REVIEW ON EXAMPLES OF NONLINEAR MAPPINGS

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ABSTRACT. In this paper, we systemically survey several examples relative to nonlinear mappings/operators and investigate strict inclusions of two different classes of nonlinear mappings/operators. Some open questions are also added.

1. INTRODUCTION

Let X be a real Banach space with its dual space X^*. Denote by $\langle \cdot, \cdot \rangle$ the duality product and it will be convenient to write $\langle x, x^* \rangle$ for $x^*(x)$ for every $x^* \in X^*$ throughout this paper. For brevity, we denote by (S) the class of *smooth* spaces, by (G) the class of spaces with *Gâteaux differentiable* norms, by (F) the class of spaces with *Fréchet differentiable* norms, by (UG) the class of spaces with *uniformly Gâteaux differentiable* norms, by (US) the class of *uniformly smooth* spaces. Also, we denote the classes of, by turns, *strictly convex*, *uniformly convex*, *reflexive* Banach spaces, and Hilbert spaces by (SC), (UC), (R) and (H). Then, we often write $X \in (R) \cap (S) \cap (SC)$ if X is a reflexive, smooth and strictly convex Banach space.

Let $\emptyset \neq C \subset X$ and let $T : C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T; that is,

$$F(T) = \{x \in C : Tx = x\}.$$

Recall that the operator $J : X \to 2^{X^*}$ defined by

$$J(x) = \{j(x) \in X^* : \langle x, j(x) \rangle = \|x\|^2 = \|j(x)\|^2\}$$

is called the (normalized) *duality mapping*. Sometimes, it will be simply denoted by Jx unless distinction is needed.

In what follows we denote *strong* and *weak* convergence in X by "$\to$" and "$\rightharpoonup$", respectively. We denote by 1_C its characteristic function, which takes value 1 on C and 0 on C^c, where A^c denotes the complement of A and I denotes the identity operator on X Also, ∂f denotes the subdifferential of a function $f : X \to \mathbb{R}$. We sometimes use the notations $D(T)$ and $R(T)$ to denote the domain and the range of an operator T, respectively. We often write $\mathbb{R}$ and $\mathbb{R}_+$ in place of $(-\infty, \infty)$ and $[0, \infty)$, respectively.

Throughout this paper, we assume, unless other specified, that C is a nonempty subset of a real Banach space X (for more useful applications to the fixed-point theory and so on, it may be suitable to take C to be *closed convex* in all examples)

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and let $T : C \to C$ be a mapping. Especially, if $X \in (H)$ (the duality product $\langle \cdot, \cdot \rangle$ is just the inner product in H), we give an equivalent formulation which is represented by the inner product.

It seems interesting for beginners relative to nonlinear mappings/operators to systematically survey classes of several nonlinear mappings/operators and to classify/unify their implications concerning to their inclusion relations each other. There are lots of slight transformations about several nonlinear mappings. However, it is not easy to distinguish their differences, much hard in classes of nonLipschitzian mappings. Even if the author has already tried to investigate and unify their relations in [30], lots of mistakes there stimulated him to review on several examples, and the scopes of the previous works were also expanded to pseudocontractive operators of several types. In this paper, some equivalent transformations for proving some examples will be suggested. Also, we investigate strictness for inclusion of two classes of nonlinear mappings. Finally, some open questions are added. This work is rewritten as beginners's guide in this fields in place of my earlier work [30].

2. Classes of Lipschitzian mappings of nonexpansive type

Definition 2.1. A mapping $T : C \to C$ is said to be *Lipschitzian* if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall\, x, y \in C,$$

where $L := L_T$ denotes the *Lipschitz constant* of T. Obviously, it is equivalent to the following property: for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall\, x, y \in C. \tag{2.1}$$

For a Lipschitzian mapping T, we say:

(a) T is *uniformly k-Lipschiztain* if $k_n = k$ for all $n \in \mathbb{N}$;

(b) T is *nonexpansive* if $k_n = 1$ for all $n \geq 1$; that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C.$$

(c) T is *asymptotically nonexpansive* [13] if $\lim_{n\to\infty} k_n = 1$.

Before introducing firm nonexpansivity, we review the following Lemma 3.3 in [15], which is a simple fact concerning the (proper) convex function $f(s) = \|u + sv\|$ on $\mathbb{R}$, where $u, v \in X$ are arbitrarily given:

Lemma 2.2. ([15]) *For two points $u, v \in X$, the following are equivalent:*

(a) $\|u\| \leq \|u + sv\|$ *for all* $s \in [0, 1]$*;*

(b) *the convex function* $f(s) = \|u + sv\|$ *is increasing on* $[0, 1]$*;*

(c) *there is* $j(u) \in J(u)$ *such that* $\langle v, j(u)\rangle \geq 0$.

Remark 2.3. Note that $s \in [0, 1]$ in (a) and (b) could be replaced with all $s > 0$ by convexity of f.

Given $x, y \in C$ and a mapping $T : C \to C$, consider the following *convex* functions $\varphi : [0, 1] \to [0, \infty)$ defined by

$$\varphi(s) = \|(x - y) - s[(I - T)x - (I - T)y]\|.$$

Definition 2.4. A mapping $T : C \to X$ is *firmly nonexpansive* [15] if φ is non-increasing on $[0, 1]$; equivalently,

$$\begin{aligned} \|Tx - Ty\| &\leq \|(Tx - Ty) + s[(I - T)x - (I - T)y]\|, \\ &= \|s(x - y) + (1 - s)(Tx - Ty)\|, \quad \forall\, x, y \in C,\ s \in [0, 1], \end{aligned} \tag{2.2}$$

which is equivalent to the following: for each x, $y \in C$, there is $j(Tx - Ty) \in J(Tx - Ty)$ such that

$$\|Tx - Ty\|^2 \leq \langle x - y, j(Tx - Ty)\rangle. \tag{2.3}$$

In case that $X \in (H)$, since $J = I$, the inequality (2.3) becomes

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall\, x, y \in C,$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall\, x,\, y \in C.$$

Remark 2.5. Note that $\varphi(s)$ is non-increasing if and only if

$$f(s) = \varphi(1 - s) = \|(Tx - Ty) + s[(I - T)x - (I - T)y]\|$$

is increasing; as a consequences of Lemma 2.2, two subsequent equivalents (2.2) and (2.3) can be immediately obtained. Note also that (2.2) holds for all $s > 0$ in view of Remark 2.3; see also Lemma 11.1 in [15].

Notation 2.6. *Denote the classes of mappings which are firmly nonexpanisve, nonexpansive, asymptotically nonexpansive, uniformly Lipschitzian, Lipschitzian, uniformly continuous, and continuous by* (FN), (N), (AN), (UL), (L), (UC) *and* (C) *in order.*

Remark 2.7. Note that the following hold:

$$(FN) \subsetneq (N) \subsetneq (AN) \subsetneq (UL) \subsetneq (L) \subsetneq (UC) \subsetneq (C). \tag{2.4}$$

It is obvious that if we take $Tx = x^2$ for $x \in C = \mathbb{R}$, then $T \in (C) \setminus (UC)$; if we take $Tx = \sqrt{x}$, $\forall\, x \in C = [0, \infty) \subset \mathbb{R}$, $T \in (UC) \setminus (L)$; furthermore, if we take $Tx = 2x$ for $x \in C$, then $T \in (L) \setminus (UL)$ with its Lipschitz constant $L_T = 2$.

3. Classes of non-Lipschitzian mappings of quasi-nonexpansive type

Now consider the special cases when $F(T) \neq \emptyset$.

Definition 3.1. A mapping $T : C \to C$ is said to be

(b)$'$ *quasi-nonexpansive* [12] if $F(T) \neq \emptyset$ and (b) in Definition 2.1 is satisfied for all $(x, q) \in C \times F(T)$: namely,

$$\|Tx - q\| \leq \|x - q\|, \quad \forall\, (x, q) \in C \times F(T).$$

(c)$'$ *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and (c) in Definition 2.1 is remained true for all $(x, q) \in C \times F(T)$.

(d)$'$ $T : C \to C$ is said to be *firmly quasi-nonexpansive* if $F(T) \neq \emptyset$ and Definition 2.4 is satisfied for each $(x, q) \in C \times F(T)$: more explicitly, for each $(x, q) \in C \times F(T)$, there is $j(Tx - q) \in J(Tx - q)$ such that

$$\|Tx - q\|^2 \leq \langle x - q, j(Tx - q)\rangle; \tag{3.1}$$

equivalently,

$$\|Tx - q\| \leq \|s(x - q) + (1 - s)(Tx - q)\|, \quad \forall\, s \geq 0,\ (x, q) \in C \times F(T). \tag{3.2}$$

In particular, in case that $X \in (H)$, (3.1) becomes

$$\|Tx - q\|^2 \leq \langle x - q, Tx - q\rangle \Leftrightarrow \langle Tx - x, Tx - q\rangle \leq 0; \tag{3.3}$$

equivalently,

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2, \quad \forall\ (x, q) \in C \times F(T). \tag{3.4}$$

Remark 3.2. Note also that if $X \in (H)$, then

$$T \in (Fq\text{-}N) \;\Leftrightarrow\; 2T - I \in (q\text{-}N) \tag{3.5}$$

because $T \in (Fq\text{-}N) \;\Leftrightarrow\; \langle Tx - q, Tx - x\rangle \leq 0$ from (3.3), and

$$\|(2T - I)x - q\|^2 = 4\langle Tx - q, Tx - x\rangle + \|x - y\|^2, \quad \forall\,(x, q) \in C \times F(T)$$

by the analogous proof of the following equivalence:

$$T \in (FN) \;\Leftrightarrow\; 2T - I \in (N); \tag{3.6}$$

see Theorem 12.1 in [14] for more equivalent forms.

Remark 3.3. Whenever $X \in (H)$ and (3.4) holds, T is often called *directed* operator; see [11, 48].

Notation 3.4. *Denote the classes of mappings which are firmly quasi-nonexpanisve, quasi-nonexpansive, and asymptotically quasi-nonexpansive by* $(Fq\text{-}N)$, $(q\text{-}N)$, *and* $(Aq\text{-}N)$ *in turns.*

Then it is immediate that

$$(Fq\text{-}N) \subsetneq (q\text{-}N) \subsetneq (Aq\text{-}N). \tag{3.7}$$

and if $F(T) \neq \emptyset$, we readily see that

$$(FN) \subsetneq (Fq\text{-}N), \;\; (N) \subsetneq (q\text{-}N), \;\; (AN) \subset (Aq\text{-}N), \tag{3.8}$$

Recall that the following notation

$$H(x, y) = \{u \in H : \langle u - y, x - y\rangle \leq 0\}$$

is originally due to Haugazeau [17].

