作用素値内積を用いた一般化シュワルツの不等式について
The generalized Schwarz inequality via operator-valued inner product

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1. Introduction

This report is based on [8].

Let $B(\mathcal{H})$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(\mathcal{H})$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For selfadjoint operators A and B, the order relation $A \geq B$ means that $A - B$ is positive.

The Cauchy-Schwarz inequality is one of the most useful and fundamental inequalities in functional analysis: For all vectors x and y in a Hilbert space H

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle. \tag{1.1}$$

We want to study a non-commutative version of the Cauchy-Schwarz inequality (1.1). Since the product AB of positive operators A and B is not always positive, we need to deform the Cauchy-Schwarz inequality (1.1) to be convenient. For example, the Cauchy-Schwarz inequality is transformed as follows: Dividing both sides in (1.1) by $\langle y, y\rangle (\neq 0)$

$$\overline{\langle x, y\rangle}\langle y, y\rangle^{-1}\langle x, y\rangle \leq \langle x, x\rangle \tag{1.2}$$

and taking square root of both sides in (1.1)

$$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle\langle y, y\rangle}. \tag{1.3}$$

Firstly, we consider the case of (1.2): Regarding a sesquilinear map $B(X, Y) = Y^*X$ for $X, Y \in B(\mathcal{H})$ as an operator-valued inner product, several operator versions for the Schwarz inequality are discussed by many researchers. For example, if $X, Y \in B(\mathcal{H})$, then the Schwarz inequality for operators

$$X^*Y(Y^*Y)^{-1}Y^*X \leq X^*X \tag{1.4}$$

holds. Indeed, since $Y(Y^*Y+\varepsilon I)^{-1}Y^* \leq I$ for all $\varepsilon > 0$ and $Y(Y^*Y+\varepsilon I)^{-1}Y^*$ is increasing for $\varepsilon \downarrow 0$, there exists the strong operator limit of $Y(Y^*Y + \varepsilon I)^{-1}Y^*$ as $\varepsilon \to 0$ and we define

$$X^*Y(Y^*Y)^{-1}Y^*X = \operatorname*{s-lim}_{\varepsilon\to 0} X^*Y(Y^*Y + \varepsilon I)^{-1}Y^*X$$

and write $X^*Y(Y^*Y)^{-1}Y^*X \in B(\mathcal{H})$. This formulation for matrices is firstly given by Marshall and Olkin in [10]. Let T be a positive operator and X, Y any two operators in $B(\mathcal{H})$. Replacing X and Y in (1.4) by $T^{1/2}X$ and $T^{1/2}Y$, respectively, we obtain $X^*TY(Y^*TY)^{-1}Y^*TX \in B(\mathcal{H})$ and

$$X^*TY(Y^*TY)^{-1}Y^*TX \leq X^*TX. \tag{1.5}$$

In [3], Bhatia and Davis showed some new operator versions of the Schwarz inequality for a positive linear map, which is a generalization of (1.5): A map Φ on $B(\mathcal{H})$ is called 2-positive if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0 \qquad \text{implies} \qquad \begin{pmatrix} \Phi(A) & \Phi(B) \\ \Phi(C) & \Phi(D) \end{pmatrix} \geq 0.$$

It is known that if A is positive, X is positive invertible and B is any operators in $B(\mathcal{H})$, then

$$\begin{pmatrix} A & B \\ B^* & X \end{pmatrix} \geq 0 \quad \Longleftrightarrow \quad A \geq BX^{-1}B^*.$$

From this it follows that if T is positive and Φ is a 2-positive linear map on $B(\mathcal{H})$, then

$$\Phi(X^*TY)\Phi(Y^*TY)^{-1}\Phi(Y^*TX) \leq \Phi(X^*TX) \tag{1.6}$$

for every $X, Y \in B(\mathcal{H})$.

In the framework of an operator-valued inner product, the formulation of the Schwarz operator inequality is very important, but the left-hand sides of the Schwarz inequalities (1.4) and (1.6) for operators are expressed as the strong-operator limits unless Y^*Y and $\Phi(Y^*TY)$ are invertible. This fact is a cause of difficulty in application. Thus, we consider the case of (1.3). For this, we recall the geometric operator mean, also see [2, 11]. Let A and B be two positive operators in $B(\mathcal{H})$. The geometric operator mean $A \sharp B$ of A and B is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}}$$

if A is invertible. The geometric operator mean has the monotonicity:

$$0 \leq A \leq C \quad \text{and} \quad 0 \leq B \leq D \quad \text{implies} \quad A \sharp B \leq C \sharp D$$

and the subadditivity:

$$A \sharp B + C \sharp D \leq (A + C) \sharp (B + D).$$

By monotonicity, we can uniquely extend the definition of $A \sharp B$ for all positive operators A and B by setting

$$A \sharp B = \operatorname*{s\text{-}lim}_{\varepsilon \to 0} (A + \varepsilon I) \sharp (B + \varepsilon I).$$

