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Invariant measure of perturbed graph-directed IFS with degeneration

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1 Introduction

We consider perturbed graph iterated function systems in which some perturbed functions converge to constant functions. In our system, the unperturbed system has several Gibbs measures $\mu_1, \mu_2, \ldots, \mu_m$ associated with the dimensions of the limit sets while the perturbed system has a unique Gibbs measure $\mu(\epsilon, \cdot)$ for each $\epsilon > 0$. We also investigate the case when a limit point of $\mu(\epsilon, \cdot)$ in the sense of the weak topology has the convex combination $\sum_{k=1}^{m} p(k)\mu_k$ for some probability vector $(p(k))_{k=1}^{m}$. Such a system relates to a metastable system or a system with holes (e.g. [3, 4]).

Our interests in this situation is how the coefficient $(p(k))$ is specified when $\mu(\epsilon, \cdot)$ converges to a measure $\mu = \sum_{k=1}^{m} p(k)\mu_k$ weakly. We proved in our previous investigation [10] that if $m = 2$ or 3, then the coefficient $(p(k))$ is expressible by the limit of a sequence composed of the Peron eigenvalues of the sub Ruelle operators of certain suitable perturbed potentials (see Theorem 3.3 and Theorem 3.4). However, there is a difficulty in extending this result to the case $m \geq 4$ [10]. In our recent result [12] (2017), we give another characterization of the coefficient $(p(k))$ using the notion of extended Ruelle operators in all cases $m \geq 2$. In this paper, we summarize our previous results and a recent result concerning perturbed graph IFS with degeneration.

In the next section 2, we give the definition of graph iterated function systems and a formulation of perturbation of this system. We mention in Section 3 our previous results. The main theorem is described in Section 4. In the finial section 5, we shall present two concrete examples.

2 Graph iterated function systems

2.1 Definition

Let $D \geq 1$ be an integer. We consider a set $(G, (J_v), (O_v), (T_e))$ satisfying the following conditions (1)-(4):

(1) $G = (V, E, i, t)$ is a finite directed multigraph which consists of a vertices set V, a directed edges set E and two functions $i, t : E \to V$. For each $e \in E$, $i(e)$ is called the initial vertex of e and $t(e)$ called the terminal vertex of e.

(2) For each $v \in V$, a subset J_v of D-dimensional Euclidean space $\mathbb{R}^D$ is compact and connected such that the interior $\operatorname{int} J_v$ of J_v is not empty, and $\operatorname{int} J_{v'}$ and $\operatorname{int} J_v$ are disjoint for $v' \neq v$.

(3) For each $v \in V$, O_v is an open and connected subset of $\mathbb{R}^D$ such that $J_v \subset O_v$.

(4) For each $e \in E$, a function T_e from $O_{t(e)}$ into $O_{i(e)}$ is a conformal $C^{1+\beta}$-diffeomorphism with $\beta \in (0,1]$ and satisfies $0 < \|T_e'(x)\| < 1$ for $x \in J_{t(e)}$ and $T_e(\operatorname{int} J_{t(e)}) \subset \operatorname{int} J_{i(e)}$ for $e \in E$. Moreover, an open set condition (OSC) is satisfied, namely $T_e \operatorname{int} J_{t(e)} \cap T_{e'} \operatorname{int} J_{t(e')} = \emptyset$ with $e' \neq e$ and $i(e') = i(e)$. Here $\|T_e'(x)\|$ denotes the operator norm of $T_e'(x)$ on $\mathbb{R}^D$.

We call such a set $(G, (J_v), (O_v), (T_e))$ a graph iterated function systems (GIFS for short). Such a system is studied by many authors [2, 5, 6, 7, 9].

A subgraph H of G is said to be strongly connected if for any two vertices v_1, v_2 of H there is a path on H from v_1 to v_2. A subgraph $H = (V_H, E_H)$ of G is called a strongly connected component of G if this is strongly connected and for any strongly connected subgraph $H' = (V_{H'}, E_{H'})$ of G with $E_H \subset E_{H'}$, H' is equal to H. Denoted by $SC(G)$ the set of all strongly connected components of G.

