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Identification of random functions from the SFCs defined by the Ogawa integral regarding regular CONSs

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1. abstract

Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Our concern is whether and how a noncausal type stochastic differential $dX_t = a(t,\omega)\,dB_t + b(t,\omega)\,dt$ is identified from its stochastic Fourier coefficients (SFCs for short) $(e_n, dX) := \int_0^1 \overline{e_n(t)}\,dX_t$ with respect to a CONS $(e_n)_{n\in\mathbb{N}}$ of $L^2([0,1]\,;\mathbb{C})$. This problem has been studied by S.Ogawa and H.Uemura (Ogawa (2013)[9], (2014)[10]; Ogawa,Uemura (2014)[12], [13], (2015)[14]). In this note we explain the result we obtained on the problem for the stochastic differentials by the Ogawa integral for regular CONSs. This note is an announcement of the author's full paper on this result.

2. Introduction

Let $(e_n)_{n\in\mathbb{N}}$ be a CONS of $L^2([0,L]\,;\mathbb{C})$, $(B_t)_{t\in[0,\infty)}$ a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $a, b : [0,L]\times\Omega \to \mathbb{C}$ jointly-measurable functions, which we call random functions. Here we regard the symbol $[0,L]$ as the infinite interval $[0,\infty)$ when $L=\infty$. We consider the SFC (short for stochastic Fourier coefficient)

$$(e_n, dY) := \int_0^L \overline{e_n(t)}a(t)\,dB_t + \int_0^L \overline{e_n(t)}b(t)\,dt$$

of the stochastic differential $dY_t = a(t)\,dB_t + b(t)\,dt$ with respect to $(e_n)_{n\in\mathbb{N}}$, which is originally introduced by S.Ogawa [6]-[8]. We note that the SFC (e_n, dY) doesn't make sense unless the stochastic integral $\int_0^L dB$ is specified, $\overline{e_n}a$ is stochastic integrable and $\overline{e_n}b$ is Lebesgue integrable on $[0,L]$. Specifically, the SFC is called of Skorokhod type (SFC-S) if the stochastic integral $\int_0^L dB$ is the Skorokhod integral ([17]) and of Ogawa type (SFC-O) if the stochastic integral $\int_0^L dB$ is the Ogawa integral ([5], see also Definition 3.1).

We'd like to answer the following questions:

Question 1: Are random functions a and b identified from the sequence of SFCs $((e_n, dY))_{n\in\mathbb{N}}$ or a subsequence of it? In other words, letting $\mathrm{N}=\mathbb{N}$ or $\mathrm{N}\subset\mathbb{N}$, is the map which associates a pair (a,b) with $((e_n, dY))_{n\in\mathrm{N}}$ injective?

If yes,

Question 2: how is the inverse map in Question 1 ? Specifically, is the inverse map "constructive in certain senses"?

Furthermore, we also ask the following question for purely mathematical interest:

Question 3: Are the random functions identified "without using information" of the underlying Brownian motion $(B_t)_{t\in[0,\infty)}$?

Question 1 was originally posed by Ogawa [9] after a series of studies [6]-[8] of a stochastic integral equation of Fredholm type, for Question 1 is closely connected with the existence and uniqueness of solutions for stochastic integral equations of Fredholm type ([6]-[8], [10]). On the other hand, Question 2 is related to the study of the volatility estimation problem proposed by P. Malliavin et al. ([3],[4]) and conducted by Ogawa and H.Uemura ([10], [14], [15], [16]). After the publications of [3],[4], the importance of re-considering Question 1 from the application viewpoint was recognized between Ogawa and Uemura. More precisely, they assumed the situation of estimating values of the diffusion coefficient a from SFCs as given data. In this situation, you can't use values of the underlying Brownian motion nor any other function properly depending on $\omega \in \Omega$ to estimate a. Then, in [1] the notion of constructiveness in a first-order language was introduced and Question 2 was taken up from the purely mathematical viewpoint. Further, from the purely mathematical interest, in the same article the author also introduced the notion of independent identification for Brownian motion, which can be said identification without using information that the underlying Brownian motion is $(B_t)_{t\in[0,\infty)}$, and took up Question 3. See [1] for the precise meanings of "constructive in certain senses" and "without using information" in the questions.

Up to the present, affirmative answers to these questions are given ([9]-[16],[2],[1]).