Definition 3.5. ([2]). $\mathfrak{T} = \{T : H \to H \,|\, \operatorname{dom} T = H, \; F(T) \subset H(x, Tx), \; \forall x \in H\}$

Remark 3.6. Note that if $\operatorname{dom} T = H$, $\mathfrak{T} = (Fq\text{-}N)$, by immediately combining (3.5) with the well known fact $T \in \mathfrak{T} \;\Leftrightarrow\; 2T - I \in (q\text{-}N)$; see Proposition 2.3 in [2]. Furthermore, we could observe that if T is a *subgradient projector* relative to a continuous convex function $f : H \to \mathbb{R}$, such that the level set $S(f, 0) = \{x \in H : f(x) \leq 0\} \neq \emptyset$, i.e.,

$$Tx = \begin{cases} x - \frac{f(x)}{\|g(x)\|^2}\, g(x) & \text{if } f(x) > 0 \\ x & \text{if } f(x) \leq 0, \end{cases}$$

where g is a selection of ∂f, then $T \in \mathfrak{T} = (Fq\text{-}N)$; see also Proposition 2.3 in [2], while every (metric) projector $T \in (FN)$.

Let $X \in (S)$. Since the normalized duality mapping J from X to X^* is single-valued, the Lyapunov functional $\phi : X \times X \to [0, \infty)$ is well defined as

$$\phi(x, y) = \|x\|^2 - 2(x, Jy) + \|y\|^2, \quad \forall\, x, y \in X.$$

In 2008, Kohsaka and Takahashi [33] say that a mapping $T : C \to C$ is *non-spreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(y, Tx), \quad \forall\, x, y \in C.$$

In particular, if $X \in (H)$, it reduces to

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall\, x, y \in C;$$

equivalently,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C; \tag{3.9}$$

see [20].

Notation 3.7. *Denote the class of nonspreading mappings by* $(nonS)$.

Remark 3.8. Obviously, $(FN) \subsetneq (nonS)$. Furthermore, if $X \in (SC) \cap (UG)$ and $F(T) \neq \emptyset$, $(nonS) \subsetneq (q\text{-}N)$; see [33].

4. Examples for mappings of nonexpansive/quasi-nonexpansive type

From now on, we will survey strict inclusions for classes of nonlinear mappings stated in section 2 and 3. We begin with an example of a mapping $T \in (AN)\backslash(q\text{-}N)$ with $F(T) \neq \emptyset$.

Example 4.1. *Let* $X = \ell^p$, *where* $1 < p < \infty$. *Obviously,* X *is uniformly convex and uniformly smooth. Let* $T : X \to X$ *be defined by*

$$Tx = (0, x_1^2, a_2x_2, a_3x_3, \cdots), \quad \forall x = (x_1, x_2, x_3, x_4, \cdots) \in X, \tag{4.1}$$

where $\{a_n\}_{n=0}^{\infty}$ *is a sequence of real numbers such that* $a_2 > 0$, $a_n \in (0,1)$ *for* $n \neq 2$ *and* $\sum_{n=2}^{\infty} a_n = \frac{1}{2}$ *(e.g., if* $a_2 < 1$, *consider* $a_n := 1 - \frac{1}{n^2}$ *for* $n \ge 2$*); if* $a_2 > 1$, *then we take* $a_3 = {a_2}^{-1}$, *and* $a_n := 1 - \frac{1}{(n-2)^2}$ *for* $n \ge 4$*). Then* $F(T) \neq \emptyset$ *and* $T \in (AN) \setminus (q\text{-}N)$.

Proof. Clearly, $F(T) = \{0\}$, where $0 = (0, 0, \cdots)$, and T is Lipschitzian, i.e., $\|Tx - Ty\| \le 2\|x - y\|$ for all $x, y \in X$. Since $a_n \in (0,1)$ for $n \neq 2$, we firstly see

$$\prod_{i=j}^{n+j} a_i \le 1 = 2\sum_{i=2}^{\infty} a_i \le 2\prod_{i=2}^{n} a_i, \quad \forall j \ge 1,\ n \ge 2. \tag{4.2}$$

Taking $k_n := 2\prod_{i=2}^{n} a_i \downarrow 1$, as a simple calculation, we have

$$T^n x = \Big(\overbrace{0, \cdots, 0}^{n}, \prod_{i=2}^{n} a_i x_1^2, \prod_{i=2}^{n+1} a_i x_2, \prod_{i=3}^{n+2} a_i x_3, \cdots\Big)$$

for $x = (x_1, x_2, x_3, \cdots) \in X$ and $n \ge 2$. Then it follows from (4.2) that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall n \ge 1;$$

thus $T \in (AN)$. However, if we take $x = (0, 1, \cdots)$, then

$$\|Tx - 0\| = a_0 > 1 = \|x - 0\|;$$

hence $T \notin (q\text{-}N)$. □

Remark 4.2. (a) Example 4.1 is originally due to Goebel and Kirk [13] in the Hilbert space ℓ^2, where $a_n = A_n \in (0,1)$ for all $n \ge 2$ and $\sum_{n=2}^{\infty} A_n = \frac{1}{2}$ (in this case, we can define $T : C \to C$, where $C = B$ the unit ball of $X = \ell^2$). In 2007, their example was slightly modified by Oslike et al.; see Example 2 in [38], where $T : B \to \ell^2$ was defined as in (4.1). Generally, since $Tx \notin B$ for $x \in B$, it should be modified as in Example 4.1 for their further argument. In 2008, it was carried over the Banach space ℓ^p, $p > 1$; see Example 3.13 in [27].

(b) Due to (3.8), if T is defined as in Example 4.1, then $T \in (Aq\text{-}N) \setminus (N)$ and also $T \in (AN) \setminus (N)$, which clarifies the strictness of "$(N) \subsetneq (AN)$" in (2.4).

Moreover, $T \in (Aq\text{-}N) \setminus (q\text{-}N)$, which ensures the strictness of " $(q\text{-}N) \subsetneq (Aq\text{-}N)$" in (3.7).

(c) Since $1 + x \le e^x := E(x)$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} 1 &< \prod_{n=2}^{n}\Big(1+\frac{1}{(n-1)^2}\Big) \le \prod_{n=2}^{\infty}\Big(1+\frac{1}{(n-1)^2}\Big) \\ &\le \prod_{n=2}^{\infty} E\Big(\frac{1}{(n-1)^2}\Big) = E\Big(\sum_{n=2}^{\infty}\frac{1}{(n-1)^2}\Big) = E\Big(\frac{\pi^2}{6}\Big). \end{aligned}$$

If T is defined as (4.1) and $a_n = 1 + \frac{1}{(n-1)^2}$ for $n \ge 2$, since

$$2 < k_n = 2\prod_{i=2}^{n} a_i \le 2\prod_{n=2}^{\infty} a_n \le 2E\Big(\frac{\pi^2}{6}\Big) := k,$$

$T \in (UL) \setminus (AN)$, which proves strictness of "$(AN) \subsetneq (UL)$" in (2.4).

(d) If we take $b_n = \exp((-1)^n \frac{1}{n-1})$ for $n \ge 2$, then $b_{2n} \downarrow 1$ on $[1, e]$ while $b_{2n+1} \uparrow 1$ on $[\exp(-\frac{1}{3}), 1]$ but $\prod_{n=2}^{\infty} b_n = \exp(\prod_{n=2}^{\infty}(-1)^n \frac{1}{n-1}) = \exp(\ln 2) = 2$. Now define

$$Tx = (0, \sin x, a_2 b_2 x_2, a_3 b_3 x_3, \cdots), \quad \forall x = (x_1, x_2, x_3, \cdots) \in X.$$

Since $\prod_{i=2}^{n} a_n \downarrow \frac{1}{2}$ and $\prod_{i=2}^{n} b_n \to 2$, if we take $k_n := \prod_{i=2}^{n} a_i b_i$, then it is obvious that $T \in (AN) \setminus (Fq\text{-}N)$, too.

Consider the following easy example of a mapping $T \in (AN) \setminus (N)$.

Example 4.3. ([28]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} kx, & \text{if } 0 \le x \le 1/2; \\ \frac{k}{2k-1}(k-x), & \text{if } 1/2 \le x \le k; \\ 0, & \text{if } k \le x \le 1, \end{cases}$$

where $1/2 < k < 1$. Then $F(T) = \{0\}$ and $T \in (AN) \setminus (N)$.

Proof. Clearly, $F(T) = \{0\}$. Since $\frac{k^n}{2k-1} \to 0$, there exists $K \in \mathbb{N}$ such that $\frac{k^n}{2k-1} < \frac{1}{2}$ for all $n \ge K$. Now we show that $T \in (AN)$. Indeed, if $0 \le x \le 1/2$ and $1/2 \le y \le k$, then $T^n x = k^n x$ and $T^n y = \frac{k^n}{2k-1}(k-y)$. by noticing $\frac{k}{2k-1} > 1$, $\frac{1}{2k-1} > 1 \Leftrightarrow k < 1$, we observe

$$\begin{aligned} |T^n x - T^n y| &= \Big|k^n x - \frac{k^n}{2} + \frac{k^n}{2} - \frac{k^n}{2k-1}(k-y)\Big| \\ &= \Big|k^n\Big(x - \frac{1}{2}\Big) + \frac{k^n}{2k-1}\Big[\Big(k - \frac{1}{2}\Big) - (k-y)\Big]\Big| \\ &\le k^n\Big|x - \frac{1}{2}\Big| + \frac{k^n}{2k-1}\Big|y - \frac{1}{2}\Big| \\ &\le \frac{k^n}{2k-1}|x-y| \le k_n|x-y|, \end{aligned}$$

if we take $k_n \to 1$ such that $k_n \ge \frac{1}{2}$ for all $n \ge K$ and $k_n = \frac{k^n}{2k-1}$ for $n < K$. The remaining cases are obvious. Hence $T \in (AN)$. However, if we take $1/2 \le x, y \le k$, then

$$|Tx - Ty| = \frac{k}{2k-1}|x-y| > |x-y|,$$

which claims $T \notin (N)$. □

Example 4.4. *Under the same hypotheses of $B \subset X = \ell^p$ with Example 4.1, consider a shifting operator $T : B \to B$ defined by*

$$Tx = (0, x_1, x_2, \cdots), \quad \forall x = (x_1, x_2, \cdots) \in B.$$

Then, $F(T) = \{0\}$ and $T \in (q\text{-}N) \setminus (Fq\text{-}N)$.

Proof. Clearly, $F(T) = \{0\}$. Since $\|Tx\| = \|x\|$, $T \in (q\text{-}N)$. However, if we take $x = (1, 0, \cdots) \in B$, $q = 0$ and $s = \frac{1}{2}$ in (3.2), we readily see that $\|x + Tx\| = \|(1, 1, 0, \cdots)\| = 2^{\frac{1}{p}}$ and

$$\|Tx - 0\| = 1 > \frac{1}{2} 2^{\frac{1}{p}} = \|sx + (1 - s)Tx\| \Leftrightarrow p > 1,$$

which yields $T \notin (Fq\text{-}N)$ by (3.2). □

Remark 4.5. Note that Example 4.4 ensures strictness of $(Fq\text{-}N) \subsetneq (q\text{-}N)$.

