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AN ITERATIVE SEQUENCE FOR A FINITE NUMBER OF METRIC PROJECTIONS ON A COMPLETE GEODESIC SPACE

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ABSTRACT. In this paper, we prove convergence of an iterative sequence to a common point of a finite number of closed convex subsets of a complete geodesic space with curvature bounded above by one.

1. INTRODUCTION AND PRELIMINARIES

Let X be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic if c satisfies

$$c(0) = x, c(l) = y, \text{ and } d(c(u), c(v)) = |u - v|$$

for every $u, v \in [0, l]$. An image $[x, y]$ of c is called a geodesic segment joining x and y. If a geodesic segment exists for every $x, y \in X$, then we call X a geodesic space.

We consider to find a common point of a finite number of closed convex subsets by using a sequence generated by a finite number of metric projections on a complete geodesic space. Focusing on two properties of a sequence of mappings, we can handle the sequence of mapping more effectively; see [1]. We know that Halpern's iterative method [3] is an efficient method to find a fixed point of a mapping. We also know that the Halpern iteration method can be applied with two metric projections on a complete CAT(1) space [6]. In this paper, we propose a sequence approximating a common point of a finite number of closed convex sets based on this method.

Let X be a geodesic space. For a triangle $\triangle(x, y, z) \subset X$ such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$, let a comparison triangle $\triangle(\overline{x}, \overline{y}, \overline{z})$ in two-dimensional unit sphere $\mathbb{S}^2$ be such that each corresponding edge has the same length as that of the original triangle. X is called a CAT(1) space if for every $p, q \in \triangle(x, y, z)$ and their corresponding points $\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})$ satisfy that

$$d(p, q) \leq d_{\mathbb{S}^2}(\overline{p}, \overline{q}),$$

where $d_{\mathbb{S}^2}$ is the spherical metric on $\mathbb{S}^2$.

Let X be a CAT(1) space and let T be a mapping from X to X such that the set $F(T) = \{z \in X : z = Tz\}$ of fixed points of T is not empty. If $d(Tx, p) \leq d(x, p)$ for every $x \in X$ and $p \in F(T)$, then we call T a quasinonexpansive mapping. Let T be a quasinonexpansive mapping, and suppose that for every $p \in F(T)$ and $\{x_n\} \subset X$ with $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, p)/\cos d(Tx_n, p)) = 1$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Such a mapping T is called a strongly quasinonexpansive mapping. We also define a strongly quasinonexpansive sequence. $\{T_n\}$ is said to be a strongly quasinonexpansive sequence if it is quasinonexpansive, and suppose that for every $p \in \bigcap_{n=1}^{\infty} F(T_n)$ and $\{x_n\} \subset X$ with $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and

$\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(T_nx_n,p)) = 1$, it follows that $\lim_{n\to\infty} d(x_n,T_nx_n) = 0$; see [1].

Let X be a metric space. An element $z \in X$ is said to be an asymptotic center of $\{x_n\} \subset X$ if

$$\limsup_{n\to\infty} d(x_n,z) = \inf_{x\in X} \limsup_{n\to\infty} d(x,x)$$

Moreover, $\{x_n\}$ Δ-converges to a Δ-limit z if z is the unique asymptotic center of any subsequences of $\{x_n\}$.

Let X be a CAT(1) space and let T be a mapping from X to X such that $F(T) \neq \emptyset$. Suppose that for every $\{x_n\} \subset X$ with $\{x_n\}$ Δ-converges to z and $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, it follows that $z \in F(T)$. Such a mapping T is called a Δ-demiclosed mapping. We also define a Δ-demiclosed sequence. $\{T_n\}$ is said to be a Δ-demiclosed sequence if for every $\{x_n\} \subset X$ with $\{x_n\}$ Δ-converges to z and $\lim_{n\to\infty} d(x_n,T_nx_n) = 0$, it follows that $z \in \bigcap_{n=1}^{\infty} F(T_n)$; see [1].