The main result in this study is Theorem 5.1 which gives affirmative answers to the questions in the case that SFC is of Ogawa type as extensions of the previous results in [14], [15], [2], [1].

2.1. Summary of previous and main results

We roughly summarize the previous and main results. Each result gives an affirmative answer to the question for a specific diffusion coefficient a under certain assumptions on $(e_n)_{n\in\mathbb{N}}$ and b. In Table 1 (resp. Table 2), these specific diffusion coefficients a identified from SFC-Os in the previous results (resp. the main results) are listed. In each table we distinguish between the case the inverse map in Question 2 is constructive with the underlying Brownian motion B and the case it is constructive without it, in a certain language.

Table 1 ⟪ **Previous results:** Diffusion coefficients a identified from SFC-Os ⟫

With the Brownian motion B	Without the Brownian motion B
Random function of bounded variation A ([1])	$\lvert A\rvert$ ([14], [15], [1])
$\mathbb{C}$-valued random function Z which satisfies (C) ([16])	$(\operatorname{sgn} Z)Z$ ([16])
Skorokhod integral process $W = \int_0^{\cdot} f\,\delta B$ $(f \in \mathcal{L}_1^{2,2})$ ([2])	

Table 2 ⟪ **Main results:** Diffusion coefficients a identified from SFC-Os ⟫

With the Brownian motion B	Without the Brownian motion B
	$\lvert W\rvert$
Local martingale $M = \int_0^{\cdot} g\,dB$	$\lvert M\rvert$
$A+M+W$	$\lvert A+M+W\rvert$
$R\quad(\operatorname{Re} R, \operatorname{Im} R \in \mathscr{L}_{\mathrm{PC}}^{*,e})$	$(\operatorname{sgn} R)R$

Remark 1 $L = 1$ except the results in [1].
Remark 2 (C): There exists a CONS $(\chi_k)_{k\in\mathbb{N}}$ of $L^2([0,1];\mathbb{C})$ which satisfies $\sup_{k\in\mathbb{N},t\in[0,1]} |\chi_k(t)| < \infty$ and $(\lambda_k)_{k\in\mathbb{N}}$ which satisfies $\forall k\ \lambda_k > 0$, $\sum_{k=1}^{\infty} \lambda_k < \infty$ such that $E\Big(\sum_{k=1}^{\infty} \frac{1}{\lambda_k}\langle \chi_k, Z\rangle_{L^2[0,1]}^2\Big) < \infty$.
Remark 3 $\mathscr{L}_{\mathrm{PC}}^{*,e}$ is defined by using the Ogawa integral at the beginning of Section 5.
Remark 4 $A + M + W \in \mathscr{L}_{\mathrm{PC}}^{*,e}$.
Remark 5 For the other notation and terminology used in the tables, we follow those introduced in Subsection 3.1.

3. Preliminaries

3.1. Notation and terminology

Let $(B_t)_{t\in[0,\infty)}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, λ the Lebesgue measure on $\mathbb{R}$ and L a constant which satisfies $0 < L \le \infty$. If $L = \infty$ we regard the symbol $[0, L]$ as the infinite interval $[0,\infty)$. $\mathcal{L}([0, L])$ denotes the σ-field of Lebesgue measurable sets on $[0, L]$. We say $f : [0, L] \times \Omega \to \mathbb{C}$ is a random function (or measurable stochastic process) on $[0, L]$ if f is $\mathcal{L}([0, L]) \otimes \mathcal{F}$-measurable and we say a sequence $(X_t)_{t\in[0,L]}$ is a (weak) stochastic process on $[0, L]$ if X_t is a $\mathbb{C}$-valued random variable for each $t \in [0, L]$. Note that a random function $f(t)$ and a stochastic process X_t are noncausal, namely, not necessarily adapted to some filtration for $(B_t)_{t\in[0,L]}$, for that matter, not necessarily $\mathcal{F}^B$-measurable for $t \in [0, L]$, where $\mathcal{F}^B$ is the σ-field generated by $(B_t)_{t\in[0,L]}$. $\langle f, g\rangle = \langle f, g\rangle_{L^2([0,L]\,;\mathbb{C})}$ means the inner product of $f, g \in L^2([0, L]\,; \mathbb{C})$ defined by $\int_0^L \overline{f} g \, d\lambda$, where $\overline{f}$ represents the complex conjugate of f. By $\mathcal{L}_i^{r,2}$ we denote the Sobolev space with respect to the H-derivative D in $L^2((\Omega, \mathcal{F}^B, P)\,;\, L^2([0, L]^i\,; \mathbb{C}))$ with differentiability index r for each $i \in \{0\} \cup \mathbb{N}$. $T_K f$ denotes the Hilbert-Schmidt integral transform by the kernel $K \in L^2([0, L]^2\,; \mathbb{C})$ of $f \in \mathcal{L}_1^{1,2}$ defined by $T_K f(t) = \int_0^L K(t, s) f(s)\, ds$. The symbol $\int_0^L \delta B$ and $\int_0^L dB$ denote the Skorokhod integral and the Itô integral, respectively. For $z \in \mathbb{C}$, $z^{(1)}$ and $z^{(2)}$ mean $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively, and $\operatorname{sgn} z$ means 1 if $0 \le \arg z < \pi$ and -1 otherwise, where $\arg 0 := 0$. Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. In this note, by a CONS we mean an ordered CONS.

