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Estimations of the weighted power mean by the Heron mean

前橋工科大学　伊藤 公智 (Masatoshi Ito)
(Maebashi Institute of Technology)

Abstract

As the means generalizing the arithmetic and the geometric ones, the power mean and the Heron mean are known. For positive real numbers a and b, the weighted power mean $P_{t,q}(a,b)$ and the weighted Heron mean $K_{t,q}(a,b)$ for $t \in [0,1]$ and $q \in \mathbb{R}$ are defined by $P_{t,q}(a,b) = \{(1-t)a^q + tb^q\}^{\frac{1}{q}}$ and $K_{t,q}(a,b) = (1-q)a^{1-t}b^t + q\{(1-t)a + tb\}$, respectively.

In this report, as a generalization of Wu and Debnath's result on non-weighted means (the case $t = \frac{1}{2}$), we get estimations of the weighted power mean by the weighted Heron mean. We also obtain the results for bounded linear operators on a Hilbert space, and some determinant and trace inequalities of matrices by using our main results.

1 Introduction

This report is based on [5]. As means of two positive real numbers a and b, the following are well known.

$$A(a,b) = \frac{a+b}{2} \text{ (arithmetic mean)}, \quad G(a,b) = \sqrt{ab} \text{ (geometric mean)},$$
$$H(a,b) = \frac{2ab}{a+b} \text{ (harmonic mean)}, \quad LM(a,b) = \frac{a-b}{\log a - \log b} \text{ (logarithmic mean)}.$$

We also know some generalizations of these means. For example, for $q \in \mathbb{R}$,

$$P_q(a,b) = \begin{cases} \left(\dfrac{a^q + b^q}{2}\right)^{\frac{1}{q}} & \text{if } q \neq 0, \\ \sqrt{ab} & \text{if } q = 0, \end{cases} \quad \text{(power mean)},$$

$$J_q(a,b) = \begin{cases} \dfrac{q}{q+1}\dfrac{a^{q+1} - b^{q+1}}{a^q - b^q} & \text{if } q \neq 0, -1, \\ \dfrac{a-b}{\log a - \log b} & \text{if } q = 0, \\ \dfrac{ab(\log a - \log b)}{a-b} & \text{if } q = -1, \end{cases} \quad \text{(power difference mean)},$$

$$L_q(a,b) = \frac{a^{q+1}+b^{q+1}}{a^q+b^q} \text{ (Lehmer mean)},$$

$$K_q(a,b) = (1-q)\sqrt{ab} + q\frac{a+b}{2} \text{ (Heron mean)}.$$

These means are symmetric, that is, $A(a,b) = A(b,a)$, $G(a,b) = G(b,a)$ and so on. We note that $J_q(a,a) \equiv \lim_{b\to a} J_q(a,b) = a$. It is well known that

$$\begin{aligned}
&H(a,b) \le G(a,b) \le LM(a,b) \le A(a,b),\\
&A(a,b) = P_1(a,b) = J_1(a,b) = L_0(a,b) = K_1(a,b),\\
&LM(a,b) = J_0(a,b),\\
&G(a,b) = P_0(a,b) = J_{\frac{-1}{2}}(a,b) = L_{\frac{-1}{2}}(a,b) = K_0(a,b),\\
&H(a,b) = P_{-1}(a,b) = J_{-2}(a,b) = L_{-1}(a,b),
\end{aligned}$$

and also $P_q(a,b)$, $J_q(a,b)$, $L_q(a,b)$ and $K_q(a,b)$ are monotone increasing on $q \in \mathbb{R}$.

It is well known that some of the above means have their weighted version as follows: For $t \in [0,1]$ and $q \in \mathbb{R}$,

$$\begin{aligned}
&A_t(a,b) = (1-t)a + tb \quad \text{(arithmetic mean)},\\
&G_t(a,b) = a^{1-t}b^t \quad \text{(geometric mean)},\\
&H_t(a,b) = \{(1-t)a^{-1} + tb^{-1}\}^{-1} \quad \text{(harmonic mean)},\\
&P_{t,q}(a,b) = \begin{cases} \{(1-t)a^q + tb^q\}^{\frac{1}{q}} & \text{if } q \ne 0,\\ a^{1-t}b^t & \text{if } q = 0, \end{cases} \quad \text{(power mean)},\\
&K_{t,q}(a,b) = (1-q)a^{1-t}b^t + q\{(1-t)a + tb\} \quad \text{(Heron mean)}.
\end{aligned}$$

If the weight t is equal to $\frac{1}{2}$, then the weighted means coincide with the original (non-weighted) ones as $A(a,b) = A_{\frac{1}{2}}(a,b)$ and $P_q(a,b) = P_{\frac{1}{2},q}(a,b)$. The weighted means have the properties that $A_t(a,b) = A_{1-t}(b,a)$, $G_t(a,b) = G_{1-t}(b,a)$ and so on.

