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# Weak Mean Stability in Random Holomorphic Dynamical Systems

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We consider random holomorphic dynamical systems. We introduce the notion "weak mean stability" and show several results of such systems. Also, we show that in many holomorphic families of rational maps, generic systems are weak mean stable. We apply the theory of weak mean stable systems to "random relaxed Newton's methods" to find roots of any polynomial. We find many new phenomena in random holomorphic dynamical systems which cannot hold in deterministic iteration dynamics of a single holomorphic map on the Riemann sphere.

**Definition 1.**

(1) Let $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$ be the Riemann sphere endowed with the spherical distance $d$.

(2) Let $\mathrm{Rat} := \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f \text{ is non-constant and holomorphic}\}$ endowed with the distance $\eta$, where $\eta(f,g) = \sup_{z\in\hat{\mathbb{C}}} d(f(z), g(z))$. Note that $(\mathrm{Rat}, \eta)$ is a complete separable metric space.

(3) For a metric space $Y$, we denote by $\mathfrak{M}_1(Y)$ the space of all Borel probability measures on $Y$.

(4) For a subset $Y$ of Rat, we set

$$\mathfrak{M}_{1,c}(Y) := \{\tau \in \mathfrak{M}_1(Y) \mid \operatorname{supp}\tau \text{ is a compact subset of } Y\}.$$

(5) For a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we set $G_\tau := \{\gamma_n \circ \cdots \circ \gamma_1 \mid n \in \mathbb{N}, \gamma_j \in \operatorname{supp}\tau(\forall j)\}$. Note that this is a semigroup whose product is the composition of maps.

(6) We say that an element $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is weakly mean stable if there exist an $n \in \mathbb{N}$, an $m \in \mathbb{N}$, non-empty open subsets $U_1, \ldots, U_m$ of $\hat{\mathbb{C}}$, a non-empty compact subset $K$ of $\hat{\mathbb{C}}$ with $K \subset \cup_{j=1}^m U_j$, and a constant $c$ with $0 < c < 1$ such that the following (a) (b) (c) hold.

(a) For each $(\gamma_1, \ldots, \gamma_n) \in (\operatorname{supp}\tau)^n$, we have

$$\gamma_n \circ \cdots \circ \gamma_1(\cup_{j=1}^m U_j) \subset K.$$

Moreover, for each $j = 1, \ldots, m$, for all $x, y \in U_j$ and for each $(\gamma_1, \ldots, \gamma_n) \in (\operatorname{supp}\tau)^n$, we have

$$d(\gamma_n \circ \cdots \circ \gamma_1(x), \gamma_n \circ \cdots \circ \gamma_1(y)) \leq cd(x, y).$$

(b) Let $D_\tau := \bigcap_{h \in G_\tau} h^{-1}(\hat{\mathbb{C}} \setminus \cup_{j=1}^m U_j)$. Then $\sharp D_\tau < \infty$.

(c) For each minimal set $L$ of $\tau$ with $L \subset D_\tau$, there exist a $z \in L$ and an $\alpha \in G_\tau$ such that $\alpha(z) = z$ and $|\alpha'(z)| > 1$ (if $z = \infty$, then we consider $(\varphi \circ \alpha \circ \varphi^{-1})'(0)$ instead of $\alpha'(z)$ where $\varphi(z) = 1/z$).
Here, a non-empty compact subset $L$ of $\hat{\mathbb{C}}$ is said to be a minimal set of $\tau$ if for each $z \in L$, $\overline{\cup_{h \in G_\tau}\{h(z)\}} = L$.

(7) For each $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we define $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ as follows.

$$M_\tau^*(\mu)(A) := \int \mu(h^{-1}(A))\, d\tau(h)$$

for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and for each Borel subset $A$ of $\hat{\mathbb{C}}$.

**Theorem 2** ([4]). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be weakly mean stable.*
*Then there exists an $l \in \mathbb{N}$ such that for each $x \in \hat{\mathbb{C}}$, there exists an $(M_\tau^*)^l$-invariant $\mu_x \in \mathfrak{M}_1(\hat{\mathbb{C}})$ such that*

$$(M_\tau^*)^{nl}(\delta_x) \to \mu_x \quad \text{as } n \to \infty$$

*in $\mathfrak{M}_1(\hat{\mathbb{C}})$ with respect to the weak convergence topology.*
*Here, $\delta_x$ denotes the Dirac measure at $x$.*

**Theorem 3** ([4]). *Let $\tau \in \mathfrak{M}_{1,c}(Rat)$ be weakly mean stable. Let*

$$J(G_\tau) := \{z \in \hat{\mathbb{C}} \mid \textit{for any neighborhood } U \textit{ of } z \textit{ in } \hat{\mathbb{C}}, G_\tau \textit{ is not equicontinuous on } U\}.$$

*Suppose we have the following* (1) *and* (2).