3.2. Ogawa integrals

We give the definition of the Ogawa integral.

Definition 3.1 (Ogawa integral)

Let $f(t) \in L^0(\Omega\,; L^2([0, L]\,; \mathbb{C}))$ and $T \in \mathcal{L}([0, L])$.

(φ**-integrability**)

Let $(\varphi_m)_{m\in\mathbb{N}}$ be a CONS of $L^2([0, L]\,; \mathbb{C})$. We say f is integrable with respect to φ or φ-integrable on T if

$$\sum_{m=1}^{\infty} \langle \varphi_m, f 1_T\rangle \int_0^L \varphi_m \, dB \tag{1}$$

converges in probability. In this case, (1) is called the Ogawa integral of f with respect to φ or φ-integral of f on T and denoted by $\int_T f \, d_\varphi B$.

($\mathcal{C}$**-integrability**)

Let $\mathcal{C}$ be a non-empty set of CONSs of $L^2([0, L]\,; \mathbb{C})$. We say f is integrable for $\mathcal{C}$ or $\mathcal{C}$-integrable on T if f is integrable with respect to any $\varphi \in \mathcal{C}$ on T and the Ogawa integral $\int_T f \, d_\varphi B$ is independent of the particular choice of $\varphi \in \mathcal{C}$. In this case, $\int_T f \, d_\varphi B$ is called the Ogawa integral of f for $\mathcal{C}$ or $\mathcal{C}$-integral of f on T and denoted by $\int_T f \, d_{\mathcal{C}} B$. In particular, when $\mathcal{C}$ is $\mathcal{C}(\mathbb{K}) := \{\psi \,|\, \psi \text{ is a CONS of } L^2([0, L]\,; \mathbb{K})\,\}$ $\Big($resp. $\mathcal{R} := \Big\{ \psi \in \mathcal{C}(\mathbb{R}) \,\Big|\, \sup_{M\in\mathbb{N}} \Big| \sum_{m=1}^{M} \psi_m \int_0^{\cdot} \psi_m \, d\lambda \Big|_{L^2[0,L]} < \infty \Big\}\Big)$, we say f is universally integrable or u-integrable for $L^2([0, L]\,; \mathbb{K})$ (resp. Ogawa integrable for regular CONSs of $L^2[0, L]$), and $\int_T f \, d_{\mathcal{C}} B$ is called the universal Ogawa integral or u-integral for $L^2([0, L]\,; \mathbb{K})$ (resp. the Ogawa integral for regular CONSs of $L^2[0, L]$) and denoted by $\int_T f \, d_{\mathrm{u}} B$ (resp. $\int_T f \, d_* B$).

3.3. Cross and quadratic variations of random functions

We describe the concept of cross and quadratic variations of random functions, which have been introduced in Chapters 8 and 9 in [11], respectively.

