3r-1}{3r+1} \frac{x^{\frac{3r+1}{2}} - 1}{x^{\frac{3r-1}{2}} - 1}.$$

It is monotone increasing for $r \in \mathbb{R}$, which is the result of Takahasi, Tsukada, Tanahashi and Ogiwara [5]. For $r \in [-1, 1]$, they are positive normalized operator monotone functions. Moreover they are symmetric:

$$A \,\mathrm{m}_f B = B \,\mathrm{m}_f A, \qquad \text{that is,} \qquad f(x) = x f\left(\frac{1}{x}\right).$$

Typical means are listed below as numerical ones:

r	-1	$-1/3$	0	$1/3$	1
$a \,\mathbf{m}_r b = aF_r(b/a)$	$\dfrac{2ab}{a+b}$	$L^*(a,b) = \frac{b \log \frac{a}{b}}{a-b}$	$\sqrt{ab}$	$L(a,b) = \frac{a-b}{\log \frac{a}{b}}$	$\dfrac{a+b}{2}$

Transformed by $s = \frac{3r-1}{2}$, it is equivalent to the following path:

$$\tilde{F}_s(x) = \frac{s}{s+1} \frac{x^{s+1} - 1}{x^s - 1},$$

which is also discussed in [3].

Recently Wada [6] introduced the power monotonicity of representing functions f and showed the relation to the Ando-Hial inequality [7, 1]: f is called *PMI* (resp., *PMD*) if f satisfies

$$f(x^p) \leqq f(x)^p \qquad (\text{ resp., } f(x^p) \geqq f(x)^p) \qquad \text{for all } p > 1.$$

In this note, we show the power monotonicity for F_r. Incidentally we see the role of the terminal means in these inequalities.

2 Main result.

Theorem 1. *The function F_r is PMI for $r \geqq 0$ and PMD for $r \leqq 0$.*

For $\tilde{F}_s$, this result is equivalent to:

Theorem F. *The function $\tilde{F}_s$ is PMI for $s \geqq -\frac{1}{2}$ and PMD for $s \leqq -\frac{1}{2}$.*

In fact, we show Theorem 1' since $\tilde{F}_s$ has simple parameters. To show this, we need two lemmas due to Takahasi-Tsukada-Tanahashi-Hagiwara [5]. For completeness, we give each proof. First, we see the following property:

Lemma 2. *The function* $J(t) = \begin{cases} \frac{e^t}{e^t-1} - \frac{1}{t} & (t \neq 0) \\ \frac{1}{2} & (t = 0) \end{cases}$ *is monotone increasing.*

It is easy to see that

$$\text{For } x > 1,\ G_x(s) = \begin{cases} \log \dfrac{x^s - 1}{s} & (s \neq 0) \\ \log(\log x) & (s = 0) \end{cases} \text{ is monotone increasing.} \qquad (*)$$

Combining these, we have:

Corollary 3. *For $x > 1$, $G'_x(s)$ is monotone increasing.*

Incidentally, these results show the known property: *$\tilde{F}_s$ is monotone increasing for s*, which is the required result in [5]. In fact, by $\log \tilde{F}_s(x) = G_x(s+1) - G_x(s)$,

$$\frac{\partial \log \tilde{F}_s(x)}{\partial s} = G'_x(s+1) - G'_x(s) \geqq 0.$$

Thus $\log \tilde{F}_s$ is monotone increasing for s when $x > 1$. As for the case $0 < x < 1$, we have

$$\tilde{F}_s(x) = \frac{s}{s+1} \frac{x^{s+1} - 1}{x^s - 1} = \frac{s}{s+1} \frac{\left(\frac{1}{x}\right)^{s+1} - 1}{\left(\frac{1}{x}\right)^s - 1} \times x,$$

which also shows the monotonicity for s. Thus it holds for all $x > 0$.

Remark . The representing function $S_\alpha(x)$ of the *Stolarsky mean* is defined by

$$s_\alpha(t) = \left(\frac{t^\alpha - 1}{\alpha(t-1)}\right)^{\frac{1}{\alpha-1}}.$$

S.Wada [6, Prop.3.2] showed that s_α is PMD on $[-2,-1]$ and PMI on $[-1,2]$. Putting $\alpha = \frac{s+1}{s}$, $t = x^s$, we have

$$\tilde{S}_s(x) = \left(\frac{s}{s+1}\frac{x^{s+1}-1}{x^s-1}\right)^s,$$

which is closely related to our path $\tilde{F}_s$. In fact, for $s > 0$, we have $\tilde{F}_s$ is PMD (resp., PMI) on $\mathcal{I}$ if and only if $\tilde{S}_s$ is PMD (resp., PMI) on $\mathcal{I}$. For the negative case, these concepts are exchanged. Then we show Wada's result directly implies that F_r is PMD for $r \in (-\frac{1}{3}, 0)$ and that it is PMI for $r \in (0, \frac{1}{9}) \cup (1, \infty)$. Thus Wada's result does not imply all our results in the above theorem.

In fact, consider the PMD case:

$$-2 < \alpha = \frac{s+1}{s} < -1 \iff -\frac{1}{2} < s = \frac{3r-1}{2} < -\frac{1}{3} \iff 0 < r < \frac{1}{9},$$

which shows F_r is PMI for $r \in (0, \frac{1}{9})$ by $s < 0$.

Next consider the PMI case, which is divided into the negative case and the positive one: For $(-1 <)s < 0$, we have

$$-1 < \alpha = \frac{s+1}{s} < 0 \iff -1 < s = \frac{3r-1}{2} < -\frac{1}{2} \iff -\frac{1}{3} < r < 0,$$

which shows F_r is PMD for $r \in (-\frac{1}{3}, 0)$. Lastly, for $s > 0$ we have

$$0 < \alpha = \frac{s+1}{s} < 2 \iff s > 1 \iff r > 1,$$

which shows F_r is PMI for $r \in (1, \infty)$.

3 Relation to the terminal means

Restricting ourselves to the case $p = n$, integers and $|r| \geqq \frac{1}{3}$. Then, we show the following partial result of Theorem 1 via the arithmetic or harmonic means:

Theorem 4. *For any integer n and $r \geqq \frac{1}{3}$,*

$$F_r(x^n) - F_r(x)^n \geqq F_r(x)\left(\left(\frac{x+1}{2}\right)^{n-1} - F_r(x)^{n-1}\right) \geqq 0.$$

For $r \leqq -\frac{1}{3}$,

$$F_r(x)^n - F_r(x^n) \geqq F_r(x)\left(F_r(x)^{n-1} - \left(\frac{2x}{1+x}\right)^{n-1}\right) \geqq 0.$$

To see this, we give a lemma:

Lemma 5. *A function* $g_n(r) = \frac{\sum_{\ell=0}^{n-1} x^{\frac{\ell(3r+1)}{2}}}{\sum_{k=0}^{n-1} x^{\frac{k(3r-1)}{2}}} = \frac{F_r(x^n)}{F_r(x)}$ *is monotone increasing.*

4 Concluding remark

Very recently, Yamazaki extend Theorem 4 to the following:

Theorem (Yamazaki). *For* $p, q \in [-1, 1]$*, the represenming function*

$$F_{p,q}(x) = \left(\frac{p}{p+q}\frac{x^{p+q}-1}{x^p-1}\right)^{\frac{1}{q}}$$

is PMI for $2p + q \geqq 0$*, and PMD for* $2p + q \leqq 0$*.*

This theorem is shown by the following integral representation:

$$F_{p,q}(x) = \left(\int_0^1 (1-t+tx^p)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} \quad \text{Consider:}$$

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