数理解析研究所講究録 第2116巻 2019年 157-164

RECOVERING MODELLED DISTRIBUTIONS FROM PARACONTROLLED CALCULUS

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Abstract. The aim of this paper is a construction of a model space where the theory of regularity structures (RS) and the theory of paracontrolled calculus (PC) is equivalent. As a first step, we show the local behavior, namely the generalized Taylor expansion, of the iterated paraproduct. Next we construct a specific Hopf algebra associated with such local structure.

1. Introduction

1.1. **Hairer and GIP theories.** Many singular SPDEs have motivations from statistical physics, quantum field theory, etc., but they are sometimes ill-posed without "renormalization". For example, the dynamical Φ_3^4 model

$$(\partial_t - \Delta + m_0^2)\Phi(t,x) = -\Phi(t,x)^3 + \xi(t,x), \quad t>0, \quad x\in\mathbb{R}^3,$$

where ξ is a space-time white noise, is ill-posed. Indeed, since Φ is expected to have regularity $-\frac{1}{2}-$ in x, the cubic term Φ^3 cannot be defined in classical sense.

The theory of *paracontrolled calculus* by Gubinelli, Imkeller and Perkowski [5] made it possible to show the local well-posedness for such SPDEs. To be precise, letting ξ_ϵ be a smooth approximation for ξ and choosing a sequence of renormalization constants C_ϵ, we can show the convergence of the solution Φ_ϵ of

$$(\partial_t - \Delta + m_0^2)\Phi_\epsilon(t,x) = -\Phi_\epsilon(t,x)^3 + C_\epsilon\Phi_\epsilon(t,x) + \xi(t,x)$$

locally in time. We can obtain similar results by the famous theory of *regularity structures* by Hairer [7]. Compared with RS, PC has an advantage in showing detailed properties [6, 10, 11, 1, 4, 9] (global well-posedness, ergodicity, etc.) On the other hand, PC can be applicable to less number of SPDEs than RS. This is because PC is not algebraically sophisticated. Our ultimate goal is to show the equivalence of RS and PC and construct an algebraic theory describing a general version of PC.

One of the main differences between the two theories is in the definition of solutions. In PC, solutions are written by using the Bony's paraproduct [3]. For example, the solution of Φ_3^4 is written in the form

$$\Phi = \mathbf{I} - \Psi + \Phi' \varolessthan \mathbf{Y} + \Phi^{\#},$$

where $\mathbf{I}, \mathbf{Y}, \Psi$ are stochastic data defined from ξ, Φ' and $\Phi^{\#}$ are unknown functions, and $\varolessthan$ is the Bony's paraproduct. In RS, solutions are described based on local estimates. For Φ_3^4, the solution must have the local form

$$\begin{aligned}\Phi(s,y) = {}& \mathbf{I}(s,y) + \varphi(t,x) - (\Psi(s,y) - \Psi(t,x)) \\ &+ \Phi'(t,x)(\mathbf{Y}(s,y) - \mathbf{Y}(t,x)) + (\nabla\Phi)\cdot(y-x) + \cdots,\end{aligned}$$

where $\varphi, \Phi', \nabla\Phi$ are unknown functions. Therefore in order to get the relationship between these two concepts, we need local estimates of Bony's paraproduct.

1.2. **Bailleul-Hoshino's result.** Bailleul and Hoshino [2] studied the relationship between RS and PC. To describe their result, we briefly recall some notions in both theories. Stochastic data as above is called a *model* in RS. An associated notion exists in PC, but such notion seems to have no fixed name. (Since PC has been applied to only specific problems until now, such notion has been called just a family of stochastic data.) Bailleul and Hoshino introduced a notion of *paracontrolled remainder* as an abstraction of such stochastic data in the theory of PC. For the definition of solution, the distribution with local behaviors as above is generalized as a *modelled distribution* in RS. The distribution written by nonlocal operators as above is called a *paracontrolled distribution* in PC.