Let X be a CAT(1) space. For every $x,y \in X$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$, if $z \in [x,y]$ satisfies that $d(y,z) = \alpha d(x,y)$ and $d(x,z) = (1-\alpha)d(x,y)$, then we denote z by $z = \alpha x \oplus (1-\alpha)y$. A subset $C \subset X$ is called π-convex if $\alpha x \oplus (1-\alpha)y \in C$ for every $x,y \in C$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$.

Let X be a CAT(1) space. For every $x,y,z \in X$ with $d(x,y)+d(y,z)+d(z,x) < 2\pi$ and $\alpha \in [0,1]$, the following inequality holds [5]:

$$\cos d(x,w)\sin d(y,z) \geq \cos d(x,y)\sin(\alpha d(y,z)) + \cos d(x,z)\sin((1-\alpha)d(y,z))$$

where $w = \alpha y \oplus (1-\alpha)z$.

Let X be a complete CAT(1) space and let $C \subset X$ be a nonempty closed π-convex subset such that $d(x,C) = \inf_{y\in C} d(x,y) < \pi/2$ for every $x \in X$. Then for every $x \in X$, there exists a unique point $x_0 \in C$ satisfying

$$d(x,x_0) = \inf_{y\in C} d(x,y).$$

We define the metric projection P_C from X onto C by $P_Cx = x_0$. We know that the metric projection P_C is a strongly quasinonexpansive and Δ-demiclosed mapping such that $F(P_C) = C$ [2, 6]. These properties are important for our results. The following lemmas are also important.

Lemma 1.1. (Kimura-Satô [6]) *Let X be a CAT(1) space such that $d(v,v') < \pi$ for every $v,v' \in X$. Let $\alpha \in [0,1]$ and $u,y,z \in X$. Then*

$$\begin{aligned}1-&\cos d(\alpha u \oplus (1-\alpha)y, z)\\ &\leq (1-\beta)(1-\cos d(y,z)) + \beta\left(1 - \frac{\cos d(u,z)}{\sin d(u,y)\tan(\frac{\alpha}{2}d(u,y)) + \cos d(u,y)}\right),\end{aligned}$$

where

$$\beta = \begin{cases} 1 - \dfrac{\sin((1-\alpha)d(u,y))}{\sin d(u,y)} & (u \neq y),\\ \alpha & (u = y).\end{cases}$$

Lemma 1.2. (Saejung-Yotkaew [7]) *Let $\{s_n\}$ and $\{t_n\}$ be sequences of real numbers such that $s_n \geq 0$ for every $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence in $]0,1[$ such that $\sum_{n=0}^{\infty} \beta_n = \infty$. Suppose that $s_{n+1} \leq (1-\beta_n)s_n + \beta_n t_n$ for every $n \in \mathbb{N}$. If $\limsup_{k\to\infty} t_{n_k} \leq 0$ for every subsequence $\{n_k\}$ of $\mathbb{N}$ satisfying $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0$, then $\lim_{n\to\infty} s_n = 0$.*

Lemma 1.3. (He-Fang-López-Li [4]) *Let X be a complete CAT(1) space and $p \in X$. If a sequence $\{x_n\}$ in X satisfies that $\limsup_{n\to\infty} d(x_n, p) < \pi/2$ and that $\{x_n\}$ is Δ-convergent to $x \in X$, then $d(x,p) \leq \liminf_{n\to\infty} d(x_n,p)$.*

2. The main result

In this section, we propose an iterative scheme converging to a common point of a finite number of closed convex subsets with nonempty intersection. To prove its convergence property, we prepare four lemmas with their corollaries.

Lemma 2.1. *Let X be a CAT(1) space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. For a given real number a in $]0, \frac{1}{2}]$, let $\sigma^l \in [a, 1-a]$ for every $l = 0, 1, \ldots, N-1$. For given points $y, y^k \in X$ for every $k = 0, 1, \ldots, N$, define $w^l \in X$ by $w^N = y^N$ and $w^l = \sigma^l y^l \oplus (1-\sigma^l) w^{l+1}$ for every $l = 0, 1, \ldots, N-1$. Then $\cos d(w^0, y) \cos(a d(y^0, w^1)) \geq \min\{\cos d(y^0, y), \cos d(w^1, y)\}$.*

