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Fixed Point and Convergence Theorems for Two Nonlinear Mappings in Hilbert Spaces

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Abstract

In this paper, first, we introduce attractive point and fixed point theorems and mean convergence theorem for two commutative 2-generalized mappings in Hilbert spaces. Next we obtain a weak convergence theorem of Mann's type iteration, a strong convergence theorem of Halpern's type iteration, and a strong convergence theorem by the hybrid method for two commutative 2-generalized hybrid mappings in a Hilbert space.

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1 Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [23] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C\}$.

We know that $A(T)$ is closed and convex. This property is important for proving our main theorems. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon we know the first nonlinear ergodic theorem in a Hilbert space: Let C be a bounded, closed and convex subset of H and let $T : C \to C$ be nonexpansive. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$. In 2010, Kocourek, Takahashi and Yao [10] defined a broad class of nonlinear mappings in a Hilbert space: Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called $(\alpha,\ \beta)$-*generalized hybrid.* We also know the following mapping: For $\lambda \in \mathbb{R}$, a mapping $U : C \to H$ is called λ-*hybrid* if

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + 2(1-\lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

It is *nonspreading* for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous. We also know that λ-hybrid mappings are in the class of generalized hybrid mappings. The nonlinear ergodic theorem by Baillon for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao. Recently, Kohsaka [11] also proved the following theorem:

Theorem 1.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative λ and μ-hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

On the other hand, Takahashi and Takeuchi [23] proved the following attractive point and mean convergence theorem without convexity in a Hilbert space.

Theorem 1.2. *Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Assume that $\{T^n z\}$ for some $z \in C$ is bounded and define*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n} T^k x$$

for all $x \in C$ and $n \in \mathbb{N}$. Then $\{S_n x\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)} T^n x$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.

Maruyama, Takahashi and Yao also defined a more broad class of nonlinear mappings called 2-generalized hybrid which contains generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H and let T be a mapping of C into C. A mapping $T : C \to H$ is 2-*generalized hybrid* [15] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1\|T^2x - Ty\|^2 &+ \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ &\le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned} \tag{1.3}$$

for all $x, y \in C$.

In this paper, using means, we first obtain attractive point and fixed point theorems for commutative 2-generalized hybrid mappings in Hilbert spaces. Using these results, we prove weak and strong convergence theorems for commutative 2-generalized hybrid mappings in Hilbert spaces.

2 Preliminaries

In a Hilbert space, it is known that

$$\|x + y - z\|^2 - \|x - z\|^2 \geq 2\langle y, x - z\rangle \tag{2.1}$$

for all $x, y, z \in H$ and

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [21]. Furthermore, in a Hilbert space, we have that

$$2 \langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$. Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - u\| \leq \|x - u\|, \quad \forall x \in C, \quad u \in F(T).$$

If C is closed and convex and $T : C \to H$ with $F(T) \neq \emptyset$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [9]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive; $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x\rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [20, 21]. Using this inequality and (2.3), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \ y \in D. \tag{2.4}$$

The following result was proved by Takahashi and Toyoda [24].

Lemma 2.1 ([24]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $n \in \mathbb{N}$ and $u \in C$, then $\{P_C x_n\}$ converges strongly to some $z \in C$, where P_C is the metric projection of H onto C.*

Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to C$ was called 2-*generalized hybrid* [15] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ satisfying (1.3). We also call such a mapping $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-*generalized hybrid*. We know that the class of the mappings above covers well-known mappings. For example, the class of $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mappings is the class of (α_2, β_2)-generalized hybrid mappings in the sense of Kocourek, Takahashi and Yao [10]. If $x = Tx$ in (1.3), then for any $y \in C$,

$$\begin{aligned} \alpha_1 \|x - Ty\|^2 &+ \alpha_2 \|x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ &\leq \beta_1 \|x - y\|^2 + \beta_2 \|x - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2. \end{aligned}$$

Hence we have that

$$\|x - Ty\| \leq \|x - y\|, \quad \forall x \in F(T),\ y \in C. \tag{2.5}$$

Thus, a 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Hojo, Takahashi and Takahashi [6] obtained the following attractive point and fixed point theorems for two commutative 2-generalized hybrid mappings in a Hilbert space.

Theorem 2.2 ([6]). *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be commutative 2-generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

Also, we prove a mean convergence theorem for commutative 2-generalized hybrid mappings without convexity in a Hilbert space.

Let $D = \{(k, l) : k, l \in \mathbb{N} \cup \{0\}\}$. Then D is a directed set by the binary relation:

$$(k, l) \leq (i, j) \quad \text{if} \quad k \leq i \ \text{ and } \ l \leq j.$$

Theorem 2.3 ([6]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let P be the metric projection of H onto $A(S) \cap A(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l) \in D} P S^k T^l x$. In particular, if C is closed and convex, $\{S_n x\}$ converges weakly to an element q of $F(S) \cap F(T)$.

3 Weak convergence theorem of Mann's type iteration

In this section, we obtain a weak convergence theorem of Mann's type iteration for two commutative 2-generalized hybrid mappings in a Hilbert space. Before proving the theorem, we need the following lemma.

Lemma 3.1 ([5]). *Let C be a nonempty, closed and convex subset of a Hilbert space H and let S and T be commutative 2-generalized hybrid mappings of C into itself. Let $\{x_n\}$ be a bounded sequence of C. Define*

$$S_n x_n = \frac{1}{(1+n)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n$$

for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\|S_n x_n - x_n\| \to 0$. Then every weak cluster point of $\{x_n\}$ is a point of $F(S) \cap F(T)$.

Theorem 3.2 ([5]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let P be the metric projection of H onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S) \cap F(T)$, *where* $z = \lim_{n\to\infty} Px_n$.

4 Strong convergence theorem of Halpern's type iteration

Using the idea of mean convergence by Shimizu and Takahashi [18, 19], and Kurokawa and Takahashi [14], we prove the following strong convergence theorem of Halpern's type iteration for two commutative 2-generalized hybrid mappings in a Hilbert space.

Theorem 4.1 ([5]). *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* S *and* T *be commutative 2-generalized hybrid mappings of* C *into itself such that* $F(S) \cap F(T) \neq \emptyset$. *Let* $u \in C$ *and define a sequence* $\{x_n\}$ *in* C *as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to* Pu, *where* P *is the metric projection of* H *onto* $F(S) \cap F(T)$.

5 Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [17], we first prove a strong convergence theorem for two commutative 2-generalized hybrid mappings in a Hilbert space.

Theorem 5.1 ([5]). *Let* H *be a real Hilbert space, let* C *be a nonempty, convex and closed subset of* H. *Let* $S, T : C \to C$ *be commutative 2-generalized hybrid mappings such that* $F(S) \cap F(T) \neq \emptyset$. *Let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ *is the metric projection of* H *onto* $C_n \cap Q_n$ *and* $\{\alpha_n\} \subset [0,1]$ *satisfies* $0 \leq \alpha_n \leq a < 1$ *for some* $a \in \mathbb{R}$. *Then,* $\{x_n\}$ *converges strongly to* $z_0 = P_{F(S)\cap F(T)} x$, *where* $P_{F(S)\cap F(T)}$ *is the metric projection of* H *onto* $F(S) \cap F(T)$.

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