Assume that G is strongly connected. There exists a unique family $\{K_v \subset J_v : v \in V\}$ of nonempty compact subsets such that the set equation

$$K_v = \bigcup_{e \in E : i(e) = v} T_e(K_{t(e)})$$

holds for each $v \in V$. Put $K(G) = \bigcup_{v \in V} K_v$. We call this set the limit set of the GIFS $(G, (J_v), (O_v), (T_e))$. Denoted by $E^\infty = \{\omega = (\omega_n)_{n=0}^\infty \in \prod_{n=0}^\infty E : t(\omega_n) = i(\omega_{n+1})$ for all $n \geq 0\}$ a code space. The shift transformation $\sigma : E^\infty \to E^\infty$ is given by $(\sigma\omega)_n = \omega_{n+1}$ for any $n \geq 0$ and $\omega = (\omega_n)_{n=0}^\infty \in E^\infty$. Let $\pi : E^\infty \to \mathbb{R}^D$ be a coding map for the GIFS $(G, (J_v), (O_v), (T_e))$ defined by $\{\pi(\omega)\} = \bigcap_{k=0}^\infty T_{\omega_0} \cdots T_{\omega_k} J_{t(\omega_k)}$

for $\omega \in E^{\infty}$. We put the function

$$\varphi(\omega) = \log \|T'_{\omega_0}(\pi\sigma\omega)\|.$$

A σ-invariant Borel probability measure μ_G on E^{∞} is said to be a Gibbs measure of the GIFS $(G, (J_v), (O_v), (T_e))$ if this is the Gibbs measure of the potential $(\dim_H K(G))\varphi$ (see [1] for definition).

2.2 Formulation of our perturbed GIFS

Now we formulate our perturbed GIFS. We introduce the following conditions $(G.1)$-$(G.4)$:

$(G.1)$ The graph $G = (V, E, i, t)$ is strongly connected.

$(G.2)$ The set $(G, (J_v), (O_v), (T_e(\epsilon, \cdot)))$ is a GIFS for all $\epsilon > 0$.

$(G.3)$ There exists a decomposition $E = E_0 \cup E_1$ of E such that

$$T_e(\epsilon, x) \;\to\; \begin{cases} T_e(x) & e \in E_0 \\ a_e & e \in E_1 \end{cases} \text{ uniformly in } x \in J_{t(e)},$$

$$\|\tfrac{\partial}{\partial x} T_e(\epsilon, x)\| \;\to\; \begin{cases} \|T'_e(x)\| & e \in E_0 \\ 0 & e \in E_1 \end{cases} \text{ uniformly in } x \in J_{t(e)},$$

where a_e is an element in $J_{t(e)}$ for $e \in E_1$. Moreover, let $G_0 = (V_0, E_0)$ with $V_0 = i(E_0) \cup t(E_0)$. Then the set $(G_0, (J_v)_{v \in V_0}, (O_v)_{v \in V_0}, (T_e)_{e \in E_0})$ is a GIFS. Moreover, there exists a strongly connected subgraph $H = (V_H, E_H)$ of G_0 such that the limit set of the GIFS $(H, (J_v)_{v \in V_H}, (O_v)_{v \in V_H}, (T_e)_{e \in E_H})$ has positive Hausdorff dimension.

$(G.4)$ There exist constants $c_1 > 0$ and $\beta \in (0, 1]$ such that for any $e \in E$, $x, y \in O_{t(e)}$ and $\epsilon > 0$, $|\|\frac{\partial}{\partial x} T_e(\epsilon, x)\| - \|\frac{\partial}{\partial x} T_e(\epsilon, y)\|| \le c_1 \|\frac{\partial}{\partial x} T_e(\epsilon, x)\| |x - y|^{\beta}$.

By virtue of the condition $(G.1)$, the perturbed GIFS $(G, (J_v), (O_v), (T_e(\epsilon, \cdot)))$ has a unique limit set $K_\epsilon(G)$ and a unique Gibbs measure $\mu(\epsilon, \cdot)$ for each $\epsilon > 0$. On other other hand, the non-perturbed GIFS $(G_0, (J_v)_{v \in V_0}, (O_v)_{v \in V_0}, (T_e)_{e \in E_0})$ has several limit sets $K(H)$ $(H \in SC(G_0))$ and several Gibbs measures μ_H $(H \in SC(G_0))$.