	RS	PC
Stochastic data	Model	Paracontrolled remainders
Solution	Modelled distribution	Paracontrolled distribution

Bailleul and Hoshino's result is described as follows.

Theorem 1.1 (Only rough image)**.** *For the stochastic data, a model and a family of paracontrolled remainders are equivalent notions. For the definition of solution, any modelled distribution can be translated into a paracontrolled distribution.*

The remaining problem is that we do not know whether any paracontrolled distribution can be translated into a modelled distribution. Hence in this paper, we take a constructive approach, that is, to construct a situation where the equivalence between PD and MD holds and to decompose any general situation into such primary situation. Our main result is for the construction of such model space.

1.3. **Main result.** Our main result consists of two steps.

(1) *Analytic step*: To show local behaviors of iterated paraproducts.

(2) *Algebraic step*: To construct an algebra representing such local structure.

Section 2 is for the analytic part, and Section 3 is for the algebraic part.

The theory of RS consists of analytic part and algebraic part, but its analysis is not easy to combine with known real analysis. There is still no algebraic theory compatible with the theory of PC. Our result may clarify the relation between two theories and be a first step to combine them.

2. Local estimates

2.1. **Preliminaries.** We recall the Littlewood-Paley theory on $\mathbb{R}^d$. Fix smooth radial functions χ and ρ on $\mathbb{R}^d$ such that,

- $\mathrm{Supp}(\chi) \subset \{|x| < \frac{4}{3}\}$ and $\mathrm{Supp}(\rho) \subset \{\frac{3}{4} < |x| < \frac{8}{3}\}$,
- $\chi(x) + \sum_{j=0}^{\infty} \rho(2^{-j}x) = 1$ for any $x \in \mathbb{R}^d$.

Set $\rho_{-1} := \chi$ and $\rho_j := \rho(2^{-j}\cdot)$ for $j \geq 0$. We define

$$\Delta_j f := \mathcal{F}^{-1}(\rho_j \mathcal{F} f), \quad f \in \mathcal{S}'(\mathbb{R}^d),$$

where $\mathcal{F}$ is the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$. For $\alpha \in \mathbb{R}$, we define the (nonhomogeneous) Besov space

$$\mathcal{C}^\alpha := \{f \in \mathcal{S}'(\mathbb{R}^d)\ ;\ \|f\|_\alpha := \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^\infty} < \infty\}.$$

If $f \in \mathcal{C}^\alpha$, we say that f has a regularity α.

Another way to define a regularity of function is the Taylor expansion. It is well known that two definitions are (almost) equivalent.

Proposition 2.1. *Let $\alpha \in (0,\infty) \setminus \mathbb{N}$. For $f \in \mathcal{C}^\alpha$, define*

$$\Delta_{yx} f = f(y) - \sum_{|k|<\alpha} \frac{(y-x)^k}{k!} \partial^k f(x).$$

Then one has

$$|\Delta_{yx} f| \lesssim \|f\|_\alpha |y-x|^\alpha.$$

2.2. **Main results.** Our aim is to show local behaviors as above for iterated paraproducts. The *Bony's paraproduct* is defined by

$$f \olessthan g = \sum_{i \le j-2} \Delta_i f \Delta_j g.$$

The following estimate is well known.

Proposition 2.2. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then we have the bounded*

$$\|f \olessthan g\|_\beta \lesssim \|f\|_\alpha \|g\|_\beta.$$

The above estimate only says that the Taylor expansion of $f \olessthan g$ is possible up to β. One of our main results describes more detailed behavior of $f \olessthan g$.