Similarly to the non-weighted means, the weighted means have the properties that

$$\begin{aligned}
&H_t(a,b) \le G_t(a,b) \le A_t(a,b),\\
&A_t(a,b) = P_{t,1}(a,b) = K_{t,1}(a,b),\\
&G_t(a,b) = P_{t,0}(a,b) = K_{t,0}(a,b),\\
&H_t(a,b) = P_{t,-1}(a,b),
\end{aligned}$$

and also $P_{t,q}(a,b)$ and $K_{t,q}(a,b)$ are monotone increasing on $q \in \mathbb{R}$. The inequality $G_t(a,b) \le A_t(a,b)$ is sometimes called Young's inequality.

Many researchers investigate estimations of these means. For example, recently, we have obtained the results on estimations of several means by the Heron mean. The results for the power difference mean are in [13, 3], and the results for the Lehmer mean are in [4]. For the power mean, Janous [6], Wu and Debnath [12] obtained the following Theorem 1.A.

Theorem 1.A ([6, 12])**.** *Let $a, b > 0$ with $a \neq b$.*

(i) *If $0 < r < \frac{1}{2}$ or $1 < r$, then $K_{\left(\frac{1}{2}\right)^{\frac{1}{r}-1}}(a,b) < P_r(a,b) < K_r(a,b)$.*

(ii) *If $\frac{1}{2} < r < 1$. Then $K_r(a,b) < P_r(a,b) < K_{\left(\frac{1}{2}\right)^{\frac{1}{r}-1}}(a,b)$.*

(iii) *If $r < 0$. Then $K_r(a,b) < P_r(a,b) < K_0(a,b) = G(a,b)$.*

The given parameters of $K_\alpha(a,b)$ in each case are best possible.

We remark that Janous [6] has shown Theorem 1.A for $0 < r < 1$ as the results on estimations of the generalized Heronian mean $\frac{a+w\sqrt{ab}+b}{w+2}$ for $w \geq 0$, and also Wu and Debnath [12] got Theorem 1.A as the results on upper and lower bounds of $\frac{P_r(a,b)-G(a,b)}{A(a,b)-G(a,b)}$.

In this report, as an extension of Theorem 1.A, we obtain estimations of the weighted power mean by the weighed Heron mean. We also obtain the results for bounded linear operators on a Hilbert space. Moreover, related to the results in [1, 7], we get some determinant and trace inequalities of matrices.

2 Main results

In this section, we obtain estimations of the weighted power mean of two positive real numbers by the weighted Heron mean. In what follows, we define that

$$\beta(t,r) = \frac{tr}{1-t}\left\{\frac{t(1-2r)}{t-r}\right\}^{\frac{1}{r}-2} \quad \text{and} \quad \widehat{\beta}(t,r) = \min\{\beta(t,r), 1\} \tag{2.1}$$

for $t \in (0,1)$ and $r \in \mathbb{R}$ with $r \neq 0, \frac{1}{2}, t$. We need two lemmas in order to prove our main results. We omit these proofs.

Lemma 2.1. *Let $t \in (0,1)$ and $r \in \mathbb{R}$ with $r \neq 0, \frac{1}{2}$. Let $\beta(t,r)$ as in (2.1).*

(i) *If $0 < r < t < \frac{1}{2}$, then $r < \beta(t,r)$ holds.*

(ii) *If $r < 0 < t < \frac{1}{2}$, then $\beta(t,r) < r$ holds.*

(iii) *If $\frac{1}{2} < t < r < 1$, then $\beta(t,r) < r$ holds.*

(iv) *If $\frac{1}{2} < t < 1 < r$, then $r < \beta(t,r)$ holds.*

Lemma 2.2. *Let $t, r \in (0,1)$ with $r \neq \frac{1}{2}$.*

(i) *If $t \leq r \leq 1-t$, then $t^{\frac{1}{r}-1} < r < (1-t)^{\frac{1}{r}-1}$ holds.*

(ii) *If $1-t \leq r \leq t$, then $(1-t)^{\frac{1}{r}-1} < r < t^{\frac{1}{r}-1}$ holds.*

Now we state our main results.

Theorem 2.3. *Let $t, r \in (0,1)$. Let $\beta(t,r)$ and $\widehat{\beta}(t,r)$ as in (2.1). For all $a, b > 0$ with $a \neq b$, we have the following.*

(i) *If $t \leq r \leq 1-t$, then $K_{t,t^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,(1-t)^{\frac{1}{r}-1}}(a,b)$.*

(ii) *If $1-t \leq r \leq t$, then $K_{t,(1-t)^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,t^{\frac{1}{r}-1}}(a,b)$.*

(iii) *If $r < t \leq 1-t$, then $K_{t,t^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\widehat{\beta}(t,r)}(a,b)$.*

(iv) *If $r < 1-t \leq t$, then $K_{t,(1-t)^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\widehat{\beta}(1-t,r)}(a,b)$.*

(v) *If $t \leq 1-t < r$, then $K_{t,\beta(1-t,r)}(a,b) < P_{t,r}(a,b) < K_{t,(1-t)^{\frac{1}{r}-1}}(a,b)$.*

(vi) *If $1-t \leq t < r$, then $K_{t,\beta(t,r)}(a,b) < P_{t,r}(a,b) < K_{t,t^{\frac{1}{r}-1}}(a,b)$.*

The given parameters of $K_{t,\alpha}(a,b)$ in each case are best possible on α except the parts $\alpha = \beta(\,\cdot\,,r)$ and $\alpha = \widehat{\beta}(\,\cdot\,,r)$.