For each $\epsilon > 0$, $\pi(\epsilon, \cdot)$ means the coding map of the GIFS $(G, (J_v), (O_v), (T_e(\epsilon, \cdot)))$ and $\varphi(\epsilon, \omega)$ the function $\log \|\frac{\partial}{\partial x} T_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega))\|$. We put

$$SC_0 = \{H \in SC(G_0) \,:\, \dim_H K(H) = \max_{H' \in SC(G)} \dim_H K(H')\}.$$

For simplicity, we write $SC_0 = \{H(1), H(2), \dots, H(m)\}$. In these cases, we are interested in convergence of the Hausdorff dimension $\dim_H K_\epsilon(G)$ of $K_\epsilon(G)$, convergence of the Gibbs measure $\mu(\epsilon, \cdot)$ of the potential $(\dim_H K_\epsilon(G))\varphi(\epsilon, \cdot)$ and convergence of the measure-theoretic entropy $h(\mu(\epsilon, \cdot))$ of this measure.

3 Previous results

We use the notation defined in Section 2. We begin with the following results.

Theorem 3.1 ([10]) *Assume that the conditions* $(G.1)$-$(G.4)$ *are satisfied. Then*

(1) $\dim_H K_\epsilon(G)$ *converges to* $\max_k \dim_H K(H(k))$;

(2) any limit point of the Gibbs measure $\mu(\epsilon, \cdot)$ *in the sense of weakly topology has the form* $\sum_{k=1}^m p(k)\mu_{H(k)}$ *for some probability vector* $(p(k))_k$;

(3) if $\mu(\epsilon, \cdot)$ *converges to a measure* $\sum_{k=1}^m p(k)\mu_{H(k)}$ *weakly, then* $h(\mu(\epsilon, \cdot))$ *converges to* $\sum_{k=1}^m p(k)h(\mu_{H(k)})$.

Theorem 3.1(2) says that the measure $\mu(\epsilon, \Sigma_0)$ of the set $\Sigma_0 = \{\omega \in E^\infty : \omega_0 \in E \setminus \bigcup_{k=1}^m E_{H(k)}\}$ vanishes as $\epsilon \to 0$, where $E_{H(k)}$ denotes the edge set of $H(k)$. Note also that if $\sharp SC_0 = 1$ then $\mu(\epsilon, \cdot)$ converges weakly. However, in the case when $\sharp SC_0 \geq 2$, $\mu(\epsilon, \cdot)$ may do not converge in general. In the following subsections, we will focus on convergence of $\mu(\epsilon, \cdot)$ under the case $\sharp SC_0 \geq 2$.

3.1 Perturbed piecewise expanding Markov maps with holes

In this section, we consider perturbed piecewise expanding Markov maps with holes which are treated as a special perturbed GIFS. We will give a sufficient condition for convergence of the measure $\mu(\epsilon, \cdot)$ of perturbed GIFS with $D = 1$.

Assume that the conditions $(G.1)$-$(G.4)$ with $D = 1$ are satisfied. We also consider the following conditions.

(G.5) $\bigcup_{v \in V} J_v = [0, 1]$.

(G.6) For any $v \in V$ and $\epsilon > 0$, $\bigcup_{e \in E : i(e) = v} T_e(\epsilon, J_{t(e)}) = J_v$.

(G.7) For any $v \in V$, there exists a subgraph $H \in SC_0$ of G such that $\bigcup_{e \in E_H : i(e) = v} T_e(J_{t(e)}) = J_v$.

For $\epsilon \geq 0$, we define a map $f_\epsilon : [0, 1] \to [0, 1]$ by $f_\epsilon(x) = T_e(\epsilon, \cdot)^{-1}(x)$, where e is decided uniquely if $x \in \bigcup_e \mathrm{int}(T_e(\epsilon, J_{t(e)}))$, and otherwise we arbitrary choose e so that $x \in \partial T_e(\epsilon, J_{t(e)})$. In this setting, the map f_ϵ is a topologically transitive piecewise expanding map with a fixed finite Markov partition for $\epsilon > 0$, and the map f_0 consists of a finite many of topologically transitive piecewise expanding maps. The set of critical points of f_ϵ is written by $C_\epsilon = \bigcup_{e\in E} \partial T_e(\epsilon, J_{t(e)})$. It is known that the sets $\bigcup_{n=0}^{\infty} f_\epsilon^{-n} C_\epsilon$ and $\pi(\epsilon, \cdot)^{-1}(\bigcup_{n=0}^{\infty} f_\epsilon^{-n} C_\epsilon)$ are at most countable sets. Then the absolutely continuous invariant probability measure (ACIM) of f_ϵ coincides with the measure $\mu(\epsilon, \cdot) \circ \pi(\epsilon, \cdot)^{-1}$. In these setting, the volumes of the "holes" $T_e(\epsilon, J_{t(e)})$, $e \in E_1$, vanishes as $\epsilon \to 0$.

