数理解析研究所講究録 第2114巻 2019年 82-91

Weak and Strong Convergence Theorems for Two Commutative Nonlinear Mappings in Banach Spases

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Abstract. In this article, we first prove a mean convergence theorem of Baillon's type iteration for finding a common fixed point of commutative 2-generalized nonspreading mappings in a Banach space. Furthermore, we obtain a weak convergence theorem of Mann's type iteration for finding a common fixed point of the mappings in a Banach space. We also prove a strong convergence theorem of Halpern's type iteration for finding a common fixed point of the mappings in a Banach space. Using these results, we get well-known and new weak and strong convergence theorems in a Hilbert space and a Banach space.

2010 *Mathematics Subject Classification*: 47H10
Keywords and phrases: Fixed point, attractive point, generalized hybrid mapping, generalized nonspreading mapping, Mann iteration process, Halpern iteration process, Banach space.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Baillon [4] proved the first mean convergence theorem for nonexpansive mappings in a Hilbert space. In 2010, Kocourek, Takahashi and Yao [13] defined a broad class of nonlinear mappings in a Hilbert space: Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. The class of generalized hybrid mappings covers nonexpansive mappings and hybrid mappings. The mean convergence theorem by Baillon for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao. Furthermore, Takahashi and Takeuchi [29] proved the following mean convergence theorem without convexity in a Hilbert space. Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $A(T)$ the set of *attractive points* [29] of T, i.e., $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C\}$. We know that $A(T)$ is closed and convex.

Theorem 1.1. *Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Assume that $\{T^n z\}$ for some $z \in C$ is bounded and define $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ for all $x \in C$ and $n \in \mathbb{N}$. Then $\{S_n x\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)} T^n x$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.*

Maruyama, Takahashi and Yao [23] also defined a more broad class of nonlinear mappings called 2-generalized hybrid which covers generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H and let T be a mapping of C into H. A mapping $T : C \to H$ is 2-*generalized hybrid* [23] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1\|T^2x - Ty\|^2 &+ \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ &\leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned} \tag{1.2}$$

for all $x, y \in C$.

Recently, Hojo, Takahashi and Takahashi [6] proved an attractive and mean convergence theorems without convexity for commutative 2-generalized hybrid mappings in a Hilbert space. This result generalizes Takahashi and Takeuchi's theorem [29] and Kohsaka's theorem [15] which is a mean convergence theorem for commutative λ-hybrid mappings in a Hilbert space.

On the other hand, in 1953, Mann [22] introduced the following iteration process. Let C be a nonempty, closed and convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations of Mann iterative process for finding fixed points of nonexpansive mappings. Iin 1967, Halpern [5] gave an iteration process as follows: Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations of Halpern iterative process for finding fixed points of nonexpansive mappings.

We also know the concept of 2-generalized nonspreading mappings which was defined in a Banach space by Takahashi, Wong and Yao [31] and this class covers 2-generalized hybrid mappings in a Hilbert space. Furthermore, the concept of attractive points was defined in a Banach space by Lin and Takahashi [21]: Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Then we denote by $A(T)$ the set of attractive points of T, i.e., $A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C\}$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$ and J is the duality mapping on E.

In this article, we first prove a mean convergence theorem of Baillon's type iteration for finding a common fixed point of commutative 2-generalized nonspreading mappings in a Banach space. Furthermore, we obtain a weak convergence theorem of Mann's type iteration for finding a common fixed point of the mappings in a Banach space. We also prove a strong convergence theorem of Halpern's type iteration for finding a common fixed point of the mappings in a Banach space. Using these results, we get well-known and new weak and strong convergence theorems in a Hilbert space and a Banach space.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is *quasi-nonexpansive* if $F(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see [11]. Let E be a Banach space. The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. Thus J^{-1} is also a single-valued bijection and it is the duality mapping from E^* into E. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is uniformly norm to norm continuous on each bounded subset of E. For more details, see [25, 26].