Next let us introduce an easy example of a *discontinuous* mapping $T \in (Fq\text{-}N) \setminus (FN)$, $F(T) \neq \emptyset$, assuring strictness of "$(FN) \subsetneq (Fq\text{-}N)$" in (3.8), which was originally due to Bauschke and Combettes [2].

Example 4.6. ([2]) *Let $H = \mathbb{R}$ with its usual norm $|\cdot|$ and $C = [-\pi, \pi]$. Let $T : C \to C$ be defined by*

$$Tx = \frac{3}{4} 1_{\mathbb{Q}} I x = \begin{cases} \frac{3}{4} x, & x \in C \cap \mathbb{Q}; \\ 0, & x \in C \cap \mathbb{Q}^c. \end{cases}$$

Then, $F(T) = \{0\}$ and $T \in (Fq\text{-}N) \setminus (FN)$.

Proof. Clearly, $F(T) = \{0\}$. We readily see that $2T - I = \left(\frac{1}{2} 1_{\mathbb{Q}} - 1_{\mathbb{Q}}^c\right) I \in (q\text{-}N)$ for

$$|(2T - I)x - 0| = \left| \left(\frac{1}{2} 1_{\mathbb{Q}}(x) - 1_{\mathbb{Q}}^c(x)\right) I x \right| \leq |x - 0|, \quad \forall x \in C.$$

Hence $T \in (q\text{-}F)$ by (3.5). However, if we take $x = \pi$ and $y = 1$, then $2T - I \notin (N)$ for

$$|(2T - I)\pi - (2T - I)1| = \pi + \frac{1}{2} > |\pi - 1|.$$

From (3.6), we have $T \notin (FN)$. □

Question 4.7. *How about strictness of "$(AN) \subset (Aq\text{-}N)$" in (3.8)? Construct an example of a mapping $T \in (Aq\text{-}N) \setminus (AN)$ in a case when $F(T) \neq \emptyset$. It still remains open.*

Recall that if $F(T) \neq \emptyset$, then $(N) \subsetneq (q\text{-}N)$ in (3.8). Consider the following example due to Hicks and Kubicek [18] of a non-Lipschitzian mapping $T \in (q\text{-}N) \setminus (N)$, showing that the class $(q\text{-}N)$ is strictly larger than the class (N) with $F(T) \neq \emptyset$.

Example 4.8. ([18]; Example 1) *Let $X = \mathbb{R}$ and $C = [-1, 1]$. Define $T : C \to C$ by*

$$Tx = \begin{cases} \frac{2}{3} x \sin \frac{1}{x}, & x \in C \setminus \{0\}; \\ 0, & x = 0. \end{cases}$$

Then $F(T) = \{0\}$ and $T \in (q\text{-}N) \setminus (N)$.

Proof. Obviously, $F(T) = \{0\}$). For any $x(\neq 0) \in C$, since

$$|Tx| = \left|\frac{2}{3}x \sin\frac{1}{x}\right| \leq \frac{2}{3}|x| \leq |x|,$$

which proves that $T \in (q\text{-}N)$. However, if we take $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, then

$$\begin{aligned} |Tx - Ty| &= \left|\frac{2}{3}\cdot\frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{3}\cdot\frac{2}{3\pi}\sin\frac{3\pi}{2}\right| = \frac{16}{9\pi} \\ &> \frac{4}{3\pi} = |x - y| \end{aligned}$$

and so, $T \notin (N)$. Clearly, T is not Lipschitzian; see Example 4.3 of [25] for detailed proof. □

The following example of a Lipschitzian mapping $T \in (q\text{-}N) \setminus (N)$ was originally due to Chidume [8].

Example 4.9. ([8]) *Let* $X = \ell^\infty$ *and* $C = \{x \in \ell^\infty : \|x\|_\infty \leq 1\}$. *Define* $T : C \to C$ *by*

$$Tx = (0, x_1^2, x_2^2, \cdots), \quad x = (x_1, x_2, \cdots) \in C.$$

Then, $F(T) = \{0\}$ *and* $T \in (q\text{-}N) \setminus (N)$.

Proof. It is clear that $F(T) = \{0\}$ and $\|Tx - Ty\|_\infty \leq 2\|x - y\|_\infty$ for all $x, y \in C$; hence T is Lipschitzian. Moreover, since

$$\begin{aligned} \|Tx\|_\infty &= \|(0, x_1^2, x_2^2, \cdots)\|_\infty \\ &\leq \|(0, x_1, x_2, \cdots)\|_\infty = \|x\|_\infty, \quad \forall x \in C, \end{aligned}$$

it results $T \in (q\text{-}N)$. However, if we take $x = \left(\frac{3}{4}, \frac{3}{4}, \cdots\right)$, $y = \left(\frac{1}{2}, \frac{1}{2}, \cdots\right)$ in C, then

$$\|Tx - Ty\|_\infty = \left\|\left(0, \frac{5}{16}, \frac{5}{16}, \cdots\right)\right\|_\infty = \frac{5}{16} > \frac{1}{4} = \left\|\left(\frac{1}{4}, \frac{1}{4}, \cdots\right)\right\|_\infty = \|x - y\|_\infty,$$

thus $T \notin (N)$. □

The following example of $T \in (q\text{-}N) \setminus (Fq\text{-}N)$ due to [41] is not adequate for our argument because T is linear.

Example 4.10. ([41]) *Let* $H := \mathbb{R}^2$ *with the usual norm and* $C := [-1,1]\times[-1,1] \subset H$, *Define* $T : C \to C$ *by*

$$Tx = (-x_2, x_1), \quad \forall x = (x_1, x_2) \in C.$$

Then $F(T) = \{0\}$, *where* $0 = (0,0) \in C$. *Obviously,* T *is linear and isometry. Therefore,* $T \in (N) \subset (q\text{-}N)$ *because* $F(T) \neq \emptyset$. *However,* $T \notin (Fq\text{-}N)$ *from* (3.4) *because*

$$\|Tx\| = \|x\| \quad \text{and} \quad \|x - Tx\| > 0 \ \text{for } x(\neq 0) \in C.$$

Remark 4.11. Example 4.10 asserts the strictness of "$(Fq\text{-}N) \subsetneq (q\text{-}N)$" in (3.7). Note also that $T \in (N) \setminus (FN)$.

Now let us investigate an example of $T \in (q\text{-}N) \setminus [(nonS) \cup (N)]$ due to Kim [24].

Example 4.12. ([24]) *Let* $C = [-\pi, \pi] \subset X := \mathbb{R}$. *Let a mapping* $T : C \to C$ *be defined by*

$$Tx = x \cos x, \quad \forall x \in C.$$

Then $T \in (q\text{-}N) \setminus [(nonS) \cup (N)]$.

Proof. Clearly, $F(T) = \{0\}$. Since $|Tx - 0| = |x \cos x| \le |x - 0|$ for all $x \in C$, it results $T \in (q\text{-}N)$. However, if we take $x = \pi$ and $y = \frac{\pi}{2}$, then $|Tx - Ty| = |\pi \cos \pi - \frac{\pi}{2} \cos \frac{\pi}{2}| = \pi > \frac{\pi}{2} = |x - y|$; so, $T \notin (N)$. Also, if we take $x = \pi$, $y = -\pi \in C$, since $Tx = -\pi$, $Ty = \pi$,

$$|Tx - Ty| = |x - y| = 2\pi \text{ and } 2\langle x - Tx, y - Ty\rangle = 2\langle 2\pi, -2\pi\rangle = -8\pi^2,$$

we have

$$\|Tx - Ty\|^2 = 4\pi^2 > -4\pi^2 = \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle,$$

thus, $T \notin (nonS)$ by (3.9). □

Here consider an example of $T \in (nonS) \setminus (C)$ due to [21]; hence $T \in (nonS) \setminus [(FN) \cup (N) \cup (L) \cup (UL) \cup (UC)]$ from (2.4).

Example 4.13. ([21]) *Let $B_r := \{x \in H : \|x\| \le r\}$ for $r > 0$ and $C := B_3 \subset H$ and define a mapping $T : C \to C$ by*

$$Tx = \begin{cases} x, & x \in B_2; \\ P_{B_1}x, & x \in C \setminus B_2, \end{cases}$$

where P_A is the metric projection of H onto A. Then $T \in (nonS) \setminus (C)$.

Proof. Obviously, $F(T) = B_2$. Let $x, y \in C$. It suffices to check out the case $x \in C \setminus B_2$, $y \in B_2$. Then, since P_{B_1} is nonexpansive and $y - Ty = 0$, it results that

$$\begin{aligned} \|Tx - Ty\|^2 &= \|P_{B_1}x - y\|^2 = \|P_{B_1}x - P_{B_1}y\|^2 \\ &\le \|x - y\|^2 = \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle; \end{aligned}$$

hence $T \in (nonS)$. Clearly, $T \notin (C)$. In fact, fix $x_0 \in \partial B_2$, $y_0 \in \partial C$. Consider $x_n = \left(1 - \frac{1}{n}\right) x_0 + \frac{1}{n} y_0 \in C$ for each $n \ge 1$. Then $x_n \to x_0$ but $Tx_n = P_{B_1}x_n \not\to Tx_0 = x_0$ because $\|P_{B_1}x_n\| = 1$ and $\|x_0\| = 2$. □

Remark 4.14. Note that $T \notin (C)$ in Example 4.13; hence $T \notin (U)$, in other words, $T \in (nonS)$ is generally not Lipschitzian.

Finally, we give two examples of $T \in (q\text{-}N) \setminus (C)$ while $(N) \subsetneq (UC)$.

Example 4.15. ([47]; see Example 2.11) *The mapping $T : [0,1] \to [0,1]$ is defined by*

$$Tx = \begin{cases} \frac{x}{5}, & x \in [0, \frac{1}{2}], \\ x \sin \pi x, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then $T \in (q\text{-}N) \setminus (C)$.

Proof. Obviously, $F(T) = \{0\}$. For any $x \in [0, \frac{1}{2}]$, we have

$$|Tx - 0| = |\frac{x}{5} - 0| \le |x - 0|,$$

and for any $x \in (\frac{1}{2}, 1]$, we have

$$|Tx - 0| = |x \sin \pi x - 0| \le |x - 0|.$$

Thus $T \in (q\text{-}N)$. Obviously, T is continuous at $\frac{1}{2}$. □

Example 4.16. ([45]; see Example 2) *Let* $C = [0,3] \subset \mathbb{R}$ *and define* $T : C \to C$ *by*

$$Tx = \begin{cases} 0, & x \neq 3; \\ 2, & x = 3. \end{cases}$$

Then it is obvious that $F(T) = \{0\}$ *and* $T \in (q\text{-}N) \setminus (C)$.