We also introduce the following conditions for holes:

(G.8) For any $e \in E_1$, there exists a C^1 map $T_{e,1}$ on $J_{t(e)}$ such that $\|T_e'(\epsilon, \cdot)\| = \epsilon \|T_{e,1}'\| + o(\epsilon)$ in $C(J_{t(e)}, \mathbb{R})$.

(G.9) Let $Q_0 = (Q_0(kk'))$ be a matrix indexed by $\{1, 2, \ldots, m\}^2$ with

$$Q_0(kk') = \begin{cases} 1, & \text{if } T_{e,1}' \not\equiv 0 \text{ for some } e \in E_1 \text{ with } i(e) \in V_{H(k)}, t(e) \in V_{H(k')} \\ 0, & \text{otherwise} \end{cases}$$

Then Q_0 is non-zero and irreducible.

Theorem 3.2 ([10]) *Assume that the conditions* $(G.1)$-$(G.9)$ *are satisfied and* $\sharp SC_0 \geq 2$ *Then the Gibbs measure* $\mu(\epsilon, \cdot)$ *converges to the measure* $\sum_{k=1}^{m} p(k)\mu_{H(k)}$ *and the vector* $p = (p(k))$ *is characterized as the invariant measure of the continuous time Markov chain generated by an infinitesimal generator* Q*, i.e.* $pQ = 0$*. In particular,* Q *is calculated by the convergence speed of the holes.*

Note that $(G.8)$ and $(G.9)$ are conditions which contribute to convergence of $\mu(\epsilon, \cdot)$. Therefore if these conditions are not satisfy, then there is an example so that $\mu(\epsilon, \cdot)$ does not converge. As related results, there is a study of convergence of ACIMs of perturbed piecewise expanding maps with holes [3, 4].

3.2 In the case $\sharp SC_0 = 2$ or 3

As main results in [10], we gave a general convergence of $\mu(\epsilon, \cdot)$ in the case when $\sharp SC_0 = 2$ or 3. For details, let $C(E^\infty)$ be the set of all complex-valued continuous functions on E^∞. We put

$$\begin{aligned} E(k) &= E_{H(k)} \cup \left(E \setminus \bigcup_{H \in SC_0} E_H \right), \\ \eta_\epsilon(k) &= \exp(P((\dim_H K_\epsilon(G))\varphi(\epsilon, \cdot)|_{E(k)^\infty})) \text{ for } k = 1, 2, \\ p_\epsilon^2(k) &= \frac{1 - \eta_\epsilon(k')}{1 - \eta_\epsilon(1) + 1 - \eta_\epsilon(2)} \quad \text{for } \{k, k'\} = \{1, 2\}, \end{aligned}$$

where $P(\varphi)$ denotes the topological pressure of a potential φ (see [1] for definition). Note that the number $\eta_\epsilon(k)$ coincides with the Perron eigenvalue of the sub Ruelle operator $\mathcal{L}_{\epsilon,E(k)}$ acting on $C(E^\infty)$ which is defined by

$$\mathcal{L}_{\epsilon,E(k)}f(\tau) = \begin{cases} \displaystyle\sum_{e\in E(k)\,:\,t(e)=i(\tau_0)} \exp((\dim_H K_\epsilon(G))\varphi(\epsilon, e\cdot\tau))f(e\cdot\tau), & \tau_0 \in E(k) \\ 0, & \tau_0 \notin E(k) \end{cases}$$

for $f \in C(E^\infty)$ and $\tau \in E^\infty$. This operator satisfies Ruelle-Perron-Frobenius type Theorem [8]. Remark also that $\eta_\epsilon(k)$ is less than 1 from $E(k)$ and $E_{H(k')}$ $(k' \neq k)$ are disjoint and G is strongly connected. We first have the following in the case $\sharp SC_0 \geq 2$.