Let E be a smooth Banach space. The function ϕ: $E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{2.2}$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [12]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.3}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \le \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \ge 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.4}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then we have

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.5}$$

The following lemma which was by Kamimura and Takahashi [12] is well-known.

Lemma 2.1 ([12]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following lemmas are in Xu [34] and Kamimura and Takahashi [12].

Lemma 2.2 ([34]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \le \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \le \lambda \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.3 ([12]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \le \phi(x, y)$ for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \le r\}$.*

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $A(T)$ the set of *attractive points* of T, i.e., $A(T) = \{z \in E : \phi(z, Tx) \le \phi(z, x),\ \ \forall x \in C\}$; see [21].

Lemma 2.4 ([21]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $A(T)$ is a closed and convex subset of E.*

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called *generalized nonexpansive* [7] if $F(T) \neq \emptyset$ and $\phi(Tx, y) \le \phi(x, y)$ for all $x \in C$ and $y \in F(T)$; see also [33]. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \ge 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [7] for more details. The following results are in Ibaraki and Takahashi [7].

Lemma 2.5 ([7]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.6 ([7]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \le 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \le \phi(x, z)$.

In 2007, Kohsaka and Takahashi [17] proved the following results:

Lemma 2.7 ([17]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) C *is a sunny generalized nonexpansive retract of* E;
(b) C *is a generalized nonexpansive retract of* E;
(c) JC *is closed and convex.*

Lemma 2.8 ([17]). *Let* E *be a smooth, strictly convex and reflexive Banach space and let* C *be a nonempty closed sunny generalized nonexpansive retract of* E. *Let* R *be the sunny generalized nonexpansive retraction from* E *onto* C *and let* $(x, z) \in E \times C$. *Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Ibaraki and Takahashi [10] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.9 ([10]). *Let* E *be a reflexive, strictly convex and smooth Banach space and let* T *be a generalized nonexpansive mapping from* E *into itself. Then* $F(T)$ *is closed and* $JF(T)$ *is closed and convex.*

The following theorem is proved by using Lemmas 2.7 and 2.9.

Lemma 2.10 ([10]). *Let* E *be a reflexive, strictly convex and smooth Banach space and let* T *be a generalized nonexpansive mapping from* E *into itself. Then* $F(T)$ *is a sunny generalized nonexpansive retract of* E.

Using Lemma 2.7, we also have the following result.

Lemma 2.11 ([28]). *Let* E *be a smooth, strictly convex and reflexive Banach space and let* $\{C_i : i \in I\}$ *be a family of sunny generalized nonexpansive retracts of* E *such that* $\cap_{i \in I} C_i$ *is nonempty. Then* $\cap_{i \in I} C_i$ *is a sunny generalized nonexpansive retract of* E.

To prove one of our main results, we need the following lemma by Aoyama, Kimura, Takahashi and Toyoda [3].

Lemma 2.12 ([3]). *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers, let* $\{\alpha_n\}$ *be a sequence of* $[0, 1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *let* $\{\beta_n\}$ *be a sequence of nonnegative real numbers with* $\sum_{n=1}^{\infty} \beta_n < \infty$, *and let* $\{\gamma_n\}$ *be a sequence of real numbers with* $\limsup_{n \to \infty} \gamma_n \le 0$. *Suppose that* $s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$ *for all* $n = 1, 2, \ldots$. *Then* $\lim_{n \to \infty} s_n = 0$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $S : C \to C$ is called *2-generalized nonspreading* [31] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 \phi(S^2x, Sy) + \alpha_2 \phi(Sx, Sy) + (1 - \alpha_1 - \alpha_2)\phi(x, Sy) \\ &\quad + \gamma_1 \{\phi(Sy, S^2x) - \phi(Sy, x)\} + \gamma_2 \{\phi(Sy, Sx) - \phi(Sy, x)\} \\ &\le \beta_1 \phi(S^2x, y) + \beta_2 \phi(Sx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\ &\quad + \delta_1 \{\phi(y, S^2x) - \phi(y, x)\} + \delta_2 \{\phi(y, Sx) - \phi(y, x)\} \end{aligned} \tag{2.6}$$

for all $x, y \in C$; see also [32]. Such a mapping is called $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generalized nonspreading. We know that a $(0, \alpha_2, 0, \beta_2, 0, \gamma_2, 0, \delta_2)$-generalized nonspreading mapping is generalized nonspreading in the sense of [14]. We also know that a $(0, 1, 0, 1, 0, 1, 0, 0)$-generalized nonspreading mapping is nonspreading in the sense of [19]; see also [18, 27].