Question 4.17. *Find an example of a nonlinear mapping* $T \in (Aq\text{-}N) \setminus (q\text{-}N)$.

5. Class of operators of pseudo-contractive type

Definition 5.1. A mapping/an operator $T : C \to C$ is said to be

(a) *pseudocontractive* [5] if f is increasing on $[0,1]$, where $f(s) = \|(x-y) + s[(I-T)x - (I-T)y]\|$ of Lemma 2.2 with $u = x - y$ and $v = (I-T)x - (I-T)y$; equivalently,

$$\begin{aligned} \|x-y\| &\leq \|(x-y) + s[(I-T)x - (I-T)y]\| \\ &= \|(1+s)(x-y) - s(Tx - Ty)\|, \quad \forall x, y \in C,\ s > 0; \end{aligned}$$

equivalently, for each $x, y \in C$, there exist $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2. \tag{5.1}$$

Especially, in case when $X \in (H)$, the inequality (5.1) becomes

$$\langle Tx - Ty, x - y\rangle \leq \|x-y\|^2, \quad \forall x, y \in C; \tag{5.2}$$

equivalently,

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{5.3}$$

(b) *quasi-pseudocontractive* if $F(T) \neq \emptyset$ and for each $(x,q) \in C \times F(T)$, there exist $j(x-q) \in J(x-q)$ such that

$$\langle Tx - q, j(x-q)\rangle \leq \|x-q\|^2, \tag{5.4}$$

equivalently,

$$\|x-q\| \leq \|(x-q) + s(x - Tx)\|, \quad \forall s > 0.$$

In particular, in case when $X \in (H)$, the inequality (5.4) becomes

$$\langle Tx - q, x - q\rangle \leq \|x-q\|^2, \quad \forall (x,y) \in C \times F(T); \tag{5.5}$$

equivalently,

$$\|Tx - q\|^2 \leq \|x-q\|^2 + \|x - Tx\|^2, \quad \forall (x,q) \in C \times F(T). \tag{5.6}$$

(c) *asymptotically pseudocontractive*, with a sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$, [43] if each f_n is increasing on $[0,1]$, where

$$f_n(s) = \left\|(x-y) + s\Big[\Big(I - \frac{1}{k_n}T^n\Big)x - \Big(I - \frac{1}{k_n}T^n\Big)y\Big]\right\|;$$

by Lemma 2.2, which is equivalent to the following: for each $x, y \in C$ and $n \geq 1$, there exist $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2. \tag{5.7}$$

Especially, in case when $X \in (H)$, the inequality (5.7) becomes

$$\langle T^n x - T^n y, x - y\rangle \leq k_n\|x-y\|^2, \quad \forall x, y \in C; \tag{5.8}$$

equivalently, for all $(x,q) \in C \times F(T)$ and $n \geq 1$,

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x-y\|^2 + \|(I-T^n)x - (I-T^n)y\|^2.$$

(d) *asymptotically hemicontractive*, with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ [39, 35, 37] if $F(T) \neq \emptyset$ and for each $(x, q) \in C \times F(T)$ and $n \geq 1$, there exist $j(x-q) \in J(x-q)$ such that

$$\langle T^n x - q, j(x-q)\rangle \leq k_n \|x-q\|^2. \tag{5.9}$$

In particular, in case when $X \in (H)$, the inequality (5.9) becomes

$$\langle T^n x - q, x-q\rangle \leq k_n \|x-q\|^2; \tag{5.10}$$

equivalently, for all $(x, q) \in C \times F(T)$ and $n \geq 1$,

$$\|T^n x - q\|^2 \leq (2k_n - 1)\|x-q\|^2 + \|x - T^n x\|^2.$$

Notation 5.2. *Denote the classes of mappings/operators which are pseudocontractive, quasi-pseudocontractive, asymptotically pseudocontractive, and asymptotically hemicontractive by* (P), $(q\text{-}P)$, (AP) *and* (AH) *in order.*

Recall that an operator T is said to be *accretive* [4, 23] if for all $x, y \in D(T)$,

$$\langle Tx - Ty, j(x-y)\rangle \geq 0, \quad \forall j(x-y) \in J(x-y).$$

Remark 5.3. (a) Note that $T \in (P)$ iff

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq 0, \quad \forall j(x-y) \in J(x-y)$$

by (5.1), in other words, the operator $I - T$ is accretive; see [4, 23].

(b) Every asymptotically hemicontractive mapping/operator is sometimes called *asymptotically quasi-pseudocontractive*; see [11, 22]. If we denote by $(Aq\text{-}P)$ the class of asymptotically quasi-pseudocontractive mappings/operators, then $(Aq\text{-}P) = (AH)$.

(c) Note also that

$$(N) \subsetneq (P) \text{ and } (AN) \subsetneq (AP); \tag{5.11}$$

see [5, 43] or Example 7.8 for a mapping $T \in [(P) \cap (AP)] \setminus (L)$. Furthermore, it is obvious that if $F(T) \neq \emptyset$, then

$$(P) \subsetneq (q\text{-}P) \text{ and } (AP) \subsetneq (AH) = (Aq\text{-}P). \tag{5.12}$$

6. Class of operators of strictly pseudo-contractive type

Definition 6.1. An operator A with domain $D(A)$ and range $R(R)$ is said to be

(a) *α-strongly accretive* if for each $x, y \in D(A)$, there exist $j(x-y) \in J(x-y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2;$$

(b) *α-inverse strongly accretive* (in brief, α-isa) [16] if for each $x, y \in D(A)$, there exist $j(x-y) \in J(x-y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|Ax - Ay\|^2;$$

(in $X \in (H)$, note that α-isa is exactly *α-inverse strongly monotone* (in brief, *α-ism*)).

Recall that $T \in (P)$ iff $I - T$ is accretive, where $f(s) = \|u + sv\|$ in Lemma 2.2 with $u = x - y$ and $v = (I - T)x - (I - T)y$; see (a) of Definition 5.1. From now on, consider (i) $v = (I - \alpha T)x - (I - \alpha T)y$ or (ii) $v = (\alpha I - T)x - (\alpha I - T)y$ for any (fixed) $\alpha > 0$. We say that T is said to be *α-strongly pseudocontractive* if the convex function $f(s) = \|u + sv\| = \|(x - y) + s[(I - \alpha T)x - (I - \alpha T)y]\|$ is increasing, by Lemma 2.2, for each $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and $\alpha > 1$ such that

$$
\begin{aligned}
& \langle (I - \alpha T)x - (I - \alpha T)y, j(x - y)\rangle \geq 0 \\
\Leftrightarrow\quad & \langle Tx - Ty, j(x - y)\rangle \leq \frac{1}{\alpha}\|x - y\|^2 \\
\Leftrightarrow\quad & \langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \frac{\alpha - 1}{\alpha}\|x - y\|^2 \\
\Leftrightarrow\quad & I - T \text{ is } \frac{\alpha - 1}{\alpha}\text{-strongly accretive;}
\end{aligned}
$$

equivalently,

$$
\begin{aligned}
\|x - y\| &\leq \|(x - y) + s[(I - \alpha T)x - (I - \alpha T)y]\| \\
&= \|(1 + s)(x - y) - s\alpha(Tx - Ty)\|.
\end{aligned}
$$

Similarly, we say that T is said to be *α-generalized pseudocontraction* [46] if the convex function $f(s) = \|u+sv\| = \|(x-y)+s[(\alpha I-T)x-(\alpha I-T)y]\|$ is increasing, by Lemma 2.2, for each $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$
\begin{aligned}
& \langle (\alpha I - T)x - (\alpha I - T)y, j(x - y)\rangle \geq 0 \\
\Leftrightarrow\quad & \langle Tx - Ty, j(x - y)\rangle \leq \alpha\|x - y\|^2 \\
\Leftrightarrow\quad & \langle (I - T)x - (I - T)y, j(x - y)\rangle \geq (1 - \alpha)\|x - y\|^2 \\
\Rightarrow\quad & I - T \text{ is } (1 - \alpha)\text{-strongly accretive if } \alpha < 1;
\end{aligned}
$$

equivalently,

$$
\begin{aligned}
\|x - y\| &\leq \|(x - y) + s[(\alpha I - T)x - (\alpha I - T)y]\| \\
&= \|(1 + s\alpha)(x - y) - s(Tx - Ty)\|.
\end{aligned}
$$

Finally, consider the case $I - T$ is α-isa, i.e., for each $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$
\begin{aligned}
& \langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \alpha\|(I - T)x - (I - T)y\|^2 \\
\Leftrightarrow\quad & \langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \alpha\|(I - T)x - (I - T)y\|^2
\end{aligned}
$$

Definition 6.2. A mapping/an operator $T : C \to C$ is said to be

(a) *asymptotically κ-strictly pseudocontractive*, with sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ [39, 35, 37, 38, 31] if for each $x, y \in C$ and $n \geq 1$, there exist $j(x - y) \in J(x - y)$ and $\kappa \in (-\infty, 1)$ such that

$$
\langle T^n x - T^n y, j(x - y)\rangle \leq \frac{1 + k_n}{2}\|x - y\|^2 - \frac{1 - \kappa}{2}\|(I - T^n)x - (I - T^n)y\|^2. \quad (6.1)
$$

(b) *asymptotically κ-demicontractive*, with sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ [37] if $F(T) \neq \emptyset$ and for each $(x, q) \in C \times F(T)$ and $n \geq 1$, there exist

$j(x-q) \in J(x-q)$ and $\kappa \in (-\infty, 1)$ such that

$$\langle T^n x - q, j(x-q)\rangle \le \frac{1+k_n}{2}\|x-q\|^2 - \frac{1-\kappa}{2}\|x - T^n x\|^2.$$

(c) *κ-strictly pseudocontractive* [5, 36] if for each $x, y \in C$, there exist $j(x-y) \in J(x-y)$ and $\kappa \in (-\infty, 1)$ such that

$$\begin{aligned} & \langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \frac{1-\kappa}{2}\|(I-T)x - (I-T)y\|^2 \\ \Leftrightarrow\quad & \langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \frac{1-\kappa}{2}\|(I-T)x-(I-T)y\|^2 \qquad (6.2) \\ \Leftrightarrow\quad & I - T \text{ is } \alpha\text{-isa, where } \alpha = (1-\kappa)/2 > 0\ . \end{aligned}$$

(d) *κ-demicontractive* if $F(T) \neq \emptyset$ and for each $(x, q) \in C \times F(T)$, there exist $j(x-q) \in J(x-q)$ and $\kappa \in (-\infty, 1)$ such that

$$\langle Tx - q, j(x-q)\rangle \le \|x-q\|^2 - \frac{1-\kappa}{2}\|x - Tx\|^2. \qquad (6.3)$$

Remark 6.3. Note that many authors [39, 35, 37, 38] call *κ-strictly asymptotically pseudocontractive* and *κ-asymptotically demicontractive* in place of "asymptotically κ-strictly pseudocontractive" in (a) and "asymptotically κ-demicontractive" in (c), respectively.