Theorem 2.3. *Let $\alpha, \beta > 0$, $\alpha + \beta \notin \mathbb{N}$, $f \in \mathcal{C}^\alpha$, and $g \in \mathcal{C}^\beta$. Define*

$$\Delta_{yx}(f,g) = (f \olessthan g)(y) - \sum_{|k|<\alpha+\beta} \frac{(y-x)^k}{k!} \partial_*^k (f,g)(x) - \sum_{|\ell|<\alpha} \frac{(y-x)^\ell}{\ell!} \partial^\ell f(x) \Delta_{yx} g,$$

where

$$\partial_*^k (f,g) = \partial^k (f,g) - \sum_{k=\ell+m, |\ell|<\alpha, |m|\ge\beta} \frac{k!}{\ell! m!} (\partial^\ell f)(\partial^m f).$$

Then one has

$$|\Delta_{yx}(f,g)| \lesssim \|f\|_\alpha \|g\|_\beta |y-x|^{\alpha+\beta}.$$

This result can be extended to *iterated paraproducts*. For any distribution f on $\mathbb{R}^d$, we set

$$f_i = \Delta_i f, \quad f_{i-} = \sum_{j \le i-2} f_j.$$

For any $f^i \in \mathcal{S}'(\mathbb{R}^d)$, $i = 1, \dots, n$, we define

$$(f^1, \dots, f^n) = \sum_i (f^1, \dots, f^n)_i, \quad (f^1, \dots, f^n)_i = (f^1, \dots, f^{n-1})_{i-} (f^n)_i$$

Note that

$$(f^1, \dots, f^n) \ne (\dots((f^1 \olessthan f^2) \olessthan f^3) \dots \olessthan f^{n-1}) \olessthan f^n,$$

but we conjecture that they have similar local estimates.

Theorem 2.4. *Let $\alpha_1, \dots, \alpha_n > 0$, $\alpha_1 + \cdots + \alpha_n \notin \mathbb{N}$, and let $f^i \in \mathcal{C}^{\alpha_i}$, $i = 1, \dots, n$. Define*

$$\Delta_{yx}(f^1, \dots, f^n) = (f^1, \dots, f^n)(y) - \sum_{|k|<\alpha_1+\cdots+\alpha_n} \frac{(y-x)^k}{k!} \partial_*^k (f^1, \dots, f^n)(x)$$
$$- \sum_{m=1}^{n-1} \sum_{|l|<\alpha_1+\cdots+\alpha_m} \frac{(y-x)^l}{l!} \partial_*^l (f^1, \dots, f^m)(x) \Delta_{yx}(f^{m+1}, \dots, f^n)$$

with some coefficients $\partial_*^l (f^1, \dots, f^m)(x)$ defined continuously from $f^1, \dots, f^n$. Then one has

$$|\Delta_{yx}(f^1, \dots, f^n)| \lesssim \|f^1\|_{\alpha_1} \cdots \|f^n\|_{\alpha_n} |y-x|^{\alpha_1+\cdots+\alpha_n}.$$

2.3. **Proof of the main result.** We give the proof of simple cases. The following result is used as a convergence criterion.

Lemma 2.5. *Let $\{X_{yx} = \sum_{j=1}^{\infty} X_{yx}^j\}_{x,y\in\mathbb{R}^d}$ be a family of absolutely convergent series. Assume that for some $C > 0$, $\alpha > 0$, and $\epsilon > 0$, the bound*

$$|X_{yx}^j| \le C2^{j(\theta-\alpha)}|y-x|^\theta$$

holds for any $\theta \in (\alpha-\epsilon, \alpha+\epsilon)$. Then one has

$$|X_{yx}| \lesssim C|y-x|^\alpha.$$

We show Theorem 2.3 for a simple case. Let $\alpha, \beta \in (0,1)$, $\alpha+\beta > 1$, and $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$. For each Littlewood-Paley blocks and their partial sums, we recall the following estimates.

Lemma 2.6. *Let $\alpha \in (0,1)$ and $f \in \mathcal{C}^\alpha$. Set $f_{i+} := f - f_{i-}$.*

(1) $\|f_i\|_{L^\infty} \le \|f\|_\alpha 2^{-i\alpha}$.
(2) $\|\nabla f_{i-}\|_{L^\infty} \lesssim \|f\|_\alpha 2^{-i(\alpha-1)}$.
(3) $\|f_{i+}\|_{L^\infty} \lesssim \|f\|_\alpha 2^{-i\alpha}$.