Theorem 2.4. *Let $\beta(t,r)$ as in (2.1). For all $a, b > 0$ with $a \neq b$, we have the following.*

(i) *If $t \in (0, \frac{1}{2}]$ and $r > 1$, then $K_{t,(1-t)^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\beta(1-t,r)}(a,b)$.*

(ii) *If $t \in [\frac{1}{2}, 1)$ and $r > 1$, then $K_{t,t^{\frac{1}{r}-1}}(a,b) < P_{t,r}(a,b) < K_{t,\beta(t,r)}(a,b)$.*

(iii) *If $t \in (0, \frac{1}{2}]$ and $r < 0$, then $K_{t,\beta(t,r)}(a,b) < P_{t,r}(a,b) < K_{t,0}(a,b) = G_t(a,b)$.*

(iv) *If $t \in [\frac{1}{2}, 1)$ and $r < 0$, then $K_{t,\beta(1-t,r)}(a,b) < P_{t,r}(a,b) < K_{t,0}(a,b) = G_t(a,b)$.*

The given parameters of $K_{t,\alpha}(a,b)$ in each case are best possible on α except the parts $\alpha = \beta(\,\cdot\,,r)$.

Theorems 2.3 and 2.4 imply Theorem 1.A by putting $t = \frac{1}{2}$. We remark that the best possibility of the parts $\alpha = \beta(\frac{1}{2}, r) = r$ is also shown by scrutinizing the proofs of Theorems 2.3 and 2.4. The following Figure 1 shows the domains of parameters in Theorems 2.3 and 2.4.

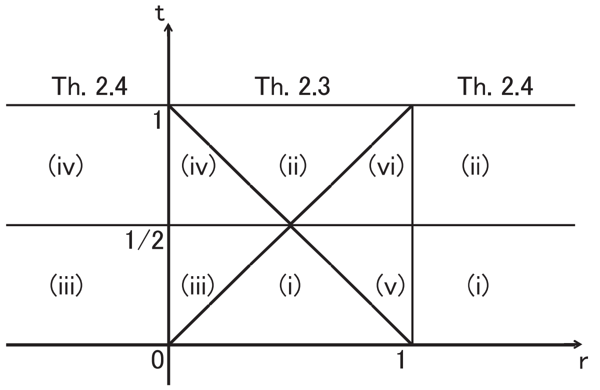

FIGURE 1

Proofs of Theorems 2.3 and 2.4. Here, we only prove (i) and (iii) in Theorem 2.3. The rest are shown by the similar way. We remark that, since $K_{t,\alpha}(a,b) = K_{1-t,\alpha}(b,a)$ and $P_{t,r}(a,b) = P_{1-t,r}(b,a)$ hold for $a, b > 0$, (ii), (iv) and (vi) are immediately obtained by (i), (iii) and (v), respectively.

We have only to consider the case $(a,b) = (1,x)$ with $x \neq 1$ by easy replacement. Let

$$\begin{aligned} f_t(x) &= P_{t,r}(1,x) - K_{t,\alpha}(1,x) \\ &= \{(1-t) + tx^r\}^{\frac{1}{r}} - (1-\alpha)x^t - \alpha\{(1-t) + tx\}. \end{aligned} \tag{2.2}$$

Now we discuss upper and lower bounds of α to hold the inequalities $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ and $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$, that is, $f_t(x) > 0$ and $f_t(x) < 0$ for all $x > 0$. Let

$$\begin{aligned} g_t(x) &= \{(1-t) + tx^r\}^{\frac{1}{r}-1}x^{r-t} - (1-\alpha) - \alpha x^{1-t}, \\ h_t(x) &= t(1-r)\{(1-t) + tx^r\}^{\frac{1}{r}-2}x^{2r-1} \\ &\quad + (r-t)\{(1-t) + tx^r\}^{\frac{1}{r}-1}x^{r-1} - \alpha(1-t) \quad \text{and} \\ k_t(x) &= t(r-1+t)x^r - (1-t)(r-t). \end{aligned} \tag{2.3}$$

Then we have

$$\begin{aligned} f_t'(x) &= tx^{t-1}g_t(x), \\ g_t'(x) &= x^{-t}h_t(x) \quad \text{and} \\ h_t'(x) &= (1-t)(1-r)\{(1-t) + tx^r\}^{\frac{1}{r}-3}x^{r-2}k_t(x) \end{aligned} \tag{2.4}$$

since

$$\begin{aligned} f_t'(x) &= tx^{r-1}\{(1-t) + tx^r\}^{\frac{1}{r}-1} - (1-\alpha)tx^{t-1} - \alpha t \\ &= tx^{t-1}\left[\{(1-t) + tx^r\}^{\frac{1}{r}-1}x^{r-t} - (1-\alpha) - \alpha x^{1-t}\right], \end{aligned}$$