Notation 6.4. *Denote the classes of mappings which are asymptotically κ-strictly pseudocontractive, asymptotically κ-demicontractive, κ-strictly pseudocontractive, and κ-demicontractive by $(A\kappa\text{-}SP)$, $(A\kappa\text{-}D)$, $(\kappa\text{-}SP)$ and $(\kappa\text{-}D)$ in order.*

Remark 6.5. (a) Note that, since $\lambda = 1 - \kappa > 0$, it suffices to only choose $\kappa \in [0, 1)$ in place of $\kappa \in (-\infty, 1)$ in (a)-(d) of Definition 6.2.

(b) From (6.2), we note that

$$\begin{aligned} & \|x-y\| \ge \frac{1-\kappa}{2}\|(I-T)x-(I-T)y\| \ge \frac{1-\kappa}{2}(\|Tx-Ty\| - \|x-y\|) \\ \Leftrightarrow\quad & \|Tx - Ty\| \le L\|x-y\|, \end{aligned}$$

where $L := \frac{3-\kappa}{1-\kappa}$. Thus, $T \in (L)$, showing that $(\kappa\text{-}SP) \subsetneq (L)$; see [36] and also Proposition 2.1 of [34] for $X \in (H)$.

(c) Similarly, it easily follows from (6.1) that

$$\begin{aligned} & (1-\kappa)\|(I-T^n)x - (I-T^n)y\|^2 \\ \le\ & 2\langle (I-T^n)x - (I-T^n)y, j(x-y)\rangle + (k_n - 1)\|x-y\|^2 \\ \le\ & 2\|(I-T^n)x - (I-T^n)y\| \cdot \|x-y\| + (k_n-1)\|x-y\|^2, \end{aligned}$$

which is a quadratic equation of $t := \|(I-T^n)x - (I-T^n)y\|$; hence it quickly follows that

$$\begin{aligned} & \|T^n x - T^n y\| - \|x-y\| \le \|(I-T^n)x - (I-T^n)y\| \\ \le\ & \frac{1+\sqrt{1+(1-\kappa)(k_n-1)}}{1-\kappa}\|x-y\| \\ \Leftrightarrow\quad & \|T^n x - T^n y\| \le L_n\|x-y\|, \end{aligned}$$

where $L_n := \frac{2-\kappa+\sqrt{1+(1-\kappa)(k_n-1)}}{1-\kappa}$; hence $T \in (UL)$, which says that $(A\kappa\text{-}SP) \subsetneq (UL)$; see also Proposition 2.6 (a) in [31] for $X \in (H)$.

As a special case that $X \in (H)$, since $J = I$ and

$$2\langle u - v, x - y\rangle = \|x - y\|^2 + \|u - v\|^2 - \|(x - u) - (y - v)\|^2, \quad \forall x, y, u, v \in H,$$

(a)-(d) in Definition 6.2 will be rewritten as follows:

Definition 6.6. Let $X \in (H)$. Given a mapping/an operator $T : C \to C$, we call

(a) $T \in (A\kappa\text{-}SP)$ [39, 31] if there exist a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and $\kappa \in (-\infty, 1)$ such that, for all $x, y \in C$,

$$2\langle T^n x - T^n y, x - y\rangle \le (1 + k_n)\|x - y\|^2 - (1 - \kappa)\|(I - T^n)x - (I - T^n)y\|^2;$$

equivalently,

$$\|T^n x - T^n y\|^2 \le k_n\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y)\|^2, \quad \forall x, y \in C.$$

(b) $T \in (A\kappa\text{-}D)$ [34] if $F(T) \neq \emptyset$ and there exist a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and $\kappa \in (-\infty, 1)$ such that

$$\langle T^n x - q, x - q\rangle \le \frac{1 + k_n}{2}\|x - q\|^2 - \frac{1 - \kappa}{2}\|x - T^n x\|^2;$$

equivalently,

$$\|T^n x - q\|^2 \le k_n\|x - q\|^2 + \kappa\|x - T^n x)\|^2, \quad \forall\ (x, q) \in C \times F(T).$$

(c) $T \in (\kappa\text{-}SP)$ [5] if there exist $\kappa \in (-\infty, 1)$ such that

$$\langle (I - T)x - (I - T)y, x - y\rangle \ge \frac{1 - \kappa}{2}\|(I - T)x - (I - T)y)\|^2, \quad \forall x, y \in C; \tag{6.4}$$

equivalently,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y)\|^2, \quad \forall x, y \in C.$$

(d) $T \in (\kappa\text{-}D)$ [5] if the exists a constant $\kappa \in (-\infty, 1)$ such that

$$\|Tx - q\|^2 \le \|x - q\|^2 + \kappa\|x - Tx\|^2, \quad \forall (x, q) \in C \times F(T); \tag{6.5}$$

equivalently,

$$\langle x - Tx, x - q\rangle \ge \frac{1 - \kappa}{2}\|x - Tx\|^2, \quad \forall (x, q) \in C \times F(T). \tag{6.6}$$

Remark 6.7. (a) Note that it suffices to choose $\kappa \in [0, 1)$ instead of $\kappa \in (-\infty, 1)$ in (a)-(d) of Definition 6.6. Note also that

$$(\kappa\text{-}SP) \subsetneq (P) \ \text{ and } \ (A\kappa\text{-}SP) \subsetneq (AP) \tag{6.7}$$

but note that two classes (P) and (AP) are also independent as well as two classes $(\kappa\text{-}SP)$ and $(A\kappa\text{-}SP)$ are independent; see (b) of Remark 7.4.

(b) Note also that if $F(T) \neq \emptyset$, then

$$(\kappa\text{-}SP) \subsetneq (\kappa\text{-}D) \subsetneq (q\text{-}P) \ \text{ and } \ (A\kappa\text{-}SP) \subsetneq (A\kappa\text{-}D) \subsetneq (AH) = (Aq\text{-}P). \tag{6.8}$$

(c) Furthermore, if $X \in (H)$, there hold the following properties:

i) Every metric projection P_C is directed; see (3.4).

ii) $(AN) = (A0\text{-}SP) \subsetneq (A\kappa\text{-}SP)$, $(Aq\text{-}N) = (A0\text{-}D) \subsetneq (A\kappa\text{-}D)$, $(N) = (0\text{-}SP) \subsetneq (\kappa\text{-}SP)$ and $(q\text{-}N) = (0\text{-}D) \subsetneq (\kappa\text{-}D)$.

iii) If $\kappa < 0$, $(\kappa\text{-}D) \subsetneq (q\text{-}N)$ by noticing that $T \in (q\text{-}N) \setminus (\kappa\text{-}D)$ in Example 4.4 for $\kappa < 0$.

iv) $((-1)\text{-}D) = (Fq\text{-}N)$, which is exactly the class of directed operators [11])

v) $(q\text{-}N) \subsetneq (\kappa\text{-}D)$ for $\kappa \in (0, 1)$ and $(Aq\text{-}N) \subsetneq (AH) = (Aq\text{-}P)$.

7. Examples of operators of pseudo-/strictly pseudo-contractive type

First, let us invoke the following two examples due to [38], showing that classes (P) and (AP) in (5.12) are independent. Also, $(\kappa\text{-}SP)$ and $(A\kappa\text{-}SP)$ in (6.7) are independent.

Example 7.1. ([38]; Example 1) *Let $C = \mathbb{R}$ and define $T : C \to C$ by*

$$Tx = -2x, \quad \forall x \in C.$$

Then $F(T) = \{0\}$ and $T \in (\kappa\text{-}SP) \setminus (AH)$ for $\kappa \in [\frac{1}{3}, 1)$.

Proof. Obviously, $F(T) = \{0\}$. Since $|(I-T)x - (I-T)y|^2 = 9|x-y|^2$ and

$$\begin{aligned} &\langle (I-T)x - (I-T)y, x-y\rangle = 3|x-y|^2 = \frac{1}{3}|(I-T)x-(I-T)y|^2 \\ \geq\ & \frac{1-\kappa}{2}|(I-T)x-(I-T)y|^2 \ \Leftrightarrow\ \frac{1}{3} \leq \mu < 1, \end{aligned}$$

it follows from (6.4) that $T \in (\kappa\text{-}SP)$ for $\kappa \in [\frac{1}{3}, 1)$. However, note that for even $n > 1$,

$$\langle T^n x, x\rangle = 2^n x^2 > 2x^2.$$

Since $k_n \to 1$, there exists $K \in \mathbb{N}$ such that $k_n < 2$ for all $k \geq K$. Therefore, for even $n \geq K$, we have

$$\langle T^n x, x\rangle = 2^n x^2 > 2x^2 > k_n |x|^2,$$

which implies that $T \notin (AH)$ by (5.10). □

Remark 7.2. Note that if T is defined as in Example 7.1, since $(\kappa\text{-}SP) \subset (\kappa\text{-}D) \cap (P)$ and $(A\kappa\text{-}SP) \cup (AP) \subset (AH)$ in (5.12) and (6.8), we have $T \in [(\kappa\text{-}D) \cap (P)] \setminus [(A\mu\text{-}SP) \cup (AP)]$ for all $\kappa \in \left[\frac{1}{3}, 1\right)$ and $\mu \in (-\infty, 1)$.

Example 7.3. ([38]; Example 2) *If $T : X \to X$ is defined as in Example 4.1, then $T \in (A\kappa\text{-}SP) \setminus (q\text{-}P)$ for $\kappa \in (0, 1)$.*

Proof. Taking $k_n := 2\left(\sum_{i=2}^n\right) \downarrow 1$, we easily see that

$$\begin{aligned} & \|T^n x - T^n y\|^2 \leq k_n\|x-y\|^2 \\ \leq\ & k_n\|x-y\|^2 + \kappa\|(I-T^n)x - (I-T^n)y\|^2 \end{aligned}$$

for $\kappa \in (0,1)$, $n \geq 2$ and $x, y \in X$. Therefore, $T \in (A\kappa\text{-}SP)$ for $\kappa \in (0,1)$. However, if we take $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \cdots\right)$ and $a_2 = 3$, since $F(T) = \{0\}$, it follows that

$$\langle Tx, x\rangle = \langle (0, 1/9, 1, a_3/3, 0, \cdots), (1/3, 1/3, 1/3, 0, \cdots)\rangle = \frac{10}{27} > \frac{1}{3} = \|x\|^2,$$

which asserts that $T \notin (q\text{-}P)$ by (5.5). □

Remark 7.4. (a) Note that if T is defined as in Example 7.3, since $(\kappa\text{-}SP) \subsetneq (P) \subsetneq (q\text{-}P)$ and $(A\kappa\text{-}SP) \subsetneq (AP) \subsetneq (AH)$ in (5.12), (6.7) and (6.8), we have $T \in [(AP) \cap (AH)] \setminus [(P) \cup (\kappa\text{-}SP)]$.