(1) $\sharp J(G_\tau) \geq 3$.
(Note: *if there exists an element* $g \in \mathrm{supp}\,\tau$ *with* $\deg(g) \geq 2$, *then* $\sharp J(G_\tau) \geq 3$.)

(2) *For each minimal set $L$ of $\tau$ with $L \subset D_\tau$, where $D_\tau$ is the set coming from Definition 1 (6), we have the following* (a)(b).

   (a) *The Lyapunov exponent $\chi(L, \tau)$ of $(L, \tau)$ is not zero.*

   (b) *If $\chi(L, \tau) > 0$, then for each $z \in L$ and for each $h \in \mathrm{supp}\,\tau$, we have $Dh_z \neq 0$.*

*Then, there exist a subset $\Omega_\tau$ of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ and a constant $c_\tau$ with $c_\tau < 0$ such that the following holds.*

- *For each $z \in \Omega_\tau$, there exists a Borel subset $B_{\tau,z}$ of* $(\mathrm{Rat})^{\mathbb{N}}$ *with* $(\otimes_{n=1}^{\infty} \tau)(B_{\tau,z}) = 1$ *such that for each $(\gamma_1, \gamma_2 \ldots, ) \in B_{\tau,z}$, we have*

$$\limsup_{n \to \infty} \frac{1}{n} \log \|D(\gamma_n \circ \cdots \circ \gamma_1)_z\| \leq c_\tau < 0.$$

**Remark 4.** Statements of Theorems 2 and 3 cannot hold for deterministic dynamics of a single $f \in \mathrm{Rat}$ with $\deg(f) \geq 2$. In fact, in the Julia set $J(f)$ of $f$, we have a chaotic phenomenon. See Mañé's paper (1988)[1] etc.

**Theorem 5** ([4]). *Let $Y$ be one of the following* (1)–(4).

(1) $\{f \in \mathrm{Rat} \mid f$ *is a polynomial with* $\deg(f) \geq 2\}$.

(2) $\{\lambda z(1-z) \in \mathrm{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$.

(3) $\{z - \lambda \frac{f(z)}{f'(z)} \in \mathrm{Rat} \mid \lambda \in \mathbb{C}, |\lambda - 1| < 1\}$ *where $f$ is a polynomial with* $\deg(f) \geq 2$. *Note that this family is related to "random relaxed Newton's methods for $f$" in which we can find roots of any polynomial $f$ more easily than deterministic Newton's method* ([4]).

(4) $\{z+\lambda f(z) \in \mathrm{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$ *where* $f$ *is a polynomial with* $\deg(f) \geq 2$ *such that for each* $z_0 \in \mathbb{C}$ *with* $f(z_0)=0$*, we have* $f'(z_0) \neq 0$.

*Then there exists an open and dense subset* $A$ *of* $\mathfrak{M}_{1,c}(Y)$ *such that each* $\tau \in A$ *is weakly mean stable and satisfies the assumptions of Theorems 2 and 3 (thus the statements of Theorems 2 and 3 hold for* $\tau$*). Here, we endow* $\mathfrak{M}_{1,c}(Y)$ *with the topology such that a sequence* $\{\tau_n\}_{n\in\mathbb{N}}$ *in* $\mathfrak{M}_{1,c}(Y)$ *tends to an element* $\tau \in \mathfrak{M}_{1,c}(Y)$ *if and only if*

(a) *for each bounded continuous function* $\varphi : Y \to \mathbb{R}$*, we have* $\int_Y \varphi d\tau_n \to \int_Y \varphi d\tau$ *as* $n \to \infty$*, and*

(b) $\mathrm{supp}\,\tau_n \to \mathrm{supp}\,\tau$ *as* $n \to \infty$ *with respect to the Hausdorff metric in the space of all non-empty compact subsets of* $Y$.