$$
\begin{aligned}
g_t'(x) &= t(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-1-t} \\
&\quad + (r-t)\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-t-1} - \alpha(1-t)x^{-t} \\
&= x^{-t}\Big[t(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-1} \\
&\qquad + (r-t)\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-1} - \alpha(1-t)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
h_t'(x) &= t^2(1-r)(1-2r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{3r-2} \\
&\quad + t(1-r)(2r-1)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-2} \\
&\quad + t(r-t)(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-2}x^{2r-2} \\
&\quad + (r-t)(r-1)\{(1-t)+tx^r\}^{\frac{1}{r}-1}x^{r-2} \\
&= (1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{r-2} \\
&\quad \times \big[(1-2r)t^2x^{2r} + (3r-1-t)\{(1-t)+tx^r\}tx^r - (r-t)\{(1-t)+tx^r\}^2\big] \\
&= (1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{r-2} \\
&\quad \times \big[tx^r - \{(1-t)+tx^r\}\big]\big[(1-2r)tx^r + (r-t)\{(1-t)+tx^r\}\big] \\
&= (1-t)(1-r)\{(1-t)+tx^r\}^{\frac{1}{r}-3}x^{r-2}\big[t(r-1+t)x^r - (1-t)(r-t)\big].
\end{aligned}
$$

Proof of (i). We may except the case $r = t = \frac{1}{2}$ since $P_{\frac{1}{2},\frac{1}{2}}(1,x) = K_{\frac{1}{2},\frac{1}{2}}(1,x)$ holds. Firstly, we consider the case $\alpha \le r$.

(i-a) The case $\alpha \le r$ and $0 < x < 1$. If $t \le r \le 1-t$ holds, then $h_t'(x) < 0$ holds for $0 < x \le 1$, that is,

$$h_t(x) \text{ is decreasing for } 0 < x \le 1 \tag{2.5}$$

by (2.3) and (2.4). Since $h_t(1) = (r-\alpha)(1-t) \ge 0$, (2.5) implies that $g_t'(x) = x^{-t}h_t(x) > 0$ holds for $0 < x < 1$, that is,

$$g_t(x) \text{ is increasing for } 0 < x \le 1.$$

Since $g_t(1) = 0$, $f_t'(x) = tx^{t-1}g_t(x) < 0$ holds for $0 < x < 1$, that is,

$$f_t(x) \text{ is decreasing for } 0 < x \le 1.$$

Therefore, since $f_t(1) = 0$, we have

$$f_t(x) > 0, \text{ that is, } K_{t,\alpha}(1,x) < P_{t,r}(1,x) \text{ for } 0 < x < 1. \tag{2.6}$$

(i-b) The case $\alpha \le r$ and $x > 1$. Noting that $K_{t,\alpha}(1,x) = xK_{1-t,\alpha}(1,x^{-1})$ and $P_{t,r}(1,x) = xP_{1-t,r}(1,x^{-1})$, we consider $f_{t_1}(y)$ for $y = x^{-1} \in (0,1)$ and $t_1 = 1-t$.

If $1 - t_1 \le r \le t_1$ holds, then $h_{t_1}'(y) > 0$ holds for $0 < y \le 1$, that is,

$$h_{t_1}(y) \text{ is increasing for } 0 < y \le 1$$

by (2.3) and (2.4). If $\alpha < r$, then there exists a $\delta_1 \in (0,1)$ such that $h_{t_1}(\delta_1) = 0$ since

$$h_{t_1}(y) = \{(1-t_1)y^{-r} + t_1\}^{\frac{1-r}{r}} \left[t_1(1-r)\{(1-t_1)y^{-r} + t_1\}^{-1} + (r-t_1)\right] - \alpha(1-t_1)$$
$$\to -\infty \quad (y \to +0)$$

and $h_{t_1}(1) = (r-\alpha)(1-t_1) > 0$. This ensures that $g'_{t_1}(y) < 0$ for $0 < y < \delta_1$ and $g'_{t_1}(y) > 0$ for $\delta_1 < y < 1$ hold, that is,

$$g_{t_1}(y) \text{ is decreasing for } 0 < y < \delta_1 \text{ and increasing for } \delta_1 < y < 1.$$

Then there exists a $\delta_2 \in (0, \delta_1)$ such that $g_{t_1}(\delta_2) = 0$ since $\lim_{y \to +0} g_{t_1}(y) = \infty$ holds and $g_{t_1}(1) = 0$ assures that $g_{t_1}(\delta_1) < 0$. So $f'_{t_1}(y) > 0$ holds for $0 < y < \delta_2$ and $f'_{t_1}(y) < 0$ holds for $\delta_2 < y < 1$ hold, that is,

$$f_{t_1}(y) \text{ is increasing for } 0 < y < \delta_2 \text{ and decreasing for } \delta_2 < y < 1.$$

If $\alpha \le (1-t_1)^{\frac{1}{r}-1}$, then $f_{t_1}(0) \ge 0$, so that $f_{t_1}(y) > 0$ holds for $0 < y < 1$ since $f_{t_1}(1) = 0$.