In this case, the geometric operator mean $A \sharp B$ for positive operators A and B always exists in $B(\mathcal{H})$ and it has all the desirable properties as geometric mean such as monotonicity, continuity from above, transformer inequality, subadditivity and self-duality so on.

J.I. Fujii in [5] studied another version of the Schwarz operator inequality in terms of the geometric operator mean, which is a main tool of our research:

Theorem A. *Let Φ be a 2-positive map on $B(\mathcal{H})$. Then*

$$|\Phi(Y^*X)| \leq \Phi(X^*X) \sharp U^*\Phi(Y^*Y)U \tag{1.7}$$

for every $X, Y \in B(\mathcal{H})$, where U is a partial isometry in the polar decomposition of $\Phi(Y^*X) = U|\Phi(Y^*X)|$.

In this paper, by virtue of the Cauchy-Schwarz operator inequality due to J.I. Fujii, we show weighted mixed Schwarz operator inequalities in terms of the geometric operator mean. As applications, we show the covariance-variance operator inequality via the geometric operator mean which differs from Bhatia-Davis's one. By our formulation, we show a Robertson type inequality associated to a unital completely positive linear maps on $B(\mathcal{H})$.

2. Weighted mixed Schwarz operator inequalities

First of all, we discuss weighted mixed Schwarz operator inequalities in terms of the geometric operator mean.

For $T \in B(\mathcal{H})$, $T = W|T|$ is the polar decomposition of T where W is a partial isometry and $|T| = (T^*T)^{1/2}$ with the kernel condition $\ker(W) = \ker(|T|)$. Note that WW^* is the projection onto the range of $|T^*|$, and W^*W is the projection onto the range of $|T|$. Then it follows from [9] that

$$W|T|^q W^* = |T^*|^q \qquad \text{for any } q > 0. \tag{2.1}$$

Furuta in [9] showed the following weighted mixed Schwarz inequalities:

Theorem B (Weighted mixed Schwarz inequalities). *For any operator T in $B(\mathcal{H})$,*

$$|\langle Tx, y\rangle|^2 \le \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2\beta}y, y\rangle$$

holds for any $x, y \in H$ and for any real number α, β with $\alpha + \beta = 1$. Moreover, for $1 < \alpha < 1$, the equality holds if and only if $|T|^{2\alpha}x$ and T^*y are linearly dependent if and only if Tx and $|T^*|^{2\beta}y$ are linearly dependent. For $\alpha = 1$, the equality holds if and only if Tx and y are linearly dependent. For $\alpha = 0$, the equality holds if and only if x and T^*y are linearly dependent.

By Theorem A, we have the following weighted mixed Schwarz operator inequality, which is an operator version of Theorem B:

Theorem 2.1 (Weighted mixed Schwarz operator inequality). *Let Φ be a 2-positive map on $B(\mathcal{H})$ and T an operator in $B(\mathcal{H})$. If $X, Y \in B(\mathcal{H})$, then*

$$|\Phi(Y^*TX)| \le \Phi(X^*|T|^{2\alpha}X) \sharp U^*\Phi(Y^*|T^*|^{2\beta}Y)U \tag{2.2}$$

for any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, where $\Phi(Y^*TX) = U|\Phi(Y^*TX)|$ is the polar decomposition of $\Phi(Y^*TX)$. In particular, in the case of $\alpha = 0, 1$

$$|\Phi(Y^*TX)| \le \Phi(X^*W^*WX) \sharp U^*\Phi(Y^*|T^*|^2Y)U$$

and

$$|\Phi(Y^*TX)| \le \Phi(X^*|T|^2X) \sharp U^*\Phi(Y^*WW^*Y)U,$$

where $T = W|T|$ is the polar decomposition of T.