3 Weak Convergence Theorems

In this section, we prove a mean convergence theorem of Baillon's type iteration and a weak convergence theorem of Mann's type iteration for finding an attractive point of commutative 2-generalized nonspreading mappings in a Banach space.

Lemma 3.1. *Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let S and T be commutative 2-generalized nonspreading mappings of C into itself. Let $\{x_n\}$ be a bounded sequence of C. Define*

$$S_n x_n = \frac{1}{(1+n)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n$$

for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\|S_n x_n - x_n\| \to 0$. Then every weak cluster point of $\{x_n\}$ is a point of $A(S) \cap A(T)$. Additionally, if C is closed and convex, then every weak cluster point of $\{x_n\}$ is a point of $F(S) \cap F(T)$.

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $B(T)$ the set of *skew-attractive points* of T, i.e., $B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z), \ \forall x \in C\}$. The following result is proved by Lin and Takahashi [21].

Lemma 3.2 ([21]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $B(T)$ is closed and $JB(T)$ is closed and convex.*

We prove a mean convergence theorem of Baillon's type iteration in a Banach space.

Theorem 3.3 ([30]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let $S, T : C \to C$ be commutative 2-generalized nonspreading mappings such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$ is bounded, $A(S) = B(S)$ and $A(T) = B(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l) \in D} RS^k T^l x$.

Using Theorem 3.3, we obtain the following theorems.

Theorem 3.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $S, T : E \to E$ be commutative $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$ and $(\alpha_1', \alpha_2', \beta_1', \beta_2', \gamma_1', \gamma_2', \delta_1', \delta_2')$-generalized nonspreading mappings such that $\alpha_1 - \beta_1 = 0$, $\gamma_1 \leq \delta_1$, $\gamma_2 \leq \delta_2$, $\alpha_2 > \beta_2$ and $\alpha_1' - \beta_1' = 0$, $\gamma_1' \leq \delta_1'$, $\gamma_2' \leq \delta_2'$, $\alpha_2' > \beta_2'$, respectively. Assume that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$ is bounded. Let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Then, for any $x \in E$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $F(S) \cap F(T)$, where $q = \lim_{(k,l) \in D} RS^k T^l x$.

Theorem 3.5 ([6]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ for some $z \in C$ is bounded. Let P be the metric projection of H onto $A(S)\cap A(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^l x$$

converges weakly to an element q of $A(S)\cap A(T)$, where $q = \lim_{(k,l)\in D} PS^kT^lx$. In particular, if C is closed and convex, $\{S_nx\}$ converges weakly to an element q of $F(S)\cap F(T)$.

Using Lemma 3.1 and the technique developed by [9], we can prove the following weak convergence theorem.

Theorem 3.6 ([2]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty and convex subset of E. Let S and T be commutative 2-generalized nonspreading mappings of C into itself such that $A(S)\cap A(T) \neq \emptyset$, $A(S) = B(S)$ and $A(T) = B(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S)\cap B(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^l x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in A(S)\cap A(T)$, where $z = \lim_{n\to\infty} Rx_n$. Additionally, if C is closed, then $\{x_n\}$ converges weakly to a point of $F(S)\cap F(T)$.

Using Theorem 3.6, we can prove the following weak convergence theorem.