If $\alpha = r$, then $f_{t_1}(y) < 0$ for $0 < y < 1$ by the similar argument. We remark that $(1-t_1)^{\frac{1}{r}-1} < r = \alpha$ for $1 - t_1 \le r \le t_1$ by (ii) in Lemma 2.2.

Therefore we have $K_{t_1,\alpha}(1,y) < P_{t_1,r}(1,y)$ for $0 < y < 1$ if $\alpha \le (1-t_1)^{\frac{1}{r}-1}$, that is,

$$K_{t,\alpha}(1,x) < P_{t,r}(1,x) \text{ for } x > 1 \text{ holds if } \alpha \le t^{\frac{1}{r}-1}. \tag{2.7}$$

Hence, by (2.6) and (2.7), we get $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ for all $x > 0$ with $x \ne 1$ if $\alpha \le t^{\frac{1}{r}-1}$. This argument also proves the best possibility of α since $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ or $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ does not always hold for $x > 0$ with $x \ne 1$ if $t^{\frac{1}{r}-1} < \alpha \le r$.

Next we consider the case $r \le \alpha$. By the similar way to (i-b), we obtain that $f_t(x) < 0$, that is, $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $0 < x < 1$ if $\alpha \ge (1-t)^{\frac{1}{r}-1}$. By applying the similar way to (i-a) for $f_{t_1}(y)$ as in (i-b), we obtain that $f_{t_1}(y) < 0$ holds for $0 < y < 1$, that is, $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $x > 1$. Hence we get $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $x > 0$ with $x \ne 1$ if $\alpha \ge (1-t)^{\frac{1}{r}-1}$. We also get the best possibility of α, that is, $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ or $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ does not always hold for $x > 0$ if $r \le \alpha < (1-t)^{\frac{1}{r}-1}$.

Proof of (iii). Firstly, we consider the case $\alpha < r$. Let $\delta_0 = \left(\frac{(1-t)(t-r)}{t(1-t-r)}\right)^{\frac{1}{r}}$. We remark that $0 < \delta_0 \le 1$ (resp. $\delta_0 \ge 1$) holds for $t, r \in (0,1)$ and $r < t \le 1-t$ (resp. $r < 1-t \le t$).

(iii-a) The case $\alpha < r$ and $0 < x < 1$. If $r < t \le 1-t$ holds, then $h'_t(x) > 0$ holds for $0 < x < \delta_0$ and $h'_t(x) < 0$ holds for $\delta_0 < x < 1$, that is,

$$h_t(x) \text{ is increasing for } 0 < x < \delta_0 \text{ and decreasing for } \delta_0 < x < 1$$

by (2.3) and (2.4). Then there exists a $\delta_1 \in (0, \delta_0)$ such that $h_t(\delta_1) = 0$ since $\lim_{x\to+0} h_t(x) = -\infty$ and $h_t(1) = (r-\alpha)(1-t) \geq 0$. This ensures that $g_t'(x) < 0$ for $0 < x < \delta_1$ and $g_t'(x) > 0$ for $\delta_1 < x < 1$, that is,

$$g_t(x) \text{ is decreasing for } 0 < x < \delta_1 \text{ and increasing for } \delta_1 < x < 1.$$

Then there exists a $\delta_2 \in (0, \delta_1)$ such that $g_t(\delta_2) = 0$ since $\lim_{x\to+0} g_t(x) = \infty$ and $g_t(1) = 0$. So $f_t'(x) > 0$ holds for $0 < x < \delta_2$ and $f_t'(x) < 0$ holds for $\delta_2 < x < 1$ hold, that is,

$$f_t(x) \text{ is increasing for } 0 < x < \delta_2 \text{ and decreasing for } \delta_2 < x < 1.$$

If $\alpha \leq (1-t)^{\frac{1}{r}-1}$, then $f_t(0) > 0$, so that $f_t(x) > 0$ holds for $0 < x < 1$ since $f_t(1) = 0$.

Therefore we have

$$K_{t,\alpha}(1,x) < P_{t,r}(1,x) \text{ for } 0 < x < 1 \text{ if } \alpha \leq (1-t)^{\frac{1}{r}-1}.$$

(iii-b) The case $\alpha < r$ and $x > 1$. Similarly to (i-b), we consider $f_{t_1}(y)$ for $y = x^{-1} \in (0,1)$ and $t_1 = 1-t$. Noting that $r < t \leq 1-t$ if and only if $r < 1-t_1 \leq t_1$, by the similar way to (i-b), we have that $K_{t_1,\alpha}(1,y) < P_{t_1,r}(1,y)$ for $0 < y < 1$ if $\alpha \leq (1-t_1)^{\frac{1}{r}-1}$, that is,

$$K_{t,\alpha}(1,x) < P_{t,r}(1,x) \text{ for } x > 1 \text{ if } \alpha \leq t^{\frac{1}{r}-1}.$$

Hence, by (iii-a) and (iii-b), we get $K_{t,\alpha}(1,x) \leq P_{t,r}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \leq t^{\frac{1}{r}-1}$ since $t^{\frac{1}{r}-1} \leq (1-t)^{\frac{1}{r}-1}$ holds. We remark that $t^{\frac{1}{r}-1} < \left(\frac{1}{2}\right)^{\frac{1}{r}-1} < r$ holds for $r, t \in (0, \frac{1}{2})$. This argument also proves the best possibility of α since $K_{t,\alpha}(1,x) < P_{t,r}(1,x)$ or $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ does not always hold for $x > 0$ with $x \neq 1$ if $t^{\frac{1}{r}-1} < \alpha < r$.