Proof. If $y^0 = w^1$, it is obvious. Otherwise, we have

$$\begin{aligned}
&\cos d(w^0, y) \sin d(y^0, w^1) \\
&\qquad \geq \cos d(y^0, y) \sin(\sigma^0 d(y^0, w^1)) + \cos d(w^1, y) \sin((1-\sigma^0) d(y^0, w^1)) \\
&\qquad \geq \min\{\cos d(y^0, y), \cos d(w^1, y)\}(\sin(\sigma^0 d(y^0, w^1)) + \sin((1-\sigma^0) d(y^0, w^1))) \\
&\qquad = 2\min\{\cos d(y^0, y), \cos d(w^1, y)\} \sin\frac{d(y^0, w^1)}{2} \cos\frac{(2\sigma^0 - 1) d(y^0, w^1)}{2}.
\end{aligned}$$

Dividing above by $2\sin(d(y^0, w^1)/2)$, we have

$$\begin{aligned}
&\cos d(w^0, y) \cos\frac{d(y^0, w^1)}{2} \\
&\qquad \geq \min\{\cos d(y^0, y), \cos d(w^1, y)\} \cos\frac{(2\sigma^0 - 1) d(y^0, w^1)}{2} \\
&\qquad \geq \min\{\cos d(y^0, y), \cos d(w^1, y)\} \cos\frac{(1-2a) d(y^0, w^1)}{2}.
\end{aligned}$$

Moreover, dividing above by $\cos((1-2a) d(y^0, w^1)/2)$, we have

$$\begin{aligned}
&\min\{\cos d(y^0, y), \cos d(w^1, y)\} \\
&\leq \cos d(w^0, y) \frac{\cos\frac{(1-2a)d(y^0,w^1)}{2} \cos(a d(y^0, w^1)) - \sin\frac{(1-2a)d(y^0,w^1)}{2} \sin(a d(y^0, w^1))}{\cos\frac{(1-2a)d(y^0,w^1)}{2}} \\
&\leq \cos d(w^0, y) \cos(a d(y^0, w^1)).
\end{aligned}$$

This completes the proof. □

Corollary 2.2. *Let X be a complete CAT(1) space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $C_k \subset X$ be a closed π-convex subset for every $k = 0, 1, \ldots, N$. Let P_{C_k} be a metric projection from X onto C_k for every $k = 0, 1, \ldots, N$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\sigma^l \in [a, 1-a]$ for every $l = 0, 1, \ldots, N-1$. Define $U^l \in X$ by $U^N = P_{C_N}$ and $U^l = \sigma^l P_{C_l} \oplus (1-\sigma^l) U^{l+1}$ for every $l = 0, 1, \ldots, N-1$. Let $x \in X$ and $p \in \bigcap_{k=0}^N C_k$. Then $\cos d(U^0 x, p) \cos(a d(P_{C_0} x, U^1 x)) \geq \cos d(x, p)$.*

Lemma 2.3. *Let X be a CAT(1) space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let T and T' be quasinonexpansive mappings from X to X such that $F(T) \cap F(T') \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\sigma \in [a, 1-a]$. Then $\sigma T \oplus (1-\sigma) T'$ is a quasinonexpansive mapping and $F(\sigma T \oplus (1-\sigma) T') = F(T) \cap F(T')$.*

Proof. At first, we will show $F(\sigma T \oplus (1-\sigma)T') = F(T) \cap F(T')$. It is obvious that $F(\sigma T \oplus (1-\sigma)T') \supset F(T) \cap F(T')$. From Corollary 2.2, for $z \in F(\sigma T \oplus (1-\sigma)T')$ and $p \in F(T) \cap F(T')$,

$$\cos(ad(Tz, T'z)) \geq \frac{\cos d(z,p)}{\cos d(\sigma Tz \oplus (1-\sigma)T'z, p)} = 1.$$

That is $Tz = T'z$, so $z = \sigma Tz \oplus (1-\sigma)T'z = Tz = T'z$. Hence $z \in F(T) \cap F(T')$. Next, we will show $\sigma T \oplus (1-\sigma)T'$ is a quasinonexpansive mapping. By Corollary 2.2, for $x \in X$ and $p \in F(T) \cap F(T')$, we have

$$\cos d(\sigma Tx \oplus (1-\sigma)T'x, p) \geq \cos d(\sigma Tx \oplus (1-\sigma)T'x, p) \cos(ad(Tx, T'x)) \geq \cos d(x,p).$$