Theorem 3.3 ([10]) *Assume the conditions* $(G.1)$-$(G.4)$ *are satisfied and* SC_0 *consists of two elements* $\{H(1), H(2)\}$. *Then* $p^2_\epsilon(k)$ *converges to a number* $p(k)$ *for all* $k = 1, 2$ *if and only if* $\mu(\epsilon, \cdot)$ *converges to the measure* $p(1)\mu_{H(1)} + p(2)\mu_{H(2)}$ *weakly.*

Next we consider the case $\sharp SC_0 = 3$. We let

$$E(k,k') = E_{H(k)} \cup E_{H(k')} \cup \left(E \setminus \bigcup_{H\in SC_0} E_H \right),$$

$$\eta_\epsilon(k,k') = \exp(P((\dim_H K_\epsilon(G))\varphi(\epsilon,\cdot)|_{E(k,k')^\infty})) \quad \text{for } k, k' \text{ with } k \neq k'$$

and define

$$q^3_\epsilon(k) = (1 - \eta_\epsilon(k',k''))(1 + \eta_\epsilon(k',k'') - \eta_\epsilon(k') - \eta_\epsilon(k'')) \text{ and}$$

$$p^3_\epsilon(k) = q^3_\epsilon(k) / \sum_{l=1}^{3} q^3_\epsilon(l) \quad \text{for } k$$

with $\{k, k', k''\} = \{1, 2, 3\}$. Note that $\eta_\epsilon(k, k')$ becomes the Perron eigenvalue of the operator $\mathcal{L}_{\epsilon,E(k,k')}$. We next obtain the following assertion.

Theorem 3.4 ([10]) *Assume the conditions* $(G.1)$-$(G.4)$ *are satisfied and* SC_0 *consists of three elements* $\{H(1), H(2), H(3)\}$. *Then* $p^3_\epsilon(k)$ *converges to a number* $p(k)$ *for all* $k = 1, 2, 3$ *if and only if* $\mu(\epsilon, \cdot)$ *converges to a measure* $p(1)\mu_{H(1)} + p(2)\mu_{H(2)} + p(3)\mu_{H(3)}$ *weakly.*

3.3 In the case $\sharp SC_0 \geq 4$

It is a natural question whether similar arguments are satisfied for the case $\sharp SC_0 \geq 4$. There is a following conjecture for $\sharp SC_0 = 4$. For k, k', k'' mutually disjoint, we put

$$E(k,k',k'') = E_{H(k)} \cup E_{H(k')} \cup E_{H(k'')} \cup \left(E \setminus \bigcup_{H \in SC_0} E_H \right)$$
$$\eta_\epsilon(k,k',k'') = \exp(P((\dim_H K_\epsilon(G))\varphi(\epsilon,\cdot)|_{E(k,k',k'')})).$$

Set

$$\begin{aligned} q_\epsilon^4(k) =& \left(1 - \eta_\epsilon(k',k'',k''')\right)\Big((1-\eta_\epsilon(k'',k'''))(1-\eta_\epsilon(k'')+\eta_\epsilon(k'',k''')-\eta_\epsilon(k''')) \\ &+ (\eta_\epsilon(k',k'',k''')-\eta_\epsilon(k',k'''))(1-\eta_\epsilon(k''')) + (1-\eta_\epsilon(k',k'''))(\eta_\epsilon(k',k''')-\eta_\epsilon(k')) \\ &+ (\eta_\epsilon(k',k'',k''')-\eta_\epsilon(k',k''))(\eta_\epsilon(k',k'',k''')-\eta_\epsilon(k')+\eta_\epsilon(k',k'')-\eta_\epsilon(k''))\Big) \end{aligned}$$

for $\{k,k',k'',k'''\} = \{1,2,3,4\}$, and $p_\epsilon^4(k) = q_\epsilon^4(k)/\sum_{l=1}^4 q_\epsilon^4(l)$.

Conjecture 3.5 ([10]) *Assume the conditions (G.1)-(G.4) and $\sharp SC_0 = 4$. Then $\mu(\epsilon,\cdot)$ converges to $\sum_{k=1}^4 p(k)\mu_{H(k)}$ weakly if and only if $p_\epsilon^4(k)$ converges to a number $p(k)$ for all $k = 1,2,3,4$.*

There is also such a similar conjecture for the case $\sharp SC_0 \geq 5$.