Next, we consider the case $r \leq \alpha$. If $\alpha \geq 1$, then we obviously get that $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ holds for all $x > 0$ with $x \neq 1$ since $K_{t,1}(1,x) = A_t(1,x)$. We remark that $r < \beta(t,r)$ holds for $0 < r < t < \frac{1}{2}$ by (i) in Lemma 2.1.

(iii-c) The case $r \leq \beta(t,r) \leq \alpha$ and $0 < x < 1$. If $r < t \leq 1-t$ holds, then $h_t'(x) > 0$ holds for $0 < x < \delta_0$ and $h_t'(x) < 0$ holds for $\delta_0 < x < 1$, that is,

$$h_t(x) \text{ is increasing for } 0 < x < \delta_0 \text{ and decreasing for } \delta_0 < x < 1.$$

by (2.3) and (2.4). Noting that $h(\delta_0) \leq 0$ if and only if $\alpha \geq \beta(t,r)$, we get that $g_t'(x) \leq 0$ for $0 < x < 1$, that is,

$$g_t(x) \text{ is decreasing for } 0 < x < 1.$$

Since $g_t(1) = 0$, $f_t'(x) > 0$ holds for $0 < x < 1$, that is,

$$f_t(x) \text{ is increasing for } 0 < x < 1.$$

Therefore, since $f_t(1) = 0$, we have

$$f_t(x) < 0, \text{ that is, } P_{t,r}(1,x) < K_{t,\alpha}(1,x) \text{ for } 0 < x < 1 \text{ if } \alpha \geq \beta(t,r).$$

(iii-d) The case $r \leq \alpha$ and $x > 1$. We consider $f_{t_1}(y)$ for $y = x^{-1} \in (0,1)$ and $t_1 = 1-t$. Noting that $r \leq t \leq 1-t$ if and only if $r \leq 1-t_1 \leq t_1$, by the similar way to (i-a), we have that $P_{t_1,r}(1,y) < K_{t_1,\alpha}(1,y)$ for $0 < y < 1$, that is,

$$P_{t,r}(1,x) < K_{t,\alpha}(1,x) \text{ for } x > 1.$$

Hence, by (iii-c) and (iii-d), we get $P_{t,r}(1,x) < K_{t,\alpha}(1,x)$ for all $x > 0$ with $x \neq 1$ if $\alpha \geq \widehat{\beta}(t,r)$. □

3 Operator inequalities

In this section, we get operator inequalities by the results in the previous section.

Here, an operator means a bounded linear operator on a Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if

$$A \leq B \text{ implies } f(A) \leq f(B)$$

for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \leq B$ means $B - A \geq 0$.

The general theory on operator means are established by Kubo and Ando [10], and they obtained in [10] that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$ and an operator monotone function $f \geq 0$ on $[0,\infty)$ with $f(1) = 1$ as follows:

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{3.1}$$

if $A > 0$ and $B \geq 0$. We remark that f is called the representing function of $\mathfrak{M}$, and also it is permitted to consider binary operations given by (3.1) even if f is a general real-valued function.

By (3.1), we can introduce the following weighted operator means for two strictly positive operators A and B. For $t \in [0,1]$ and $q \in \mathbb{R}$,

$$\mathfrak{A}_t(A,B) = (1-t)A + tB \quad \text{(arithmetic mean)},$$
$$\mathfrak{G}_t(A,B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}} \quad \text{(geometric mean)},$$
$$\mathfrak{H}_t(A,B) = \{(1-t)A^{-1} + tB^{-1}\}^{-1} \quad \text{(harmonic mean)},$$
$$\mathfrak{P}_{t,q}(A,B) = \begin{cases} A^{\frac{1}{2}}\{(1-t)I + t(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^q\}^{\frac{1}{q}} A^{\frac{1}{2}} & \text{if } q \neq 0, \\ A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}} & \text{if } q = 0, \end{cases} \quad \text{(power mean)},$$
$$\mathfrak{K}_{t,q}(A,B) = (1-q)A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}} + q\{(1-t)A + tB\} \quad \text{(Heron mean)}.$$

It is known that $\mathfrak{P}_{t,q}(A,B)$ is an operator mean if $-1 \leq q \leq 1$, and also $\mathfrak{K}_{t,q}(A,B)$ is an operator mean if $0 \leq q \leq 1$. We remark that their representing functions are $A_t(1,x)$,

$G_t(1,x)$ and so on, and also notations $A \nabla_t B$, $A \sharp_t B$, $A \,!_t\, B$ and $A \sharp_{t,q} B$ are often used instead of $\mathfrak{A}_t(A,B)$, $\mathfrak{G}_t(A,B)$, $\mathfrak{H}_t(A,B)$ and $\mathfrak{P}_{t,q}(A,B)$, respectively. (See [11], for example.)