(b) Combined Remark 7.2, Example 7.1 and 7.3, we conclude that the classes (P) and (AP) in (5.12) are also independent as well as $(A\kappa\text{-}SP)$ and $(\kappa\text{-}SP)$ in (6.8) are independent; moreover, $(\kappa\text{-}D)$ and $(A\kappa\text{-}SP)$ are independent, too.

Next, let us introduce a Lipschitzian mapping $T \in (q\text{-}P) \setminus (P)$, which asserts strictness of "$(P) \subsetneq (q\text{-}P)$" in (5.12).

Example 7.5. ([3]) *Let $C = [0,3] \subset \mathbb{R} = H$. Let $T : C \to C$ be defined by*

$$Tx = \begin{cases} 0, & x \in [0,2]; \\ 2x - 4, & x \in [2,3]. \end{cases}$$

Then, $F(T) = \{0\}$ and $T \in (q\text{-}P) \setminus (P)$.

Proof. Obviously, $F(T) = \{0\}$. In view of (5.6), we observe that $T \in$ (q-P) if and only if

$$\begin{aligned} & |Tx - 0|^2 \le |x|^2 + |x - Tx|^2 \Leftrightarrow \langle Tx, x\rangle \le |x|^2 \\ \Leftrightarrow \quad & x(2x-4) \le x^2 \Leftrightarrow x \in [0,4]. \end{aligned}$$

On the other hand, if we take $x = 3, y = 2$, then we see

$$\langle Tx - Ty, x - y\rangle = 2 \not\le 1 = |x - y|,$$

which asserts that $T \notin (P)$. □

Recall the (normalized) duality mapping in the ℓ^p space, $p > 1$, is given by

$$Jx = \|x\|^{2-p}(|x_1|^{p-1}\mathrm{sgn}\, x_1, |x_2|^{p-1}\mathrm{sgn}\, x_2, \cdots), \quad \forall x = (x_1, x_2, \cdots) \in \ell^p,$$

by combining Corollary 4.10 and Proposition 4.7 (f) in [10]; see also Example 10.2 in [14]. Now consider an example of $T \in (\kappa\text{-}D) \setminus (q\text{-}N)$ in the ℓ^p space, which ensures strictness of "$(q\text{-}N) \subsetneq (\kappa\text{-}D)$ for $\kappa \in (0,1)$" in (v) of Remark 6.7 (c); see also Example 2.5 of [47] and Example 2.5 of [9] with $k = \frac{5}{2}$ in the ℓ^2 space,

Example 7.6. *Let $C = X = \ell^p$, $p > 1$ and $k > 1$. Let $T : X \to X$ be defined by $Tx = -kx$ for all $x \in X$. Then $F(T) = \{0\}$ and $T \in [(k\text{-}L) \cap (\kappa\text{-}D)] \setminus (q\text{-}N)$, where $\frac{k-1}{1+k} \le \kappa < 1$.*

Proof. It is obvious that $F(T) = \{0\}$ and $T \in (k\text{-}L)$. For each $x \in X$, since $\|x - Tx\|^2 = (1+k)^2\|x\|^2$ and $\langle Tx - 0, J(x-0)\rangle = -k\|x\|^2$, a simple observation gives

$$\begin{aligned} & \langle Tx, Jx\rangle \le \|x\|^2 - \frac{1-\kappa}{2}\|x - Tx\|^2 \\ \Leftrightarrow \quad & -k \le 1 - \frac{(1-\kappa)(1+k)^2}{2} \Leftrightarrow \frac{k-1}{1+k} \le \kappa < 1. \end{aligned}$$

Therefore, $T \in (\kappa\text{-}D)$ for $\frac{k-1}{1+k} \le \kappa < 1$ by (6.3). On the other hand, since $k > 1$ and $\|Tx\| = k\|x\| > \|x\|$ for all $x \in C$, which asserts $T \notin (q\text{-}N)$. □

Now let us review the mapping $T \in (\kappa\text{-}D) \setminus (\kappa\text{-}SP)$ in Example 4.8, assuring strictness of "$(\kappa\text{-}SP) \subsetneq (\kappa\text{-}D)$" in (6.8).

Example 7.7. ([18]; Example 1) *Let $H; = \mathbb{R}$ and $C = [-1,1]$. Define $T : C \to C$ by*

$$Tx = \begin{cases} \frac{2}{3}x \sin\frac{1}{x}, & x \in C \setminus \{0\}; \\ 0, & x = 0. \end{cases}$$

Then $T \in (\kappa\text{-}D) \setminus (P)$ for $\kappa \in [-5,1)$; hence, $T \notin (\kappa\text{-}SP)$ for $\kappa < 1$.

Proof. Obviously, $F(T) = \{0\}$. Since $\frac{1}{9}|x|^2 \le |x - Tx|^2 \le \frac{25}{9}|x|^2$, we observe

$$\begin{aligned} |Tx|^2 &= \left|\frac{2}{3}x\sin\frac{1}{x}\right|^2 \le \frac{4}{9}|x|^2 \le |x|^2 + \frac{\kappa}{9}|x|^2 \\ (&\le |x|^2 + \kappa|Tx - x|^2) \Leftrightarrow \kappa \in [-5,1), \end{aligned}$$

which quickly implies that $T \in (\kappa\text{-}D)$ for $\kappa \in [-5, 1)$ by (6.5). However, if we take $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, then

$$\begin{aligned} |Tx - Ty|^2 &= \frac{256}{81\pi^2} \\ &> |x-y|^2 + |(I-T)x - (I-T)y|^2 = \frac{16}{9\pi^2} + \frac{16}{81\pi^2} = \frac{160}{81\pi^2}, \end{aligned}$$

which proves that $T \notin (P)$ by (5.3); hence, $T \notin (\kappa\text{-}SP)$ for $\kappa < 1$ by (6.7). □

Now we shall introduce an example in [41] of $T \in [(P) \cap (AP)] \setminus [(\kappa\text{-}D) \cup (L)]$, guaranteeing strictness of "$(\kappa\text{-}D) \subsetneq (q\text{-}P)$" in (6.8), "$(N) \subsetneq (P)$ and $(AN) \subsetneq (AP)$" in (5.11).

Example 7.8. ([41]) *Let $H = \mathbb{R}$ and $C = [0, 1] \subset \mathbb{R}$. Let an operator $T : C \to C$ be defined by*

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}, \quad x \in C,$$

Then:

(a) $F(T) = \left\{\frac{\sqrt{2}}{4}\right\}$ *and* $T \in (P) \cap (AP)$*; hence,* $T \in (q\text{-}P) \cap (AH)$.

(b) $T \notin (L)$*; hence* $T \notin (UL)$.

(c) $T \notin (\kappa\text{-}D)$ *for* $\kappa < 1 - \frac{\sqrt{2}}{2}$.

Proof. (a) Obviously, $F(T) = \left\{\frac{\sqrt{2}}{4}\right\}$ and $T^2 = I$. Observing $T'(x) = -(1 - x^{2/3})^{1/2}(1/\sqrt[3]{x}) < 0$ for $x \in (0, 1)$, it follows that T is monotone decreasing and so

$$\langle Tx - Ty, x - y\rangle = (Tx - Ty)(x - y) \le 0 \le |x - y|^2$$

$$\begin{cases} \langle T^{2n}x - T^{2n}y\rangle = \langle x - y, x - y\rangle = |x - y|^2, \\ \langle T^{2n-1}x - T^{2n-1}y\rangle = \langle Tx - Ty, x - y\rangle \le |x - y|^2 \end{cases}$$

for $x, y \in C$ and $n \ge 1$, which claims that both $T \in (P)$ by (5.2) and $T \in (AP)$ with $k_n \equiv 1$ by (5.8); hence $T \in (q\text{-}P) \cap (AH)$ by (5.12).

(b) Since $T'(x) \to \infty$ as $x \to 0+$, $T \notin (L)$; hence $T \notin (UL)$ by "$(UL) \subsetneq (L)$" in (2.4).

(c) We show that $T \notin (\kappa\text{-}D)$ for $\kappa < 1 - \frac{\sqrt{2}}{2}$. In fact, if we take $x = 0$ and $q = \frac{\sqrt{2}}{4}$ in (6.6), then $|x - Tx| = 1$ and

$$\begin{aligned} \langle x - Tx, x - q\rangle &= \langle 0 - 1, 0 - \frac{\sqrt{2}}{4}\rangle = \frac{\sqrt{2}}{4} < \frac{1-\kappa}{2} \\ &= \frac{1-\kappa}{2}|x - Tx|^2 \Leftrightarrow \kappa < 1 - \frac{\sqrt{2}}{2}, \end{aligned}$$

which insists that $T \notin (\kappa\text{-}D)$ for $\kappa < 1 - \frac{\sqrt{2}}{2}$ by (6.6). □

Here, we give an example of $T \in (AH) \setminus (Aq\text{-}N)$, which assures strictness of "$(Aq\text{-}N) \subsetneq (AH) = (Aq\text{-}P)$" in (v) of Remark 6.7 (c).

Example 7.9. ([26]). *Let $H := \mathbb{R}$, $C := [-\frac{1}{k}, 1]$, where $1 < k < 2$. Define a mapping $T : C \to C$ by*

$$Tx := \begin{cases} -kx, & \text{if } -\frac{1}{k} \le x \le 0; \\ -\frac{1}{k}x, & \text{if } 0 \le x \le 1. \end{cases}$$

Then:

(a) $F(T) = \{0\}$ *and* $T^2 = I$*; hence,* $T^{2n-1} = T$ *for all* $n \ge 1$*;*

(b) $T \in (k\text{-}L)$; *hence* $T \in (UL)$ *by (a);*
(c) $T \in (AH) \setminus (Aq\text{-}N)$.

Proof. (b) For proving the Lipschitz of T, it suffices to see that if $-\frac{1}{k} \leq x \leq 0$ and $0 \leq y \leq 1$, then

$$\begin{aligned} |Tx - Ty| &= |(-kx) - (-1/k)y| = |1/k(y-x) + (1/k - k)x| \\ &\leq 1/k|x-y| + (k - 1/k)|x| \\ &\leq 1/k|x-y| + (k-1/k)|x-y| = k|x-y|. \end{aligned}$$

In view of (a), since $T^2 = I$, it is also uniformly k-Lipschitzian.