Given $-\infty < a \leq b < \infty$, put $\Pi[a,b] = \bigsqcup_{n\in\mathbb{N}_0} \{\, (t_0, t_1, \ldots, t_n) \in [a,b]^{1+n} \,|\, a = t_0 < t_1 < \cdots < t_n = b \,\}$ and $l : \Pi[a,b] \to \mathbb{N}_0$ such that $l(\Delta) = n$, for $\Delta = (t_0, t_1, \ldots, t_n) \in \Pi[a,b]$. The function $|\cdot| : \Pi[a,b] \to [0, b-a]$ such that $|\Delta| = \max_{1\leq j\leq l(\Delta)} (t_j - t_{j-1})$, for $\Delta = (t_0, t_1, \ldots, t_n) \in \Pi[a,b]$ is called the mesh on $\Pi[a,b]$. Let f be a map from $\Pi[a,b]$ to a metric space (X,d). We mean by $\lim_{|\Delta|\to 0} f(\Delta) = x$ the assertion that a limit in X of $f(\Delta)$ is $x \in X$ with respect to $|\Delta|$ at 0, namely, the assertion

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall \Delta \in \Pi[a,b] \; (\, |\Delta| < \delta \to d(f(\Delta), x) < \epsilon \,).$$

Definition 3.2 (cross and quadratic variations)

Let $X, Y : [0,1] \to L^0(\Omega\,;\mathbb{C})$, $t \in [0,1]$ and set $S_\Delta(X,Y) = \sum_{j=1}^{l(\Delta)} \overline{(X_{t_j} - X_{t_{j-1}})}(Y_{t_j} - Y_{t_j-1})$ for $\Delta = (t_0, \ldots, t_n) \in \Pi[0,t]$. If

$$\lim_{|\Delta|\to 0} S_\Delta(X,Y) \tag{2}$$

converges in probability, i.e. in $L^0(\Omega)$, we call (2) the cross variation at t of X and Y and denote it by $\langle X, Y\rangle_t$. Moreover, if $\langle X, X\rangle_t$ exists, it is called the quadratic variation at t of X and denoted by $[X]_t$.

4. Definition of SFC-O

From now to the end of this note, let $(e_n)_{n\in\mathbb{N}}$ be a CONS of $L^2([0,L]\,;\mathbb{C})$, $a \in L^0([0,L] \times \Omega\,;\mathbb{C})$ and $b \in L^0(\Omega\,; L^2([0,L]\,;\mathbb{C}))$. We state the definition of SFC-O of a stochastic differential with respect to $(e_n)_{n\in\mathbb{N}}$. Let $\mathcal{C}$ be a non-empty set of CONSs of $L^2([0,L]\,;\mathbb{C})$.

Definition 4.1 (SFC-O$_\mathcal{C}$ of stochastic differential)

Suppose $\overline{e_n}a$ is integrable for $\mathcal{C}$ for every $n \in \mathbb{N}$. We define the n-th SFC-O$_\mathcal{C}$ $(e_n, d_\mathcal{C} Y)$ of the stochastic differential $d_\mathcal{C} Y_t = a(t)\, d_\mathcal{C} B_t + b(t)\, dt$, $t \in [0,L]$ with respect to $(e_n)_{n\in\mathbb{N}}$ by

$$(e_n, d_\mathcal{C} Y) := \int_0^L \overline{e_n(t)}\, d_\mathcal{C} Y_t = \int_0^L \overline{e_n(t)} a(t)\, d_\mathcal{C} B_t + \int_0^L \overline{e_n(t)} b(t)\, dt$$

In particular, in the case of $b = 0$, $(e_n, d_\mathcal{C} Y) = (e_n, a\, d_\mathcal{C} B)$ is also called the SFC-O$_\mathcal{C}$ of a. Besides, when $\mathcal{C} = \{\varphi\}, \mathcal{R}$ or $\mathcal{C}(\mathbb{K})$, SFC-O$_\mathcal{C}$ is called SFC-O$_\varphi$, SFC-O$_*$ or SFC-O$_\mathrm{u}$, respectively.