Lemma 2.7. *Let $\alpha \in (0,1)$ and $f \in \mathcal{C}^\alpha$. Then for any $\theta \in [1,2]$, we have*

$$|f_i(y) - f_i(x) - (y-x)\cdot\nabla f_i(x)| \lesssim \|f\|_\alpha 2^{-i(\alpha-\theta)}|y-x|^\theta.$$

We start the proof from the usual Taylor expansion

$$T_{yx}(f,g)_i = (f,g)_i(y) - (f,g)_i(x) - (y-x)\cdot\nabla(f,g)_i(x).$$

Taking a sum over i, we have

$$T_{yx}(f,g) = (f \circledcirc g)(y) - (f \circledcirc g)(x) - (y-x)\cdot\nabla(f \circledcirc g)(x),$$

but $\nabla(f \circledcirc g)(x)$ is not defined pointwise because $f \circledcirc g \in \mathcal{C}^\beta$. Now we decompose

$$\begin{aligned} T_{yx}(f,g)_i &= (f_{i-}(y) - f_{i-}(x) - (y-x)\cdot\nabla f_{i-}(x))g_i(y) \\ &\quad + f_{i-}(x)(g_i(y) - g_i(x) - (y-x)\cdot\nabla g_i(x)) \\ &\quad + (y-x)\cdot\nabla f_{i-}(x)(g_i(y) - g_i(x)) =: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned}$$

For (I), since $\alpha - \theta < 0$ if $\theta \in [1,2]$, we have

$$\begin{aligned} |f_{i-}(y) - f_{i-}(x) - (y-x)\cdot\nabla f_{i-}(x)| &\lesssim \|f\|_\alpha \sum_{j\le i-2} 2^{-j(\alpha-\theta)}|y-x|^\theta \\ &\lesssim \|f\|_\alpha 2^{-i(\alpha-\theta)}|y-x|^\theta. \end{aligned}$$

Since $\|g_i\|_{L^\infty} \le \|g\|_\beta 2^{-i\beta}$, we have

$$|(\mathrm{I})| \lesssim \|f\|_\alpha \|g\|_\beta 2^{-i(\alpha+\beta-\theta)}|y-x|^\theta.$$

Similarly, for any $\theta \in [1,2]$,

$$|(\mathrm{III})| \lesssim \|f\|_\alpha \|g\|_\beta 2^{-i(\alpha+\beta-\theta)}|y-x|^\theta.$$

We cannot obtain the required estimates from (II) since we have only $\|f_{i-}\|_{L^\infty} \lesssim \|f\|_\alpha$. Instead, we consider the modified expansion

$$\begin{aligned} \Delta_{yx}(f,g)_i &:= T_{yx}(f,g)_i - f(x)(g_i(y) - g_i(x) - (y-x)\cdot\nabla g_i(x)) \\ &= (\mathrm{I}) + (\mathrm{III}) - f_{i+}(x)(g_i(y) - g_i(x) - (y-x)\cdot\nabla g_i(x)) \\ &=: (\mathrm{I}) + (\mathrm{III}) - (\mathrm{II})'. \end{aligned}$$

Since $\|f_{i+}\|_{L^\infty} \lesssim \|f\|_\alpha 2^{-i\alpha}$, for $\theta \in [1,2]$ we have

$$|\Delta_{yx}(f,g)_i| \lesssim \|f\|_\alpha \|g\|_\beta 2^{-i(\alpha+\beta-\theta)}|y-x|^\theta,$$

so we can apply Lemma 2.5. To show the pointwise convergence of $\sum_i \Delta_{yx}(f,g)_i$, we reorganize the terms as follows.