It follows that $d(\sigma Tx \oplus (1-\sigma)T'x, p) \leq d(x,p)$, and hence $\sigma T \oplus (1-\sigma)T'$ is a quasinonexpansive mapping. This completes the proof. □

Corollary 2.4. *Let X be a complete CAT(1) space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $C_k \subset X$ be a closed π-convex subset for every $k = 0, 1, \ldots, N$ such that $\bigcap_{k=0}^{N} C_k \neq \emptyset$. Let P_{C_k} be a metric projection from X onto C_k for every $k = 0, 1, \ldots, N$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\sigma^l \in [a, 1-a]$ for every $l = 0, 1, \ldots, N-1$. Define $U^l \in X$ by $U^N = P_{C_N}$ and $U^l = \sigma^l P_{C_l} \oplus (1-\sigma^l)U^{l+1}$ for every $l = 0, 1, \ldots, N-1$. Then $F(U^0) = \bigcap_{k=0}^{N} C_k$.*

Lemma 2.5. *Let X be a CAT(1) space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $\{U_n\}$ be a strongly quasinonexpansive sequence. Let T be a strongly quasinonexpansive mapping from X to X such that $\bigcap_{n=1}^{\infty} F(U_n) \cap F(T) \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\{\sigma_n\} \subset [a, 1-a]$. Then $\{\sigma_n T \oplus (1-\sigma_n)U_n\}$ is a strongly quasinonexpansive sequence.*

Proof. Let $V_n = \sigma_n T \oplus (1-\sigma_n)U_n$ for every $n \in \mathbb{N}$. By Lemma 2.3, V_n is a quasinonexpansive mapping for every $n \in \mathbb{N}$. By Corollary 2.2, for $\{x_n\} \subset X$ and $p \in \bigcap_{n=1}^{\infty} F(V_n)$ such that $\sup_{n \in \mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n\to\infty} \cos d(x_n,p)/\cos d(V_n x_n, p) = 1$, we have

$$\cos d(V_n x_n, p) \cos(ad(Tx_n, U_n x_n)) \geq \cos d(x_n, p)$$

and thus

$$\cos(ad(Tx_n, U_n x_n)) \geq \frac{\cos d(x_n,p)}{\cos d(V_n x_n, p)} \to 1.$$

That is, $\lim_{n\to\infty} d(Tx_n, U_n x_n) = 0$. So we have

$$\lim_{n\to\infty} d(U_n x_n, V_n x_n) = \lim_{n\to\infty} \sigma_n d(Tx_n, U_n x_n) = 0.$$

Since $1 = \lim_{n\to\infty} \cos d(x_n,p)/\cos d(V_n x_n, p) = \lim_{n\to\infty} \cos d(x_n,p)/\cos d(U_n x_n, p)$, we have

$$\lim_{n\to\infty} d(U_n x_n, x_n) = 0.$$

Hence, we obtain

$$d(V_n x_n, x_n) \leq d(V_n x_n, U_n x_n) + d(U_n x_n, x_n) \to 0.$$

This completes the proof. □

Corollary 2.6. *Let X be a complete CAT(1) space such that $d(v,v') < \pi/2$ for every $v,v' \in X$. Let $C_k \subset X$ be a closed π-convex subset for every $k = 0,1,\dots,N$ such that $\bigcap_{k=0}^{N} C_k \neq \emptyset$. Let P_{C_k} be a metric projection from X onto C_k for every $k = 0,1,\dots,N$. For a given real number $a \in \left]0,\frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a,1-a]$ for every $l = 0,1,\dots,N-1$. Define $\{U_n^l\} \subset X$ by $U_n^N = P_{C_N}$ and $U_n^l = \sigma_n^l P_{C_l} \oplus (1-\sigma_n^l)U_n^{l+1}$ for every $l = 0,1,\dots,N-1$. Then $\{U_n^0\}$ is a strongly quasinonexpansive sequence.*