4 Main result

In Theorem 3.3 and Theorem 3.4, we gave a necessary and sufficient condition for convergence of $\mu(\epsilon,\cdot)$ composed of Perron eigenvalues of sub Ruelle operators in the case when $\sharp SC_0 = 2,3$. However, it is difficult to prove similar assertion when $\sharp SC_0 \geq 4$ (see [10]).

In this section, we will give an another approach by using the notion of extended Ruelle operators in all cases including $\sharp SC_0 \geq 4$.

For details, we introduce some notation below. Let $M(E^\infty)$ be the set of all Borel complex measure on E^∞. For $0<\theta<1$, denoted by d_θ the metric on E^∞ with $d_\theta(\omega,\upsilon) = \theta^{\min\{n\geq 0\,:\,\omega_n\neq \upsilon_n\}}$, and by $F_\theta(E^\infty)$ the set of all Lipschitz continuous functions belonging in $C(E^\infty)$. For k,k' mutually disjoint and $\epsilon>0$, we define an operator $\mathcal{L}_\epsilon(k,k')$ acting on $C(E^\infty)$ which is given by $\mathcal{L}_\epsilon(k,k')f(\tau) =$

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} \sum_{\substack{w \in E_{H(k)} \times F(k,k')^n : \\ w\cdot\tau\cdot \text{ path on } G}} \exp\left(\sum_{l=0}^{n} (\dim_H K_\epsilon(G))\varphi(\epsilon,\sigma^l(w\cdot\tau))\right) f(w\cdot\tau), & \tau_0 \in E_{H(k)} \\ 0, & \tau_0 \notin E_{H(k)} \end{cases}$$

for $f \in C(E^\infty)$ and $\tau \in E^\infty$, where $F(k,k') = E \setminus (E_{H(k)} \cup E_{H(k')})$. Note that this operator is a positive bounded linear operator acting on $C(E^\infty)$. This has similar properties of Ruelle operators as follows. These exists a unique triplet $(\lambda_\epsilon^{k,k'}, h_\epsilon^{k,k'}, \nu_\epsilon^{k,k'}) \in \mathbb{R} \times C(E^\infty) \times M(E^\infty)$ such that $\lambda_\epsilon^{k,k'}$ is a simple maximal eigenvalue of $\mathcal{L}_\epsilon(k,k')$, $h_\epsilon^{k,k'}$ is the corresponding nonnegative eigenfunction, and $\nu_\epsilon^{k,k'}$ is the corresponding positive eigenvector of the dual $\mathcal{L}_\epsilon(k,k')^*$ with $\nu_\epsilon^{k,k'}(h_\epsilon^{k,k'}) = \nu_\epsilon^{k,k'}(E^\infty) = 1$. Note also that $\mathcal{L}_\epsilon(k,k')$ is well-defined as a bounded linear operator acting on $F_\theta(E^\infty)$ and this operator is quasi-compact. These assertions are proved by using standard thermodynamic formalism techniques ([11]). For $k = 1, 2, \ldots, m$, we put

$$p_\epsilon(k) = \left(1 + \sum_{k' : k' \neq k} \frac{1 - \lambda_\epsilon^{k,k'}}{1 - \lambda_\epsilon^{k',k}}\right)^{-1}.$$

Now we are in a position to state our main result.

Theorem 4.1 ([12]) *Assume that the conditions* $(G.1)$-$(G.4)$ *are satisfied and* $\sharp SC_0 \geq 2$. *Then the Gibbs measure* $\mu(\epsilon,\cdot)$ *converges to a measure* μ *weakly if and only if* $p_\epsilon(k)$ *converges to a number* $p(k)$ *for all* $k = 1, 2, \ldots, m$. *In these cases,* μ *has the form* $\mu = \sum_{k=1}^m p(k)\mu_{H(k)}$.