Proof. We only prove the case of $0 < \alpha < 1$. It follows that

$$\begin{aligned} |\Phi(Y^*TX)| &= |\Phi(Y^*W|T|X)| = |\Phi(Y^*W|T|^\beta|T|^\alpha X)| && \text{by } \alpha+\beta=1 \\ &\le \Phi(X^*|T|^{2\alpha}X) \sharp U^*\Phi(Y^*W|T|^{2\beta}W^*Y)U && \text{by Theorem A} \\ &= \Phi(X^*|T|^{2\alpha}X) \sharp U^*\Phi(Y^*|T^*|^{2\beta}Y)U && \text{by (2.1)} \end{aligned}$$

and so we have the desired inequality (2.2). □

Next, we consider the equality condition in (1.7) of Theorem A. To show it, we need some preliminaries. First of all, we recall the Moore-Penrose inverse: For a given operator $A \in B(\mathcal{H})$ having a closed range, it is well known that the equations $AGA = A$, $GAG = G$, $(AG)^* = AG$ and $(GA)^* = GA$ have a unique common solution for $G \in B(\mathcal{H})$, denoted by $G = A^\dagger$ and called the Moore-Penrose inverse of A. In [6], J.I.Fujii showed a relation between the geometric operator mean and the Moore-Penrose inverse:

$$A \sharp B \leq A^{1/2} \left((A^{1/2})^\dagger B (A^{1/2})^\dagger\right)^{1/2} A^{1/2}. \tag{2.3}$$

We show that the equality in (2.3) holds under a kernel condition:

Lemma 2.2. *Let A and B be positive operators in $B(\mathcal{H})$. If A has a closed range and* $\ker A \subset \ker B$, *then*

$$A \sharp B = A^{1/2} \left((A^{1/2})^\dagger B (A^{1/2})^\dagger\right)^{1/2} A^{1/2}.$$

Lemma 2.3. *Let A and B be positive operators in $B(\mathcal{H})$. If A has a closed range and* $\ker A \subset \ker(BA^\dagger B)$, *then $A \sharp BA^\dagger B = R(A)BR(A)$, where $R(A)$ is the range projection of A. In addition, if* $\ker A \subset \ker B$, *then $A \sharp BA^\dagger B = B$.*

Lemma 2.4. *Let A, B and C be positive operators in $B(\mathcal{H})$. If A has a closed range and* $\ker A \subset \ker B \cap \ker C$, *then $A \sharp B = A \sharp C$ implies $B = C$.*

We show the following equality condition in (1.7) of Theorem A:

Theorem 2.5. *Let Φ be a 2-positive map on $B(\mathcal{H})$. For every $X, Y \in B(\mathcal{H})$, let U be a partial isometry in the polar decomposition of $\Phi(Y^*X) = U|\Phi(Y^*X)|$. If $\Phi(X^*X)$ has a closed range, then the equality in* (1.7) *of Theorem A holds if and only if $U^*\Phi(Y^*Y)U = |\Phi(Y^*X)|\Phi(X^*X)^\dagger|\Phi(Y^*X)|$.*

We note that in the case that Φ is the identity map in Theorem 2.5, we see that if $\Phi(X^*X)$ has a closed range, then the equality condition $U^*\Phi(Y^*Y)U = |\Phi(Y^*X)|\Phi(X^*X)^\dagger|\Phi(Y^*X)|$ holds if and only if there exists $W \in B(\mathcal{H})$ such that $YU = XW$, that is, $\{YU, X\}$ is linearly dependent.

As an application, we have the following equality condition of Theorem 2.1:

Theorem 2.6. *Let Φ be a 2-positive map on $B(\mathcal{H})$ and T an operator in $B(\mathcal{H})$. For every $X, Y \in B(\mathcal{H})$, let U be a partial isometry in the polar decomposition of $\Phi(Y^*TX) = U|\Phi(Y^*TX)|$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. If $\Phi(X^*|T|^{2\alpha}X)$ has a closed range, then the equality in* (2.2) *of Theorem 2.1 holds if and only if $U^*\Phi(Y^*|T^*|^{2\beta}Y)U = |\Phi(Y^*TX)|\Phi(X^*|T|^{2\alpha}X)^\dagger|\Phi(Y^*TX)|$.*

3. Variance-Covariance inequality

We recall the notion of the covariance and the variance of operators defined by Fujii, Furuta, Nakamoto and Takahasi [7]. In 1954, the noncommutative probability theory is founded by H. Umegaki as an application of the theory of von Neumann algebra in [12]. An operator $A \in B(\mathcal{H})$ plays the role of a random variable, that is, for every unit vector $x \in \mathcal{H}$, the functional $\langle Ax, x\rangle$ on the operatyor algebra may be thought as an expectation at a state x (with $\|x\| = 1$). The covariance of operators A and B at a state x is introduced by

$$\mathrm{cov}_x(A, B) = \langle A^*Bx, x\rangle - \langle A^*x, x\rangle\langle Bx, x\rangle, \tag{3.1}$$

and the variance of A at a state x by

$$\mathrm{var}_x(A) = \langle A^*Ax, x\rangle - |\langle Ax, x\rangle|^2.$$

The following variance-covariance inequality is an application of the Cauchy-Schwarz inequality:

$$|\mathrm{cov}_x(A,B)| \leq \sqrt{\mathrm{var}_x(A)\mathrm{var}_x(B)}. \tag{3.2}$$