Lemma 2.7. *Let X be a CAT(1) space such that $d(v,v') < \pi/2$ for every $v,v' \in X$. Let $\{U_n\}$ be a Δ-demiclosed sequence. Let T be a Δ-demiclosed mapping from X to X such that $\bigcap_{n=1}^{\infty} F(U_n) \cap F(T) \neq \emptyset$. For given real number a in $\left]0,\frac{1}{2}\right]$, let σ_n be in $[a,1-a]$ for every n in $\mathbb{N}$. Then $\{\sigma_n T \oplus (1-\sigma_n)U_n\}$ is a Δ-demiclosed sequence.*

Proof. Let $V_n = \sigma_n T \oplus (1-\sigma_n)U_n$ for every $n \in \mathbb{N}$. Let $p \in \bigcap_{n=1}^{\infty} F(V_n)$, $\{x_n\} \subset X$, and $z \in X$ such that $\lim_{n\to\infty} d(V_n x_n, x_n) = 0$ and suppose that $\{x_n\}$ is Δ-convergent to z. Then

$$\cos d(V_n x_n, p)\cos(ad(Tx_n, U_n x_n)) \geq \cos d(x_n, p)$$

and thus

$$\begin{aligned} 1 \geq \cos(ad(Tx_n, U_n x_n)) &\geq \frac{\cos d(x_n,p)}{\cos d(V_n x_n, p)} \\ &\geq \frac{\cos(d(x_n, V_n x_n) + d(V_n x_n, p))}{\cos d(V_n x_n, p)} \to 1. \end{aligned}$$

Hence $\lim_{n\to\infty} d(Tx_n, U_n x_n) = 0$. Thus we have

$$\begin{aligned} d(Tx_n, V_n x_n) &= (1-\sigma_n)d(Tx_n, U_n x_n) \\ &\leq (1-a)d(Tx_n, U_n x_n) \to 0. \end{aligned}$$

Since T is a Δ-demiclosed mapping, we have $Tz = z$. Similarly,

$$\begin{aligned} d(U_n x_n, V_n x_n) &= \sigma_n d(U_n x_n, U_n x_n) \\ &\leq (1-a)d(Tx_n, U_n x_n) \to 0. \end{aligned}$$

Since $\{U_n\}$ is a Δ-demiclosed sequence, we have $U_n z = z$. Hence $V_n z = z$. This completes the proof. □

Corollary 2.8. *Let X be a complete CAT(1) space such that $d(v,v') < \pi/2$ for every $v,v' \in X$. Let $C_k \subset X$ be a closed π-convex subset for every $k = 0,1,\dots,N$ such that $\bigcap_{k=0}^{N} C_k \neq \emptyset$. Let P_{C_k} be a metric projection from X onto C_k for every $k = 0,1,\dots,N$. For a given real number $a \in \left]0,\frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a,1-a]$ for every $l = 0,1,\dots,N-1$. Define $\{U_n^l\} \subset X$ by $U_n^N = P_{C_N}$ and $U_n^l = \sigma_n^l P_{C_l} \oplus (1-\sigma_n^l)U_n^{l+1}$ for every $l = 0,1,\dots,N-1$. Then $\{U_n^0\}$ is a Δ-demiclosed sequence.*

Now we shall prove our main result showing the convergence property of the iterative sequence generated by metric projections.

Theorem 2.9. *Let X be a complete CAT(1) space such that $d(v,v') < \pi/2$ for every $v,v' \in X$. Let $C_k \subset X$ be a closed π-convex subset for every $k = 0,1,\dots,N$ such that $\bigcap_{k=0}^{N} C_k \neq \emptyset$. Let P_{C_k} be a metric projection from X onto C_k for every $k = 0,1,\dots,N$. For a given real number $a \in \left]0,\frac{1}{2}\right]$, let $\{\sigma_n^l\} \subset [a,1-a]$ for every $l = 0,1,\dots,N-1$. Define $\{U_n^l\} \subset X$ by $U_n^N = P_{C_N}$ and $U_n^l = \sigma_n^l P_{C_l} \oplus (1-\sigma_n^l)U_n^{l+1}$ for every $l = 0,1,\dots,N-1$. Let $\{\alpha_n\}$ be a real sequence in $]0,1[$ such* *that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *For given points* $u, x_0 \in X$, *let* $\{x_n\}$ *be the sequence in* X *generated by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)U_n^0 x_n$$