By Theorem 2.3, we have estimations of the weighted operator power mean by the Heron mean. Theorem 2.4 ensures the similar result, but we omit it.

Theorem 3.1. *Let $t, r \in (0,1)$. Let $\beta(t,r)$ and $\widehat{\beta}(t,r)$ as in (2.1). For all $A, B > 0$, we have the following.*

(i) *If $t \le r \le 1-t$, then $\mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B) \le \mathfrak{P}_{t,r}(A,B) \le \mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B)$.*

(ii) *If $1-t \le r \le t$, then $\mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B) \le \mathfrak{P}_{t,r}(A,B) \le \mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B)$.*

(iii) *If $r < t \le 1-t$, then $\mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B) \le \mathfrak{P}_{t,r}(A,B) \le \mathfrak{K}_{t,\widehat{\beta}(t,r)}(A,B)$.*

(iv) *If $r < 1-t \le t$, then $\mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B) \le \mathfrak{P}_{t,r}(A,B) \le \mathfrak{K}_{t,\widehat{\beta}(1-t,r)}(A,B)$.*

(v) *If $t \le 1-t < r$, then $\mathfrak{K}_{t,\beta(1-t,r)}(A,B) \le \mathfrak{P}_{t,r}(A,B) \le \mathfrak{K}_{t,(1-t)^{\frac{1}{r}-1}}(A,B)$.*

(vi) *If $1-t \le t < r$, then $\mathfrak{K}_{t,\beta(t,r)}(A,B) \le \mathfrak{P}_{t,r}(A,B) \le \mathfrak{K}_{t,t^{\frac{1}{r}-1}}(A,B)$.*

The given parameters of $\mathfrak{K}_{t,\alpha}(A,B)$ in each case are best possible on α except the parts $\alpha = \beta(\,\cdot\,,r)$ and $\alpha = \widehat{\beta}(\,\cdot\,,r)$.

Proof. Put $a = 1$ and replace b by $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$. Then we have Theorem 3.1 by applying the standard operational calculus in Theorem 2.3. □

4 Determinant and trace inequalities

In this section, we get some determinant and trace inequalities of matrices. Let $P_n(\mathbb{C})$ be the set of $n \times n$ positive definite matrices on $\mathbb{C}$.

Kittaneh and Manasrah researched improved and reversed Young's inequalities in [8, 9]. As a generalization of their results in [8, 9], for $a, b > 0$ with $a \ne b$, Alzer, da Fonseca and Kovačec [1] obtained the inequality

$$\left(\frac{\nu}{\mu}\right)^{\lambda} \le \frac{A_\nu(a,b)^\lambda - G_\nu(a,b)^\lambda}{A_\mu(a,b)^\lambda - G_\mu(a,b)^\lambda} \le \left(\frac{1-\nu}{1-\mu}\right)^{\lambda}, \tag{4.1}$$

where $\lambda \ge 1$ and $0 < \nu \le \mu < 1$. Moreover, Khosravi [7] obtained a generalization of (4.1) of the case $\lambda = 1$, that is,

$$\frac{\nu}{\mu} \le \frac{A_\nu(a,b) - P_{\nu,r}(a,b)}{A_\mu(a,b) - P_{\mu,r}(a,b)} \le \frac{1-\nu}{1-\mu}, \tag{4.2}$$

where $0 < \nu \leq \mu < 1$ and $r \in \mathbb{R}$ with $r \neq 1$.

By using (4.2), Khosravi [7] obtained a generalization of the determinant inequality in [1] as follows: Let $A, B \in P_n(\mathbb{C})$. Then

$$\left(\frac{\nu}{\mu}\right)^p \left[\det\{\mathfrak{A}_\mu(A,B) - \mathfrak{P}_{\mu,r}(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{A}_\nu(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{\nu,r}(A,B)\right]^{\frac{p}{n}} \quad (4.3)$$

holds for $0 < \nu \leq \mu < 1$, $-1 \leq r < 1$ and $p \geq 1$. We get determinant inequalities related to (4.3) by using Theorem 2.3.

Theorem 4.1. *Let $A, B \in P_n(\mathbb{C})$, $r \in (0,1)$ and $p \geq 1$. Let $\widehat{\beta}(t,r)$ as in* (2.1).

(i) *If $t \in (0, \frac{1}{2}]$ and $t \leq r$, then*

$$\left(1-(1-t)^{\frac{1}{r}-1}\right)^p \left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

(ii) *If $t \in (0, \frac{1}{2}]$ and $r < t$, then*

$$\left(1-\widehat{\beta}(t,r)\right)^p \left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

(iii) *If $t \in (\frac{1}{2}, 1)$ and $1-t \leq r$, then*

$$\left(1-t^{\frac{1}{r}-1}\right)^p \left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

(iv) *If $t \in (\frac{1}{2}, 1)$ and $r < 1-t$, then*

$$\left(1-\widehat{\beta}(1-t,r)\right)^p \left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{A}_t(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}}.$$

Theorem 4.2. *Let $A, B \in P_n(\mathbb{C})$, $r \in (0,1)$ and $p \geq 1$. Let $\beta(t,r)$ as in* (2.1).