Theorem 3.7. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $S,T : E \to E$ be commutative $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2)$ and $(\alpha_1',\alpha_2',\beta_1',\beta_2',\gamma_1',\gamma_2',\delta_1',\delta_2')$-generalized nonspreading mappings such that $\alpha_1-\beta_1 = 0$, $\gamma_1 \le \delta_1$, $\gamma_2 \le \delta_2$, $\alpha_2 > \beta_2$ and $\alpha_1'-\beta_1' = 0$, $\gamma_1' \le \delta_1'$, $\gamma_2' \le \delta_2'$, $\alpha_2' > \beta_2'$, respectively. Assume that $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ for some $z \in E$ is bounded. Let R be the sunny generalized nonexpansive retraction of E onto $F(S)\cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^l x_n x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S)\cap F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Using Theorem 3.6, we obtain the following result in a Hilbert space.

Theorem 3.8. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S,T : C \to C$ be commutative 2-generalized hybrid mappings such that $\{S^kT^lz : k,l \in \mathbb{N}\cup\{0\}\}$ for some $z \in C$ is bounded. Let P be the mertic projection of H onto $F(S)\cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^l x_n x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S)\cap F(T)$, where $z = \lim_{n\to\infty} Px_n$.

Remark We do not know whether a weak convergence theorem of Mann's type iteration for nonspreading mappings in a Banach space holds or not.

4 Strong Convergence Theorems

Let E be a smooth, strictly convex and reflexive Banach space. Ibaraki and Takahashi [8] proved the following lemma.

Lemma 4.1 ([8]). *Let E be a smooth, strictly convex and reflexive Banach space and define $V(x,x^*)=\|x\|^2-2\langle x,x^*\rangle+\|x^*\|^2$ for all $x\in E$ and $x^*\in E^*$. Then*

$$V(x,x^*)+2\langle y,Jx-x^*\rangle\leq V(x+y,x^*)$$

for all $x,y\in E$ and $x^*\in E^*$.

In this section, using the idea of mean convergence by Shimizu and Takahashi [24] and Kurokawa and Takahashi [20], we prove the following strong convergence theorem for 2-generalized nonspreading mappings in a Banach space.

Theorem 4.2 ([2]). *Let E be a smooth and uniformly convex Banach space such that the duality mapping J is weakly sequentially continuous. Let C be a nonempty and convex subset of E. Let S and T be commutative 2-generalized nonspreading mappings of C into itself such that $A(S)\cap A(T)\neq\emptyset$, $A(S)=B(S)$ and $A(T)=B(T)$. Let $u\in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1=x\in C$ and*

$$x_{n+1}=\alpha_n u+(1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n}S^kT^lx_n,\quad \forall n\in\mathbb{N},$$

where $0\leq\alpha_n\leq 1$, $\alpha_n\to 0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$. Then $\{x_n\}$ converges strongly to Ru, where R is a sunny generalized nonexpansive retraction of E onto $B(S)\cap B(T)$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a point of $F(S)\cap F(T)$.

Remark We know that the duality mappings J on $l^p, 1<p<\infty$ and smooth finite dimensional Banach spaces are weakly sequentially continuous. However, we do not know whether Theorem 4.2 holds or not without assuming that J is weakly sequentially continuous.

As in the proofs of Theorems 3.7 and 3.8, we can obtain the following strong convergence theorems from Theorem 4.2.

Theorem 4.3. *Let E be a smooth and uniformly convex Banach space such that the duality mapping J is weakly sequentially continuous. Let $S,T:E\to E$ be commutative $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2)$ and $(\alpha_1',\alpha_2',\beta_1',\beta_2',\gamma_1',\gamma_2',\delta_1',\delta_2')$-generalized nonspreading mappings such that $\alpha_1-\beta_1=0$, $\gamma_1\leq\delta_1$, $\gamma_2\leq\delta_2$, $\alpha_2>\beta_2$ and $\alpha_1'-\beta_1'=0$, $\gamma_1'\leq\delta_1'$, $\gamma_2'\leq\delta_2'$, $\alpha_2'>\beta_2'$, respectively. Assume that $\{S^kT^lz:k,l\in\mathbb{N}\cup\{0\}\}$ for some $z\in C$ is bounded. Let R be the sunny generalized nonexpansive retraction of E onto $F(S)\cap F(T)$. Let $u\in E$ and define a sequence $\{x_n\}$ in E as follows: $x_1=x\in E$ and*

$$x_{n+1}=\alpha_n u+(1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n}S^kT^lx_n,\quad \forall n\in\mathbb{N},$$

where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to* Ru, *where* R *is a sunny generalized nonexpansive retraction of* E *onto* $F(S) \cap F(T)$.