for $n \in \mathbb{N}$. *Suppose that one of the following conditions holds:*

(a) $\sup_{v,v'\in X} d(v,v') < \pi/2$;
(b) $d(u, P_{\bigcap_{k=0}^N C_k}u) < \pi/4$ *and* $d(u, P_{\bigcap_{k=0}^N C_k}u) + d(x_0, P_{\bigcap_{k=0}^N C_k}u) < \pi/2$;
(c) $\sum_{n=0}^{\infty}\alpha_n^2 = \infty$.

Then $\{x_n\}$ *converges to* $P_{\bigcap_{k=0}^N C_k}u$.

We employ the technique proposed in [6] for the proof of this theorem. For the sake of completeness, we shall show the whole proof.

Proof. Let $p = P_{\bigcap_{k=0}^N C_k}u$ and let

$$s_n = 1 - \cos d(x_n, p),$$
$$t_n = 1 - \frac{\cos d(u,p)}{\sin d(u, U_n^0x_n)\tan(\frac{\alpha_n}{2}d(u,U_n^0x_n)) + \cos d(u, U_n^0 x_n)},$$
$$\beta_n = \begin{cases} 1 - \dfrac{\sin((1-\alpha_n)d(u,U_n^0x_n))}{\sin d(u,U_n^0x_n)} & (u \neq U_n^0x_n), \\ \alpha_n & (u = U_n^0x_n) \end{cases}$$

for $n \in \mathbb{N}$. Since U_n^0 is a quasinonexpansive mapping, it follows from Lemma 1.1 that

$$s_{n+1} \le (1-\beta_n)(1-\cos d(U_n^0x_n, p)) + \beta_n t_n \le (1-\beta_n)s_n + \beta_n t_n$$

for $n \in \mathbb{N}$. We have

$$\begin{aligned}\cos d(x_{n+1},p) &= \cos d(\alpha_n u \oplus (1-\alpha_n)U_n^0x_n, p) \\ &\ge \alpha_n \cos d(u,p) + (1-\alpha_n)\cos d(U_n^0x_n, p) \\ &\ge \alpha_n \cos d(u,p) + (1-\alpha_n)\cos d(x_n,p) \\ &\ge \min\{\cos d(u,p), \cos d(x_n,p)\}\end{aligned}$$

for $n \in \mathbb{N}$. So we have

$$\cos d(x_n,p) \ge \min\{\cos d(u,p), \cos d(x_0,p)\} = \cos\max\{d(u,p), d(x_0,p)\} > 0$$

for $n \in \mathbb{N}$. Hence $\sup_{n\in\mathbb{N}} d(x_n,p) \le \max\{d(u,p), d(x_0,p)\} < \pi/2$. Next, we will show each of the conditions (a),(b) and (c) implies that $\sum_{n=0}^{\infty}\beta_n = \infty$. For the conditions (a) and (b), let $M = \sup_{n\in\mathbb{N}} d(u, U_n^0x_n)$. Thus we will show $M < \pi/2$. In the case of (a), it is obvious. In the case of (b), since $\sup_{n\in\mathbb{N}} d(x_n,p) \le \max\{d(u,p), d(x_0,p)\}$, we have

$$\begin{aligned}M &\le \sup_{n\in\mathbb{N}}(d(u,p) + d(U_n^0x_n, p)) \\ &\le \sup_{n\in\mathbb{N}}(d(u,p) + d(x_n,p)) \\ &\le \max\{2d(u,p), d(u,p) + d(x_0,p)\} < \pi/2.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, each of the conditions (a) and (b) implies that $\sum_{n=0}^{\infty}\beta_n = \infty$. In the case of (c), we have

$$\beta_n \ge 1 - \sin\frac{(1-\alpha_n)\pi}{2} = 1 - \cos\frac{\alpha_n}{2} \ge \frac{\alpha_n^2\pi^2}{16}$$