(c) Since $F(T) = \{0\}$, we observe

$$\langle T^{2n}x, x\rangle = \langle x, x\rangle = |x|^2,$$
$$\langle T^{2n-1}x, x\rangle = \langle Tx, x\rangle = x(Tx) \leq 0 \leq |x|^2;$$

it follows from (5.10) that $T \in (AH)$ for $k_n \equiv 1$. However, if $-\frac{1}{k} \leq x \leq 0$, since $k > 1$, we have

$$|T^{2n-1}x|^2 = |Tx|^2 = k^2|x|^2 > |x|^2,$$

which concludes that $T \notin (Aq\text{-}N)$. □

8. Classes of nonlinear mappings of asymptotically nonexpansive type

Definition 8.1. A mapping $T : C \to C$ is said to be

(a) *asymptotically nonexpansive type* (in brief, ANT) [32]; if T^N is continuous for some $N \geq 1$ and satisfies the following property of ANT: for each $x \in C$,

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^nx - T^ny\| - \|x-y\|) \leq 0. \tag{8.1}$$

(b) *asymptotically nonexpansive in the intermediate sense* (in brief, ANis) [6] if T is uniformly continuous and satisfies the following property of ANis:

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^nx - T^ny\| - \|x-y\|) \leq 0. \tag{8.2}$$

(c) *totally asymptotically nonexpansive* (in brief, TAN) ([1]) if there exist two nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n, \beta_n \to 0$, $\tau \in \Gamma(\mathbb{R}_+)$ and $n_0 \in \mathbb{N}$ such that

$$\|T^nx - T^ny\| \leq \|x-y\| + \alpha_n\tau(\|x-y\|) + \beta_n, \quad \forall\, x, y \in C,\ n \geq n_0, \tag{8.3}$$

where $\tau \in \Gamma(\mathbb{R}_+)$ iff τ is strictly increasing, continuous on R_+ and $\tau(0) = 0$.

(d) *square totally asymptotically nonexpansive* (in brief, sTAN) if (8.3) in (c) can be replaced by

$$\|T^nx - T^ny\|^2 \leq \|x-y\|^2 + \widetilde{\alpha}_n\widetilde{\tau}(\|x-y\|^2) + \widetilde{\beta}_n,$$

for all $x, y \in C$ and $n \geq m_0$, where $m_0 \in \mathbb{N}$, $\widetilde{\alpha}_n, \widetilde{\beta}_n \to 0$ and $\widetilde{\tau} \in \Gamma(\mathbb{R}^+)$.

As analogous concepts defined in sections 5 and 6, we can define as follows:

Definition 8.2. Let $X \in (H)$. A mapping $T : C \to C$ is said to be

(a) *asymptotically κ-strictly pseudocontractive in the intermediate sense* (in brief, AκSPis) [42] if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{k_n\}$, $k_n \to 1$, such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} [\|T^nx - T^ny\|^2 - k_n\|x-y\|^2 - \kappa\|(I-T^n)x - (I-T^n)y\|^2] \leq 0. \tag{8.4}$$

(c) *asymptotically pseudocontractive in the intermediate sense* (in brief, APis) if there exists a sequence $\{k_n\}$, $k_n \to 1$, such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} [\|T^n x - T^n y\|^2 - k_n\|x-y\|^2 - \|(I-T^n)x - (I-T^n)y\|^2] \le 0. \quad (8.5)$$

(b) *asymptotically κ-demicontractive in the intermediate sense* (in brief, ADis) [42] if $F(T) \neq \emptyset$ and there exist a constant $\kappa \in [0,1)$ and a sequence $\{k_n\}$, $k_n \to 1$, such that

$$\limsup_{n\to\infty} \sup_{x\in C, q\in F(T)} [\|T^n x - q\|^2 - k_n\|x-q\|^2 - \kappa\|x - T^n x\|^2] \le 0. \quad (8.6)$$

(c) *asymptotically hemicontractive in the intermediate sense* (in brief, AHis) [42] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$, $k_n \to 1$, such that

$$\limsup_{n\to\infty} \sup_{x\in C, q\in F(T)} [\|T^n x - q\|^2 - k_n\|x-q\|^2 - \|x - T^n x\|^2] \le 0. \quad (8.7)$$

Recall that $\{f_n\}$ converges uniformly to f on C if

$$\|f_n - f\| := \sup_{x\in C} \|f_n(x) - f(x)\| \to 0$$

as $n \to \infty$, where $f_n, f : C\,(\subset X) \to C$. We say that the mapping T satisfies the *uniform convergence to a point* on C if $\{T^n x\}$ converges uniformly to p on C, i.e.,

$$\sup_{x\in C} \|T^n x - p\| \to 0 \quad \text{as } n \to \infty.$$

Notation 8.3. *We denote classes of mappings of ANT, ANis, TAN, sTAN, Aκ-SPis, APis, Aκ-Dis, and AHis, in turns, by (ANT), (ANis), (TAN), and (sTAN), (Aκ-SPis), (APis), (Aκ-Dis), and (AHis),*

Also, we denote by (pUC) the class of mappings satisfying the uniform convergence to a point on C, by (pANT) the class of mappings satisfying the property of ANT (8.1) *and by (pANis) the class of mappings satisfying the property of ANis* (8.2).

Remark 8.4. (a) Note that if we define

$$c_n := \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0,$$

where $a \vee b := \max\{a,b\}$, then (8.2) ensures that $c_n \to 0$ and

$$\|T^n x - T^n y\| \le \|x-y\| + c_n \quad (8.8)$$

for all $x, y \in C$ and $n \ge 1$. Obviously, (8.8) implies (8.2) in case $c_n \to 0$. Therefore, we conclude that $T \in$ (*p-ANis*) iff (8.8) holds for some sequence $\{c_n\}$ with $c_n \to 0$.

(b) Similarly, if we define

$$d_n := \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - k_n\|x-y\|^2 - \kappa\|(I-T^n)x - (I-T^n)y\|^2) \vee 0,$$

then (8.4) ensures that $d_n \to 0$ and

$$\|T^n x - T^n y\|^2 \le k_n\|x-y\|^2 + \kappa\|(I-T^n)x - (I-T^n)y\|^2 + d_n \quad (8.9)$$

for all $x, y \in C$ and $n \ge 1$. Obviously, (8.9) implies (8.4) in case $d_n \to 0$. Therefore, we conclude that $T \in$ (*Aκ-SPis*) iff (8.9) holds for some sequence $\{d_n\}$ with $d_n \to 0$; see also [42].

(c) Note that the property (8.3) with $\alpha_n = 0$ for all $n \geq 1$ reduces to (8.8) with $\beta_n = c_n$; moreover, if we take $\tau(t) = t$ for all $t \geq 0$ and $\beta_n = 0$ for all $n \geq 1$ in (8.3), the class (TAN) is consistent with the class (AN) by (2.1) with $k_n = 1 + \alpha_n \to 1$.

(d) Note that the following strict inclusions hold:

$$[(pANis) \cup (AN)] \subsetneq (TAN), \quad (ANis) = (UC) \cap (pANis), \tag{8.10}$$
$$[(pANis) \cup (A\kappa\text{-}SP)] \subsetneq (A\kappa\text{-}SPis) \subsetneq (APis), \tag{8.11}$$
$$(A\kappa\text{-}Dis) \subsetneq (AHis). \tag{8.12}$$

(e) Assume $\delta := \mathrm{diam}(C) < \infty$ and $T \in (TAN)$. Then

$$\begin{aligned} \|T^n x - T^n y\| &\leq \|x-y\| + \alpha_n \sup_{x,y\in C} \tau(\|x-y\|) + \beta_n \\ &\leq \|x-y\| + \alpha_n \tau(\delta) + \beta_n \\ &= \|x-y\| + c_n, \quad \forall\, x, y \in C,\ n \geq 1, \end{aligned}$$

where $c_n := \alpha_n \tau(\delta) + \beta_n \delta \to 0$. Hence we conclude: if C is bounded, then $(TAN) \subset (ANis)$.

(f) If C is bounded, then $(TAN) = (sTAN)$.

Proposition 8.5. *$(pUC) \subsetneq (pANis) \cap (A\kappa\text{-}SPis)$.*

Proof. Let c_n, (d_n) in (a), (b) of Remark 8.4 and assume $T \in (pUC)$, i.e., $\sup_{x\in C} \|T^n x - p\|$ for some $p \in C$. Then we observe

$$0 \leq c_n \vee d_n \leq \sup_{x,y\in C} \|T^n x - T^n y\| \leq \sup_{x\in C} \|T^n x - p\| + \sup_{y\in C} \|p - T^n y\| \to 0.$$

From constructions of c_n and d_n, (8.8) and (8.9) are immediately obtained. Therefore, $T \in (pANis) \cap (A\kappa\text{-}SPis)$. For strict inclusion, see Example 9.7. □

9. Examples of nonlinear mappings of asymptotically nonexpansive type

As immediate consequences of Proposition 8.5, we introduce examples of mappings $T \in [(UC) \cap (pUC)] \setminus (L) \subsetneq (ANis) = (UC) \cap (pANis)$ by Proposition 8.5.

Example 9.1. ([25]). *Let $C := \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$ and $0 < |k| < 1$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx := \begin{cases} kx \sin \frac{1}{x}, & x \in C \setminus \{0\}; \\ 0, & x = 0. \end{cases}$$

Then $F(T) = \{0\}$ and $T \in [(UC) \cap (pUC)] \setminus (L)$.

Proof. Obviously, $F(T) = \{0\}$ and $\{T^n x\}$ converges uniformly to 0 on C; hence $T \in (pUC)$ and $T \in (UC)$ is clear. However, $T \notin (L)$; see Example 4.3 of [25] for detailed proof. □

Example 9.2. ([29]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} \alpha, & x \in [0, \alpha]; \\ \frac{\alpha}{\sqrt{1-\alpha}} \sqrt{1-x}, & x \in [\alpha, 1], \end{cases}$$

where $\alpha \in (0,1)$. Then $F(T) \neq \emptyset$ and $T \in [(UC) \cap (pUC)] \setminus (L)$.

Proof. Clearly, $F(T) = \{\alpha\}$. Since $Tx = \frac{\alpha}{\sqrt{1-\alpha}}\sqrt{1-x} \leq \alpha \Leftrightarrow x \geq \alpha$, it results that $T^n x = \alpha$ for all $x \in C$, $n \geq 2$; hence $T \in (pUC) \cap (UC)$ as in Example 9.1. However, $T \notin (L)$; see Example 3.9 of [29] for detailed proof. □

Example 9.3. ([19]; Example 1.2 with $k = 1/4$). *Let $X = \mathbb{R}$, $C = [0,1]$ and $0 < k < \sqrt{2} - 1$). For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} k\sqrt{\frac{1}{2} - x} + \frac{1}{\sqrt{2}}, & 0 \leq x \leq 1/2; \\ \sqrt{x}, & 1/2 \leq x \leq 1. \end{cases}$$

Then $F(T) \neq \emptyset$ and $T \in [(UC) \cap (pUC)] \setminus (L)$.