(i) *If $t \in (0, \frac{1}{2}]$ and $1-t \leq r$, then*

$$\beta(1-t,r)^p \left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

(ii) *If $t \in (0, \frac{1}{2}]$ and $r < 1-t$, then*

$$t^{\left(\frac{1}{r}-1\right)p} \left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

(iii) *If $t \in (\frac{1}{2}, 1)$ and $t \leq r$, then*

$$\beta(t,r)^p \left[\det\{\mathfrak{A}_t(A,B)-\mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

(iv) *If* $t \in (\frac{1}{2}, 1)$ *and* $r < t$, *then*

$$(1-t)^{(\frac{1}{r}-1)p}\left[\det\{\mathfrak{A}_t(A,B) - \mathfrak{G}_t(A,B)\}\right]^{\frac{p}{n}} \leq \left[\det \mathfrak{P}_{t,r}(A,B)\right]^{\frac{p}{n}} - \left[\det \mathfrak{G}_t(A,B)\right]^{\frac{p}{n}}.$$

We omit these proofs. We remark that we use the following Lemma 4.A, a generalization of Minkowski's product inequality (see [2]) in order to prove Theorems 4.1 and 4.2.

Lemma 4.A ([7]). *Let* $a_i, b_i > 0$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{p}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{p}{n}} \leq \left(\prod_{i=1}^{n} (a_i + b_i)\right)^{\frac{p}{n}}$$

holds for $p \geq 1$.

On the other hand, by using (4.2) for $\lambda = 1$, Alzer, da Fonseca and Kovačec [1] obtained the trace inequality as follows: Let $A, B \in P_n(\mathbb{C})$. Then

$$\frac{\nu}{\mu}\left\{\operatorname{tr}\mathfrak{A}_\mu(A,B) - (\operatorname{tr}A)^{1-\mu}(\operatorname{tr}B)^{\mu}\right\} \leq \operatorname{tr}\mathfrak{A}_\nu(A,B) - \operatorname{tr}A^{1-\nu}B^{\nu}. \tag{4.4}$$

holds for $0 < \nu \leq \mu < 1$. We also get trace inequalities related to (4.4) by using Theorem 2.3.

Theorem 4.3. *Let* $A, B \in P_n(\mathbb{C})$, $r \in (0, 1)$ *and* $p \geq 1$. *Let* $\beta(t, r)$ *as in* (2.1).

(i) *If* $t \in (0, \frac{1}{2}]$ *and* $1 - t \leq r$, *then*

$$\beta(1-t, r)\left\{\operatorname{tr}\mathfrak{A}_t(A,B) - (\operatorname{tr}A)^{1-t}(\operatorname{tr}B)^{t}\right\} \leq \{\operatorname{tr}\mathfrak{A}_t(A^r, B^r)\}^{\frac{1}{r}} - \operatorname{tr}A^{1-t}B^{t}.$$

(ii) *If* $t \in (0, \frac{1}{2}]$ *and* $r < 1 - t$, *then*

$$t^{\frac{1}{r}-1}\left\{\operatorname{tr}\mathfrak{A}_t(A,B) - (\operatorname{tr}A)^{1-t}(\operatorname{tr}B)^{t}\right\} \leq \{\operatorname{tr}\mathfrak{A}_t(A^r, B^r)\}^{\frac{1}{r}} - \operatorname{tr}A^{1-t}B^{t}.$$

(iii) *If* $t \in (\frac{1}{2}, 1)$ *and* $t \leq r$, *then*

$$\beta(t, r)\left\{\operatorname{tr}\mathfrak{A}_t(A,B) - (\operatorname{tr}A)^{1-t}(\operatorname{tr}B)^{t}\right\} \leq \{\operatorname{tr}\mathfrak{A}_t(A^r, B^r)\}^{\frac{1}{r}} - \operatorname{tr}A^{1-t}B^{t}.$$

(iv) *If* $t \in (\frac{1}{2}, 1)$ *and* $r < t$, *then*

$$(1-t)^{\frac{1}{r}-1}\left\{\operatorname{tr}\mathfrak{A}_t(A,B) - (\operatorname{tr}A)^{1-t}(\operatorname{tr}B)^{t}\right\} \leq \{\operatorname{tr}\mathfrak{A}_t(A^r, B^r)\}^{\frac{1}{r}} - \operatorname{tr}A^{1-t}B^{t}.$$

We omit this proof. We remark that we use fundamental properties of the singular values, Hölder's inequality and the inequality $\sum_{i=1}^{n} a_i^p \leq (\sum_{i=1}^{n} a_i)^p$ for $a_i > 0$ ($i = 1, \ldots, n$) and $p \geq 1$ in order to prove Theorem 4.3.

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(Masatoshi Ito) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN
E-mail address: m-ito@maebashi-it.ac.jp