5. Identification of random functions from SFC-O$_*$'s

In this section, we explain the main result about identification of random functions from SFC-O$_*$'s. Hereafter, we assume $L = 1$, real and imaginary parts of each e_n are of bounded variation. Let $(\mathcal{F}_t)_{t\in[0,1]}$ be a filtration for $(B_t)_{t\in[0,1]}$ and put $L^0_{\mathrm{ad}}(\Omega\,;L^2[0,1]) = \{\, f \in L^0(\Omega\,;L^2([0,1]\,;\mathbb{C})) \mid f \text{ is } (\mathcal{F}_t)_{t\in[0,1]} \text{ -progressively measurable} \}$. As subsets of the linear space $L^0([0,1]\times\Omega) = L^0([0,L]\times\Omega\,;\mathbb{C})$, we set

$$\mathscr{L}^{*,e}_{\mathrm{PC}} = \Big\{ a \in L^0([0,1]\times\Omega) \,\Big|\, \Big(\int_0^t a\, d_*B = \sum_{n=1}^{\infty} \int_0^t e_n\, d\lambda\, (e_n,\, a\, d_*B) \text{ in prob. },\quad \int_0^t |a|^2\, d\lambda = \Big[\int_0^{\cdot} a\, d_*B \Big]_t,\quad \int_0^{t\wedge s} a\, d\lambda = \Big\langle \int_0^{\cdot} a\, d_*B,\, B_{\cdot\wedge s} \Big\rangle_t \Big)\ \forall s,t \in [0,1] \Big\}$$

and $\mathscr{L} = \mathcal{A} + \mathcal{M} + \mathcal{W}$, where

$$\mathcal{A} = \{ a \in L^0([0,1]\times\Omega) \mid \mathrm{Re}\, a, \mathrm{Im}\, a \text{ are of bounded variation a.s. } \},$$

$$\mathcal{M} = \Big\{ \int_0^{\cdot} f\, dB \,\Big|\, f \in L^0_{\mathrm{ad}}(\Omega\,;L^2[0,1]) \Big\},$$

$$\mathcal{W} = \Big\{ \int_0^{\cdot} f\, \delta B \,\Big|\, f \in \mathcal{L}^{2,2}_1 \Big\} + \mathrm{span} \Big\{ T_K f \,\Big|\, f \in \mathcal{L}^{1,2}_1,\ \sup_{t\in[0,1]} |K(t,\cdot)|_{L^2[0,1]} < \infty \Big\}.$$

First, we present a necessary and sufficient condition for a random function in $\mathscr{L}$ to be identified from SFC-Os.

Proposition 5.1

For any $a \in \mathscr{L}$ we have

(a) $\lim\limits_{n\to\infty} \int_0^1 v_n a\, d_*B = 0$ in probability for any sequence $(v_n)_{n\in\mathbb{N}}$ of functions on $[0,1]$ of bounded variation which converges to 0 in $L^2([0,1]\,;\mathbb{R})$.

In particular, we have

(b) $\mathcal{P}((e_n, d_*Y))_{n\in\mathbb{N}}(t) := \underset{N\to\infty}{\text{l.i.p.}} \sum\limits_{n=1}^{N} \int_0^t e_n\, d\lambda\, (e_n, d_*Y) = Y_t,\ \forall t \in [0,1]$, where d_*Y denotes the stochastic differential $d_*Y_t = a(t)\, d_*B_t + b(t)\, dt$.

Corollary 5.1

For $\mathscr{S} \subset \mathscr{L}$ and a dense subset S of $[0,1]$ and a map h over $\mathscr{S}$, the following are equivalent:

(i) $h(a)$ is identified for $a \in \mathscr{S}$ from $((e_n, d_*Y))_{n\in\mathbb{N}}$.

(ii) $h(a)$ is identified for $a \in \mathscr{S}$ from $(Y_t)_{t\in S}$.

Here d_*Y denotes the stochastic differential $d_*Y_t = a(t)\, d_*B_t + b(t)\, dt$.

Remark Refer to [1] for the precise meanings of the statements (i) and (ii).

Next, we introduce the space Q_{c} of weak stochastic processes with continuous cross variation processes, defined by

$$Q_{\mathrm{c}} = \Big\{ X : [0,1] \to L^0(\Omega\,;\mathbb{K}) \,\Big|\, \exists \hat{X} \in L^0(\Omega\,;L^2([0,1]\,;\mathbb{K}))\ \forall s,t \in [0,1]\ [X]_t = \int_0^t |\hat{X}|^2\,d\lambda \text{ and } \langle B_{\cdot\wedge s}, X\rangle_t = \int_0^{t\wedge s} \hat{X}\,d\lambda \Big\}.$$

Proposition 5.2

Q_{c} is a subspace of the vector space of $\mathbb{K}$-valued weak stochastic processes $X : [0,1] \to L^0(\Omega\,;\mathbb{K})$.