Theorem 4.4. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* S, T *be commutative 2-generalized hybrid mappings of* C *into itself such that* $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ *for some* $z \in C$ *is bounded. Let* $u \in C$ *and define a sequence* $\{x_n\}$ *in* C *as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{(1+n)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n$$

for all $n \in \mathbb{N}$, *where* $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to* Pu, *where* P *is the metric projection of* H *onto* $F(S) \cap F(T)$.

Acknowledgements. The author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

References

[1] Y. I. Alber, *Metric and generalized projections in Banach spaces: Properties and applications*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 15–50.

[2] S. M. Alsulami, A. Latif and W. Takahashi, *Weak and strong convergence theorems for commutative 2-generalized nonspreading mappings in Banach spaces*, to appear.

[3] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[4] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert,* C.R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511-1514.

[5] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957-961.

[6] M. Hojo, S. Takahashi and W. Takahashi, *Attractive point and ergodic theorems for two nonlinear mappings in Hilbert spaces*, Linear Nonlinear Anal. **3** (2017), 275–286.

[7] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[8] T. Ibaraki and W. Takahashi, *Weak and strong convergence theorems for new resolvents of maximal monotone operators in Banach spaces*, Adv. Math. Econ. **10** (2007), 51–64.

[9] T. Ibaraki and W. Takahashi, *Weak convergence theorem for new nonexpansive mappings in Banach spaces and its applications*, Taiwanese J. Math. **11** (2007), 929–944.

[10] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, in Nonlinear Analysis and Optimization I: Nonlinear Analysis, Contemporary Mathematics, Vol. 513, Amer. Math. Soc., Providence, RI, 2010, pp. 169–180.

[11] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[12] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach apace*, SIAM J. Optim. **13** (2002), 938–945.

[13] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[14] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.

[15] F. Kohsaka, *Existence and approximation of common fixed points of two hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **16** (2015), 2193-2205.

[16] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in Banach spaces*, Abstr. Appl. Anal. **2004** (2004), 37-47.

[17] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197-209.

[18] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[19] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.

[20] Y. Kurokawa and W. Takahashi, *Weak and strong convergence theorems for nonlspreading mappings in Hilbert spaces*, Nonlinear Anal. **73** (2010), 1562–1568.

[21] L.-J. Lin and W. Takahashi, *Attractive point theorems for generalized nonspreading mappings in Banach spaces*, J. Convex Anal. **20** (2013), 265–284.

[22] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[23] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185–197.

[24] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71–83.

[25] W. Takahashi, *Nonlinear Functional Analysis,* Yokohoma Publishers, Yokohoma, 2000.

[26] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.

[27] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[28] W. Takahashi, *Weak convergence theorems for two generalized nonspreading mappings in Banach spaces*, J. Nonlinear Convex Anal. **18** (2017), 1207–1223.

[29] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.

[30] W. Takahashi, C.-F Wen and J.-C. Yao, *Attractive and mean convergence theorems for two commutative nonlinear mappings in Banach spaces*, Dynam. Systems Appl. **26** (2017) 327–346.

[31] W. Takahashi, N.-C Wong and J.-C. Yao, *Fixed point theorems for three new nonlinear mappings in Banach spaces*, J. Nonlinear Convex Anal. **13** (2012), 363–381.

[32] W. Takahashi, N.-C Wong and J.-C. Yao, *Fixed point theorems and convergence theorems for generalized nonspreading mappings in Banach spaces*, J. Fixed Point Theory Appl. **11** (2012), 159–183.

[33] W. Takahashi and J.-C. Yao, *Weak and strong convergence theorems for positively homogeneous nonexpansive mappings in Banach spaces*, Taiwanese J. Math. **15** (2011), 961–980.

[34] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1981), 1127–1138.