Proof. Obviously, $F(T) = \{1\}$. Since $Tx \in [1/\sqrt{2}, 1]$ and $T^n x = (Tx)^{(1/2)^{n-1}}$ for all $x \in C$, it follows that $\{T^n x\}$ converges uniformly to 1 on C; hence $T \in (pUC)$. It is clear that $T \in (U) = (UC)$. However, $T \notin (L)$; see Example 1.2 of [19] for more details. □

Now consider an example of *discontinuous* mapping $T \in (pUC)$ due to Sahu et al. [42].

Example 9.4. ([42]; Example 1.6.) *Let $X = \mathbb{R}$, $C = [0,1]$, and $k \in (0,1)$. Define*

$$Tx = \begin{cases} kx, & x \in [0, 1/2]; \\ 0, & x \in (1/2, 1]. \end{cases}$$

Then $F(T) = \{0\}$ and $T \in (pUC) \setminus (C)$.

Proof. Since $\sup_{x \in C} |T^n x| = k^n \to 0$, $\{T^n x\}$ converges uniformly to 0 on C; hence $T \in (pUC)$. Obviously, T is not continuous at 1/2. □

Here consider an interesting example of a mapping $T \in [(pUC) \cap (UL)] \setminus (A\kappa\text{-}SP)$ due to Hu and Gai [19].

Example 9.5. ([19]; Example 1.3 with $k = 4$). *For any $k > 1$, let $\{a_n\}$ be a sequence of positive numbers such that $a_1 \in (0,1]$, $a_n \downarrow 0$, and $\prod_{n=1}^{\infty}(1 + a_n) = k$ (for an example, consider $a_n = \frac{1}{n^2}$ by (c) of Remark 4.2). Set*

$$b_n := \frac{1}{2^{n+1}(1 + a_n)} < 1, \quad \forall n \geq 1.$$

Let $T : C \to C$ be defined by

$$Tx = \begin{cases} (1 + a_1)x + 1/2, & x \in [0, b_1]; \\ 1/2 + 1/4, & x \in [b_1, 1/2] \end{cases}$$

and

$$Tx = \begin{cases} (1 + a_n)\left(x - \sum_{i=1}^{n-1} \frac{1}{2^i}\right) + \sum_{i=1}^{n} \frac{1}{2^i}, & x \in [\sum_{i=1}^{n-1} \frac{1}{2^i}, \sum_{i=1}^{n-1} \frac{1}{2^i} + b_n]; \\ \sum_{i=1}^{n+1} \frac{1}{2^i}, & x \in [\sum_{i=1}^{n-1} \frac{1}{2^i} + b_n, \sum_{i=1}^{n} \frac{1}{2^i}], \quad n \geq 2 \end{cases}$$

and $T1 = 1$. Then $F(T) = \{1\}$, $T \in (pUC) \cap (UL) \subsetneq (ANis) \subsetneq (A\kappa\text{-}SPis)$, but $T \notin (A\kappa\text{-}SP)$.

Proof. Noticing that, for each $x \in C$, there exists $j := j(x) \in \mathbb{N}$ such that

$$1 \leftarrow \sum_{i=1}^{n+j} 2^i \leq T^n x \leq \sum_{i=1}^{n+j+1} 2^i \to 1,$$

it follows that $\{T^n x\}$ converges uniformly to 1 on C; hence $T \in (pUC)$. Now we claim: $|T^n x - T^n y| \leq k|x-y|$ for all $x, y \in C$. Indeed, for given $n \geq 1$, firstly, consider the case of $x, y \in [0, \bar{x}]$, where $\bar{x} := \prod_{i=1}^n b_i < b_k$ for $1 \leq k \leq n$. Since

$$T^k \bar{x} = \prod_{i=1}^k (1+a_i)\bar{x} + \sum_{i=1}^k \frac{1}{2^i} < \sum_{i=1}^k \frac{1}{2^i} + b_{k+1}$$

$$\Leftrightarrow \quad \prod_{i=1;\, i\neq k+2}^n 2^{i+1} \prod_{i=k+2}^n (1+a_i) > 1$$

for all $k = 1, 2, \ldots, n-1$, it follows that

$$T^n \bar{x} = \prod_{i=1}^n (1+a_i)\bar{x} + \sum_{i=1}^n \frac{1}{2^i};$$

hence,

$$|T^n x - T^n y| = \prod_{i=1}^n (1+a_i)|x-y|, \tag{9.1}$$

which implies that $|T^n x - T^n y| \leq k|x-y|$ for all $x, y \in [0, \bar{x}]$ by $\prod_{i=1}^n (1+a_i) \uparrow k$. Next, if we take $x \in [0, \bar{x}]$ and $y \in (\bar{x}, b_1]$ satisfying the following property:

$$Ty = (1+a_1)y + \frac{1}{2} > \frac{1}{2} + b_1 \Leftrightarrow y > \frac{b_1}{1+a_1},$$

since $T^n y = \sum_{i=1}^{n+1} \frac{1}{2^i}$, $\frac{1}{2^{n+1}} = b_n(1+a_n)$, and

$$x \leq \bar{x} < \frac{b_n}{\prod_{i=1}^{n-1}(1+a_i)} = \frac{1}{2^{n+1}\prod_{i=1}^n (1+a_i)} < \frac{1}{(1+a_1)^2 2^2} = \frac{b_1}{1+a_1} < y$$

$$\Leftrightarrow \quad 2^{n+1}\prod_{i=2}^n (1+a_i) > 2^{n+1} \geq 2^3 \geq (1+a_1)2^2, \quad \forall n \geq 2,$$

it follows that

$$|T^n x - T^n y| = \Big|\prod_{i=1}^n (1+a_1)x + \sum_{i=1}^n \frac{1}{2^i} - \sum_{i=1}^{n+1} \frac{1}{2^i}\Big|$$

$$= \Big|\prod_{i=1}^n (1+a_1)x - \frac{1}{2^{n+1}}\Big| = \Big|\prod_{i=1}^n (1+a_1)x - b_n(1+a_n)\Big|$$

$$= \prod_{i=1}^n (1+a_1)\Big|x - \frac{b_n}{\prod_{i=1}^{n-1}(1+a_i)}\Big| \leq \prod_{i=1}^n (1+a_1)|x-y|,$$

which implies $|T^n x - T^n y \leq k|x-y|$ for $x \in [0, \bar{x}]$ and such a $y \in (\bar{x}, b_1]$. Since the remaining cases are obvious or similar to two previous cases, $T \in (UL)$ with its Lipschitz constant $L_T = k > 1$. Finally, we claim that $T \notin (A\kappa\text{-}SP)$; for this end, assume $T \in (A\kappa\text{-}SP)$, i.e., there exist $\kappa < 1$ and a sequence $\{k_n\}$, $k_n \to 1$ satisfying (6.6). Choosing two different $x, y \in (0, \bar{x}]$, (9.1) is obviously satisfied, and if we set $c := \prod_{i=1}^n (1+a_i) > 1$, since $\frac{1+k_n}{2} \to 1$, for $\epsilon = c - 1 > 0$, there exists $K \in \mathbb{N}$ such that

$$\frac{1+k_n}{2} < c, \quad \forall n \geq K. \tag{9.2}$$

Observing

$$|(I - T^n)x - (I - T^n)y|^2 = |(1-c)(x-y)|^2 = (1-c)^2|x-y|^2,$$

we have

$$
\begin{aligned}
|T^n x - T^n y|^2 = c^2|x-y|^2 &\le k_n|x-y|^2 + \kappa|(I-T^n)x - (I-T^n)y|^2 \\
&= [k_n + \kappa(1-c)^2]|x-y|^2 \\
&\Leftrightarrow \kappa \ge \frac{c^2 - k_n}{(1-c)^2} > 1, \quad \forall n \ge K,
\end{aligned}
$$

by (9.2), which contradicts to $\kappa < 1$. This contradiction proves $T \notin (A\kappa\text{-}SP)$. □

Remark 9.6. Since $(AN) = (A0\text{-}SP)$, it follows $T \notin (AN)$; hence $T \in (UL) \setminus (AC)$ asserts strictness of "$(AN) \subsetneq (UL)$" in (2.4). Also, $T \in (A\kappa\text{-}SPis) \setminus (A\kappa\text{-}SP)$ ensures strictness of "$(A\kappa\text{-}SP) \subsetneq (A\kappa\text{-}SPis)$ " in (8.11).

As a slight modification of Example 9.5, we shall give an example of $T \in [(UL) \cap (pANis)] \setminus (pUC)$, defined on a nonempty closed convex and unbounded subset C.

Example 9.7. *Consider $C := [0,\infty) \subset \mathbb{R}$. Let T be defined on $[0,1]$ as in Example 9.5 and define $Tx = x$ on $[1,\infty)$. Then $T \in (pANis) \setminus (pUC)$.*

Proof. Note that $\{T^n x\}$ converges uniformly to 1 on $[0,1]$ by Example 9.5. Now let c_n in (a) of Remark 8.4. Since $k_n \in [1,\infty) \downarrow 1$ and $T = I$ on $[1,\infty)$, it follows that

$$
c_n = \sup_{x,y\in[0,1]} [\|T^n x - T^n y\| - \|x-y\|] \vee 0,
$$

and so $c_n \to 0$ by the similar proof of Proposition 8.5. From constructions of c_n, (8.8) is quickly derived; hence $T \in (pANis)$. However, it is obvious that $T \notin (pUC)$ because $T = I$ on $[1,\infty)$. Since T is k-Lipschitzian on $[0,1]]$ by Example 9.5 and $T = I$ on $[1,\infty)$, it easily follows that $T \in (UL)$ with its Lipschitz constant $L_T = k+1$. □

Finally we recall Example 7.9, which is also a mapping of $T \in (UL) \setminus (TAN)$.

Example 9.8. ([26]). *Let T be defined as in Example 7.9. Then T does not satisfy (8.1); hence $T \notin (ANT)$.*

Proof. For $x(=0) \in C$, since $T^2 = I$, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \sup_{y\in C} \{|T^{2n-1}x - T^{2n-1}y| - |x-y|\} \\
= \;& \sup\{|Ty| - |y| : y \in [-1/k, 1]\} \\
= \;& \sup\{(k-1)|y| : -1/k \le y \le 0\} \\
= \;& (k-1)\frac{1}{k} = 1 - \frac{1}{k} > 0 \Leftrightarrow k > 1,
\end{aligned}
$$

and so T does not satisfy (8.1); hence $T \notin (ANT)$. □

Finally, we raise a question as follows.

Question 9.9. *Find either uniformly Lipschitzian or non-Lipschitzian mappings of $T \in (TAN) \setminus (pANis)$.*

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