In [3], Bhatia and Davis studied a noncommutative analogue of variance and covariance in statistics, which is a generalization of the covariance (3.1) at a state: Let Φ be a unital completely positive linear map on $B(\mathcal{H})$. The convariance $\mathrm{cov}(A,B)$ between two operators A and B is defined by

$$\mathrm{cov}(A,B) = \Phi(A^*B) - \Phi(A)^*\Phi(B).$$

The variance of A is defined by

$$\mathrm{var}(A) = \mathrm{cov}(A,A) = \Phi(A^*A) - \Phi(A)^*\Phi(A).$$

Since Φ is completely positive, then the variance of A is positive, i.e., $\mathrm{var}(A) \geq 0$. Bhatia and Davis showed the following counterpart of the variance-covariance inequality in the context of noncommutative probability, which is a generalization of the variance-covariance inequality (3.2): For all $A, B \in B(\mathcal{H})$,

$$\mathrm{cov}(A,B)\mathrm{var}(B)^{-1}\mathrm{cov}(A,B)^* \in B(\mathcal{H})$$

and

$$\mathrm{cov}(A,B)\mathrm{var}(B)^{-1}\mathrm{cov}(A,B)^* \leq \mathrm{var}(A).$$

By virtue of the geometric operator mean, we show the following variance-covariance inequality:

Theorem 3.1. *Let Φ be a unital completely positive linear map on $B(\mathcal{H})$ and A, B two operators in $B(\mathcal{H})$. Then*

$$|\mathrm{cov}(A,B)| \leq U^*\mathrm{var}(A)U \,\sharp\, \mathrm{var}(B), \tag{3.3}$$

where $\mathrm{cov}(A,B) = U|\mathrm{cov}(A,B)|$ *is the polar decomposition of* $\mathrm{cov}(A,B)$.

Proof. It follows from [3, Theorem 1] that the 2×2 operator matrix

$$\begin{pmatrix} \mathrm{var}(A) & \mathrm{cov}(A,B) \\ \mathrm{cov}(A,B)^* & \mathrm{var}(B) \end{pmatrix}$$

is positive. Then we have

$$\begin{aligned} 0 &\leq \begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{var}(A) & \mathrm{cov}(A,B) \\ \mathrm{cov}(A,B)^* & \mathrm{var}(B) \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} U^*\mathrm{var}(A)U & U^*U|\mathrm{cov}(A,B)| \\ |\mathrm{cov}(A,B)|U^*U & \mathrm{var}(B) \end{pmatrix} = \begin{pmatrix} U^*\mathrm{var}(A)U & |\mathrm{cov}(A,B)| \\ |\mathrm{cov}(A,B)| & \mathrm{var}(B) \end{pmatrix}. \end{aligned}$$

Since $A \,\sharp\, B = \max\{X \geq 0 : \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\}$, we have the desired inequality (3.3). □

4. Comutation relation and covariance

In this section, we discuss the near relation of the variance-covariance inequality with the Heisenberg uncertainty principle in quantum physics. In [4], Enomoto pointed out that the variance-covariance inequality (3.2) is exactly the generalized Schrödinger inequality: Let A and B be (not necessarily bounded) selfadjoint operators on a Hilbert space $\mathcal{H}$. Let $D(AB)$ and $D(BA)$ be the domain of AB and BA, respectively. Let $\{A,B\}$ and $[A,B]$ be the Jordan product $AB+BA$ and the commutator $AB-BA$, respectively. Then

$$|\mathrm{cov}_x(A,B)|^2 = \left(\frac{1}{2}\langle\{A,B\}x,x\rangle - \langle Ax,x\rangle\langle Bx,x\rangle\right)^2 + \left(\frac{1}{2i}\langle[A,B]x,x\rangle\right)^2$$

for every unit vector $x \in D(AB)\cap D(BA)$. In particular, the following Robertson type inequality holds:

$$\sqrt{\mathrm{var}_x(A)\mathrm{var}_x(B)} \geq \frac{1}{2}\left|\langle[A,B]x,x\rangle\right|$$

and the following Schrödinger type inequality holds:

$$\mathrm{var}_x(A)\mathrm{var}_x(B) \geq \left|\frac{1}{2}\langle\{A,B\}x,x\rangle - \langle Ax,x\rangle\langle Bx,x\rangle\right|^2 + \frac{1}{4}\left|\langle[A,B]x,x\rangle\right|^2.$$