**Theorem 6** ([4]). (Random relaxed Newton's methods)
*Let* $f$ *be a polynomial with* $\deg(f) \geq 2$. *Let* $1/2 < r < 1$. *Let* $\tau$ *be the normalized Lebesgue measure on*

$$Y_0 = \{z - \lambda \frac{f(z)}{f'(z)} \in \mathrm{Rat} \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq r\} \cong \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq r\}.$$

*Then* $\tau$ *is weakly mean stable and satisfies the assumptions of Theorems 2 and 3.*
*Also, for each* $z_0 \in \mathbb{C} \setminus \{z \in \mathbb{C} \mid f'(z) = 0\}$,
*there exists a Borel subset* $B_{z_0}$ *of* $(Y_0)^{\mathbb{N}}$ *with* $(\otimes_{n=1}^{\infty}\tau)(B_{z_0}) = 1$
*satisfying the following.*

- *For each* $\gamma = (\gamma_1, \gamma_2, \ldots) \in B_{z_0}$,
  *there exists a* $x = x(z_0, \gamma)$ *with* $f(x) = 0$ *such that*

$$\gamma_n \circ \cdots \circ \gamma_1(z_0) \to x \text{ as } n \to \infty \text{ exponentially fast.}$$

**Remark 7.** The statement of Theorem 6 cannot hold for deterministic Newton's method.

**Idea of Proofs of Theorems 2,3.**

(1) Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be weakly mean stable and let $n \in \mathbb{N}$, $\{U_j\}_j$, $D_\tau = \bigcap_{h \in G_\tau} h^{-1}(\hat{\mathbb{C}} \setminus \cup_j U_j)$ be as in the definition of weak mean stability.

(2) In each $U_j$, all maps $\gamma_n \circ \cdots \circ \gamma_1$ ($\forall \gamma_j \in \operatorname{supp} \tau$) are uniformly contracting. Thus there are only finitely many minimal sets of $\tau$ which meet $\cup_j U_j$ and they are "attracting".

(3) For each $y \in \hat{\mathbb{C}}$, let

$$A_{y,1} := \{\gamma = (\gamma_1, \gamma_2, \ldots,) \in (\operatorname{supp} \tau)^{\mathbb{N}} \mid \exists n \in \mathbb{N} \text{ s.t.} \gamma_n \circ \cdots \circ \gamma_1(y) \in \cup_j U_j\}$$

and let $A_{y,2} := (\operatorname{supp} \tau)^{\mathbb{N}} \setminus A_{y,1}$.

For elements in $A_{y,1}$, we have the nice things (see (2)).

Regarding $A_{y,2}$, we show that for $(\otimes_{n=1}^{\infty} \tau)$-a.e. $(\gamma_1, \gamma_2, \ldots,) \in A_{y,2}$, we have $d(\gamma_n \circ \cdots \circ \gamma_1(y), D_\tau) \to 0$ as $n \to \infty$.

**Idea of Proofs of Theorems 5,6.**

(1) We use complex analysis, Montel's theorem (a family of uniformly bounded holomorphic functions on a domain is equicontinuous on the domain), hyperbolic metric.

(2) We classify minimal sets and analyze the bifurcation of minimal sets. etc. By using these, enlarging the support of the original $\tau$ a little bit, we destroy non-attracting minimal sets which do not meet $D_\tau$.

(3) Regarding the proof Theorem 6, by using some technical argument, we destroy any minimal set which contains an attracting periodic cycle of $N_f(z) = z - f(z)/f'(z)$ with period $\geq 2$.

## Summary

(1) We introduce the notion of weak mean stability in i.i.d. random (holomorphic) 1-dimensional dynamical systems.

(2) If a random holomorphic dynamical system on $\hat{\mathbb{C}}$ is weakly mean stable, then for any $x \in \hat{\mathbb{C}}$, the orbit of the Dirac measure at $x$ under the iterations of the dual map of the transition operator converges to a periodic cycle of probability measures.

(3) If a random holomorphic dynamical system on $\hat{\mathbb{C}}$ is weakly mean stable and satisfies some mild assumtions, then for all but countably many $z \in \hat{\mathbb{C}}$, for

a.e. orbit starting with $z$, the Lyapunov exponent is negative. Note that the statements of (2) and (3) cannot hold for deterministic dynamics of a single rational map $f$ with $\deg(f) \geq 2$.

(4) In many holomorphic families of rational maps (including random relaxed Newton's methods family), generic random dynamical systems satisfy the statements of (2) and (3). We can apply this to random relaxed Newton's method to find a root of any polynomial.

**References:**


[1] R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, Dynamical Systems (Valparaiso, 1986) (Lecture Notes in Mathematics vol 1331) (Berlin: Springer) pp 86-117, 1988.

[2] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (2011) 102(1), pp 50–112.

[3] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, Adv. Math., 245 (2013) pp 137–181.

[4] H. Sumi, *Negativity of Lyapunov Exponents and Convergence of Generic Random Polynomial Dynamical Systems and Random Relaxed Newton's Methods*, 61 pages, https://arxiv.org/abs/1608.05230.