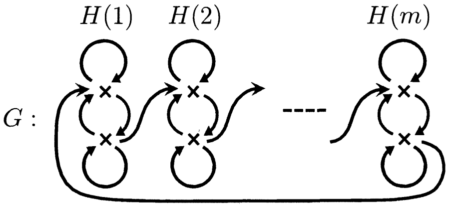
5 Concrete examples

5.1 A convergent case

Assume the following conditions (i)-(iii):

(i) For each $k = 1, 2, \ldots, m$, a graph $H(k) = (\{v_k^1, v_k^2\}, \{e_k^1, e_k^2, e_k^3, e_k^4\}, i, t)$ satisfies $i(e_k^1) = t(e_k^1) = v_k^1$, $i(e_k^2) = v_k^1$, $t(e_k^2) = v_k^2$, $i(e_k^3) = v_k^2$, $t(e_k^2) = v_k^1$, and $i(e_k^4) = t(e_k^4) = v_k^2$.

(ii) The graph $G = (V, E, i, t)$ has the vertex set $V = \bigcup_{k=1}^m V_{H(k)}$ and the edge set $E = E_0 \cup E_1$ with $E_0 = \bigcup_{k=1}^m E_{H(k)}$ and $E_1 = \{e_{12}, e_{23}, \ldots, e_{m-1m}, e_{m1}\}$ with $i(e_{kk'}) = v_k^2$ and $t(e_{kk'}) = v_{k'}^1$ (see the following figure).



(iii) GIFSs $(G,(J_v),(O_v),(T_e(\epsilon,\cdot)))$ satisfy the two conditions $(G.3)$ and $(G.4)$, namely, $\|\frac{\partial}{\partial x}T_e(\epsilon,\cdot)\| := \sup_{x\in J_{t(e)}}\|\frac{\partial}{\partial x}T_e(\epsilon,x)\| \to 0$ as $\epsilon \to 0$ for any $e \in E_1$. Moreover, $T_e(\epsilon,\cdot) \equiv T_e$ and $\|\frac{\partial}{\partial x}T_e(\epsilon,\cdot)\| \equiv 1/10$ for any $e \in E_0$.

In these cases, we notice $SC(G) = SC_0 = \{H(1), H(2), \dots, H(m)\}$. We also obtain that the operator $\mathcal{L}_\epsilon(k,k')$ becomes the sub Ruelle operator $\mathcal{L}_{\epsilon,E_{H(k)}}$ for each k,k' with $k \neq k'$. Note that this operator does not depend on k'. We see that the Perron eigenvalue $\lambda_\epsilon^{k,k'}$ of this operator is equal to $2(1/10)^{\dim_H K(\epsilon)}$ for any $k \neq k'$. Therefore $p_\epsilon(k) = 1/m$ for any k. By virtue of Theorem 4.1, the Gibbs measure $\mu(\epsilon,\cdot)$ of $(\dim_H K_\epsilon(G))\varphi(\epsilon,\cdot)$ converges to $\sum_{k=1}^m \mu_{H(k)}/m$ weakly.

5.2 A non convergent case

Assume the following (i),(ii),(iii)$'$:

(i) The same condition as (i) in Section 5.1.

(ii) The same condition as (ii) in Section 5.1.

(iii)$'$ GIFSs $(G,(J_v),(O_v),(T_e(\epsilon,\cdot)))$ satisfies the conditions $(G.3)$, $(G.4)$ and

$$\|\tfrac{\partial}{\partial x}T_e(\epsilon,\cdot)\| = \begin{cases} \epsilon, & e \in E_1 \\ 1/10, & e \in E_0 \setminus E_{H(1)} \\ 1/10 + \epsilon^{s(0)\exp(\sin(1/\epsilon))}, & e \in E_{H(1)}, \end{cases}$$

where $s(0) = \dim_H K(H(1)) = \log 2/\log 10$.

In these cases, $(1-\lambda_\epsilon^{1,k})/(1-\lambda_\epsilon^{k,1})$ has the form

$$\frac{1-\lambda_\epsilon^{1,k}}{1-\lambda_\epsilon^{k,1}} = \frac{1-2(\epsilon^{s(0)\exp(\sin(1/\epsilon))}+1/10)^{\dim_H K_\epsilon(G)}}{1-2(1/10)^{\dim_H K_\epsilon(G)}} =: a(\epsilon)$$

for all $k = 2,3,\dots,m$, and this number $a(\epsilon)$ does not converge as $\epsilon \to 0$. Therefore so is for $p_\epsilon(1) = 1/(1+(m-1)a(\epsilon))$. From Theorem 4.1, the Gibbs measure $\mu(\epsilon,\cdot)$ does not converge.

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