for $n \in \mathbb{N}$. Hence the condition (c) also implies that $\sum_{n=0}^{\infty} \beta_n = \infty$. For $\{s_{n_i}\} \subset \{s_n\}$ such that $\liminf_{i\to\infty}(s_{n_i+1} - s_{n_i}) \geq 0$, we have

$$\begin{aligned}
0 &\leq \liminf_{i\to\infty}(s_{n_i+1} - s_{n_i}) \\
&= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(x_{n_i+1}, p)) \\
&\leq \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - (\alpha_{n_i} \cos d(u, p) + (1 - \alpha_{n_i}) \cos d(U_{n_i}^0 x_{n_i}, p))) \\
&= \liminf_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}^0 x_{n_i}, p)) \\
&\leq \limsup_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}^0 x_{n_i}, p)) \leq 0.
\end{aligned}$$

Hence $\lim_{i\to\infty}(\cos d(x_{n_i}, p) - \cos d(U_{n_i}^0 x_{n_i}, p)) = 0$. Since $\sup_{n\in\mathbb{N}} d(U_n^0 x_n, p) < \pi/2$, we have $\lim_{i\to\infty}(\cos d(x_{n_i}, p)/\cos d(U_{n_i}^0 x_{n_i}, p)) = 1$. Since $\{U_{n_i}^0\}$ is strongly quasinonexpansive sequence, it follows that $\lim_{i\to\infty} d(x_{n_i}, U_{n_i}^0 x_{n_i}) = 0$. Let $\{x_{n_j}\} \subset \{x_{n_i}\}$ be a Δ-convergent subsequence such that $\lim_{j\to\infty} d(u, x_{n_j}) = \liminf_{i\to\infty} d(u, x_{n_i})$. Since $\{U_n^0\}$ is a Δ-demiclosed sequence and $\lim_{j\to\infty} d(x_{n_j}, U_{n_j}^0 x_{n_j}) = 0$, the Δ-limit $z \in \{x_{n_j}\}$ belongs to $\bigcap_{k=0}^N C_k$. By Lemma 1.3, we have

$$\liminf_{i\to\infty} d(u, U_{n_i} x_{n_i}) = \liminf_{i\to\infty} d(u, x_{n_i}) = \lim_{j\to\infty} d(u, x_{n_j}) \geq d(u, z) \geq d(u, p).$$

Hence

$$\begin{aligned}
\limsup_{i\to\infty} t_{n_i} &= \limsup_{i\to\infty}\left(1 - \frac{\cos d(u,p)}{\sin d(u, U_{n_i} x_{n_i}) \tan(\frac{\alpha_{n_i}}{2} d(u, U_{n_i} x_{n_i}) + \cos d(u, U_{n_i} x_{n_i})}\right) \\
&= \limsup_{i\to\infty}\left(1 - \frac{\cos d(u,p)}{\cos d(u, U_{n_i} x_{n_i})}\right) \leq 0.
\end{aligned}$$

From Lemma 1.2, we have $\lim_{n\to\infty} s_n = 0$. Therefore $\{x_n\}$ converges to p. This completes the proof. □

References

[1] K. Aoyama and Y. Kimura, *Strong convergence theorems for strongly nonexpansive sequences*, Appl. Math. Comput. **217** (2011), 7537-7545.

[2] R. Espínola and A. Fernández-León, *CAT(k)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl. **353** (2009), 410-427.

[3] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Am. Math. Soc. **73** (1967), 957-961.

[4] JS. He, DH. Fang, G. López and C. Li, *Mann's algorithm for nonexpansive mappings in CAT(κ) spaces*, Nonlinear Anal. **75** (2012), 445-452.

[5] Y. Kimura and K. Satô, *Convergence of subsets of a complete geodesic space with curvature bounded above*, Nonlinear Anal. **75** (2012), 5079-5085.

[6] Y. Kimura and K. Satô, *Halpern iteration for strongly quasinonexpansive mappings on a geodesic space with curvature bounded above by one*, Fixed Point Theory Appl. **2013** (2013), 14pages.

[7] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal. **75** (2012), 742-750.

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