We show a Robertson type inequality associated to a unital completely positive linear map on $B(\mathcal{H})$:

Theorem 4.1 (Robertson type inequality)**.** *Let Φ be a unital completely positive linear map on $B(\mathcal{H})$. Then for every selfadjoint operators $A,B \in B(\mathcal{H})$, there exists an isometry $V \in B(\mathcal{H})$ such that*

$$U^*\mathrm{var}(A)U \ \sharp\ \mathrm{var}(B) \geq V^*\left(\frac{\Phi([A,B]) - [\Phi(A),\Phi(B)]}{2i}\right)_+ V,$$

where $\mathrm{cov}(A,B) = U|\mathrm{cov}(A,B)|$ is the polar decomposition of $\mathrm{cov}(A,B)$ and X_+ is the positive part of a selfadjoint operator $X \in B(\mathcal{H})$.

Proof. It follows from [1, Proposition 2.1] that there exists an isometry $V \in B(\mathcal{H})$ such that $\mathrm{Re}(-i\ \mathrm{cov}(A,B))_+ \leq V|-i\ \mathrm{cov}(A,B)|V^*$ and so

$$V^*\mathrm{Re}(\mathrm{cov}(A,B))_+V \leq |\mathrm{cov}(A,B)|.$$

Since $\mathrm{Im}(\mathrm{cov}(A,B)) = \frac{1}{2i}\left(\Phi(AB-BA) - (\Phi(A)\Phi(B) - \Phi(B)\Phi(A))\right)$, we have

$$\begin{aligned} |\mathrm{cov}(A,B)| &= |-i\ \mathrm{cov}(A,B)| \\ &\geq V^*\mathrm{Re}(-i\ \mathrm{cov}(A,B))_+V \\ &= V^*\mathrm{Im}(\mathrm{cov}(A,B))_+V \\ &= V^*\left(\frac{\Phi([A,B]) - [\Phi(A),\Phi(B)]}{2i}\right)_+ V. \end{aligned}$$

Hence we have the desired inequality by Theorem 3.1. □

Under the restricted condition, we have a Schrödinger type inequality associated to a unital completely positive linear map on $B(\mathcal{H})$:

Corollary 4.2 (Schrödinger type inequality). *Let Φ be a unital completely positive linear map on $B(\mathcal{H})$ and $A, B \in B(\mathcal{H})$ two selfadjoint operators. If $\Phi(AB) - \Phi(A)\Phi(B)$ is normal, then*

$$\begin{aligned} &U^*\mathrm{var}(A)U \sharp \mathrm{var}(B) \\ &\geq \left(\frac{1}{4} \left(\Phi(\{A,B\}) - \{\Phi(A), \Phi(B)\} \right)^2 + \left(\frac{\Phi([A,B]) - [\Phi(A), \Phi(B)]}{2i} \right)^2 \right)^{\frac{1}{2}} \\ &\geq \frac{1}{2} \left| \Phi([A,B]) - [\Phi(A), \Phi(B)] \right|, \end{aligned}$$

where $\mathrm{cov}(A,B) = U|\mathrm{cov}(A,B)|$ *is the polar decomposition of* $\mathrm{cov}(A,B)$.

Proof. For every selfadjoint $A, B \in B(\mathcal{H})$, we have

$$\begin{aligned} &|\mathrm{cov}(A,B)| = |\mathrm{Re}(\mathrm{cov}(A,B)) + i\ \mathrm{Im}(\mathrm{cov}(A,B))| \\ &= \left(\frac{1}{4} \left(\Phi(\{A,B\}) - \{\Phi(A), \Phi(B)\} \right)^2 + \left(\frac{\Phi([A,B]) - [\Phi(A), \Phi(B)]}{2i} \right)^2 + \frac{1}{2}(X^*X - XX^*) \right)^{\frac{1}{2}}, \end{aligned}$$

where $X = \Phi(AB) - \Phi(A)\Phi(B)$. Since X is normal, it follows that $X^*X = XX^*$ and so we have the desired inequality by Theorem 3.1. □

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