Corollary 5.2

$\mathscr{L}_{\mathrm{PC}}^{*,e}$ is a linear space.

Proposition 5.3

The following hold for $f, g \in \mathscr{L}$:

$$\int_0^t g\,d\lambda = \Big\langle B, \int_0^{\cdot} g\,d_*B \Big\rangle_t, \tag{3}$$

and

$$\int_0^t \overline{f} g\,d\lambda = \Big\langle \int_0^{\cdot} f\,d_*B, \int_0^{\cdot} g\,d_*B \Big\rangle_t, \tag{4}$$

more generally.

Corollary 5.3

$\mathscr{L} \subset \mathscr{L}_{\mathrm{PC}}^{*,e}$.

Now, we describe the main theorem.

Theorem 5.1

Assume real and imaginary parts of each e_n are of bounded variation. Let $\mathscr{L}_{\mathrm{PC}}^{*,e}$ and $\mathscr{L}$ be the linear spaces defined at the beginning of this section and justified in Corollary 5.2. Let $\mathcal{P}$ be the map defined in Proposition 5.1 and $\mathcal{L}_0$ the first-order language defined in [1]. Given $a \in L^0([0,1] \times \Omega\,;\mathbb{C})$ and $b \in L^0(\Omega\,;L^2([0,1]\,;\mathbb{C}))$, the following two assertions hold:

(**Assertion 1**)

Suppose the following conditions 1 and 2 on a:

1. $a^{(1)} = \operatorname{Re} a \in \mathscr{L}_{\mathrm{PC}}^{*,e}$.
2. $a^{(2)} = \operatorname{Im} a \in \mathscr{L}_{\mathrm{PC}}^{*,e}$.

Then, letting $d_*Y := a(t)\, d_*B + b(t)\, dt$, the following hold:

(A) a is identified constructively in $\mathcal{L}_0$ and B from $((e_n, d_*Y))_{n\in\mathbb{N}}$ by

$$a(t) = \frac{d}{dt}\big\langle B, \mathcal{P}((e_n, d_*Y))_{n\in\mathbb{N}}\big\rangle_t.$$

(B) $|a^{(j)}|$, $j \in \{1,2\}$, $a^{(1)}\, a^{(2)}$ and $(\operatorname{sgn} a)a$ are identified constructively in $\mathcal{L}_0$ and independently for Brownian motion from $((e_n, d_*Y))_{n\in\mathbb{N}}$ by

$$|a^{(j)}|(t) = \Big(\frac{d}{dt}[\mathcal{P}((e_n, d_*Y))^{(j)}_{n\in\mathbb{N}}]_t\Big)^{\frac{1}{2}},\ j \in \{1,2\},$$

$$a^{(1)}\, a^{(2)}(t) = \frac{d}{dt}\big\langle \mathcal{P}((e_n, d_*Y))^{(1)}_{n\in\mathbb{N}}, \mathcal{P}((e_n, d_*Y))^{(2)}_{n\in\mathbb{N}}\big\rangle_t$$

and

$$(\operatorname{sgn} a)a(t) = \begin{cases} |a^{(1)}(t)| & \text{, if } a^{(2)}(t) = 0 \\ \frac{a^{(1)}a^{(2)}(t)}{|a^{(2)}(t)|} + \sqrt{-1}|a^{(2)}(t)| & \text{, if } a^{(2)}(t) \neq 0, \end{cases}$$

respectively.

(**Assertion 2**)

Suppose $a \in \mathscr{L}$. Then, the assumptions 1 and 2 in Assertion 1 holds and therefore, (A) and (B) in Assertion 1 hold.

Remark 1 See Subsection 3.1 for $z^{(1)}$, $z^{(2)}$ and $\operatorname{sgn} z$ for $z \in \mathbb{Z}$.
Remark 2 Refer to [1] for the precise meanings of the statements (A) and (B).
Remark 3 The same assertions as mentioned in this theorem hold, even if finite elements (e_n, d_*Y) of $((e_n, d_*Y))_{n\in\mathbb{N}}$ are replaced with different random variables.
Remark 4 b is identified constructively in $\mathcal{L}_0$ and B from $((e_n, d_*Y))_{n\in\mathbb{N}}$ since a is identified so.
Remark 5 The cross variation method for identification is presented in [16], where Ogawa and Uemura obtained the results mentioned in Subsection 2.1.

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