$$\begin{aligned}&\Delta_{yx}(f,g)_i\\ &= (f_{i-}g_i)(y) - (f_{i-}g_i)(x) - (y-x)\cdot\nabla(f_{i-}g_i)(x) - f(x)(g_i(y) - g_i(x) - (y-x)\cdot\nabla g_i(x))\\ &= (f_{i-}g_i)(y) - (f_{i-}g_i)(x) - f(x)(g_i(y) - g_i(x)) - (y-x)\cdot(\nabla(f_{i-}g_i) - f\nabla g_i)(x).\end{aligned}$$

Since

$$\begin{aligned}\|\nabla(f_{i-}g_i) - f\nabla g_i\|_{L^\infty} &= \|(\nabla f_{i-})g_i - f_{i+}\nabla g_i\|_{L^\infty}\\ &\lesssim \|f\|_\alpha\|g\|_\beta 2^{-i(\alpha-1)}2^{-i\beta} + \|f\|_\alpha\|g\|_\beta 2^{-i\alpha}2^{-i(\beta-1)}\\ &\lesssim \|f\|_\alpha\|g\|_\beta 2^{-i(\alpha+\beta-1)}\end{aligned}$$

and $\alpha+\beta>1$, we see that $\sum_i(\nabla(f_{i-}g_i) - f\nabla g_i)(x)$ absolutely converges for any x. Finally applying Lemma 2.5, we obtain

$$|\Delta_{yx}(f,g)| = \left|\sum_{i=-1}^\infty \Delta_{yx}(f,g)_i\right| \lesssim \|f\|_\alpha\|g\|_\beta|y-x|^{\alpha+\beta}.$$

3. Hopf algebra structure

The algebraic part of RS is a generalization of the *rough path theory*. Their algebraic structures are defined by the Hopf algebra. First we recall the definition of Hopf algebra.

3.1. **Definition of the Hopf algebra.** A unital algebra $(A, m, \mathbf{1})$ is a triplet of a linear space A, a linear map $m : A\otimes A \to A$ satisfying the associativity

$$m(m\otimes \mathrm{Id}_A) = m(\mathrm{Id}_A \otimes m),$$

and a linear map $\mathbf{1} : \mathbb{R} \to A$ satisfying $m(\mathrm{Id}_A\otimes \mathbf{1}) = m(\mathbf{1}\otimes\mathrm{Id}_A) = \mathrm{Id}_A$. (Here we identify $A\otimes\mathbb{R}\simeq\mathbb{R}\otimes A\simeq A$.) Its dual $(C,\Delta,\mathbf{1}^*)$ is a linear space called a *coalgebra*, i.e., C is a linear space, $\Delta : C\to C\otimes C$ (called *coproduct*) is a linear map satisfying the coassociativity

$$(\Delta\otimes\mathrm{Id}_C)\Delta = (\mathrm{Id}_C\otimes\Delta)\Delta,$$

and $\mathbf{1}^* : C\to\mathbb{R}$ is a linear map satisfying $(\mathrm{Id}_C\otimes\mathbf{1}^*)\Delta = (\mathbf{1}^*\otimes\mathrm{Id}_C)\Delta = \mathrm{Id}_C$. If a linear space B has both an algebra structure with $(m,\mathbf{1})$ and a coalgebra structure with $(\Delta,\mathbf{1}^*)$, and Δ and $\mathbf{1}^*$ are algebra homomorphisms, then $(B,m,\Delta,\mathbf{1},\mathbf{1}^*)$ is called a *bialgebra*.

Let $(H,m,\Delta,\mathbf{1},\mathbf{1}^*)$ be a bialgebra. If an algebra homomorphism $\mathcal{A} : H\to H$ exists and satisfies

$$m(\mathrm{Id}_H\otimes\mathcal{A})\Delta = m(\mathcal{A}\otimes\mathrm{Id}_H)\Delta = \mathbf{1}\mathbf{1}^*,$$

then $(H,m,\Delta,\mathbf{1},\mathbf{1}^*,\mathcal{A})$ is called a *Hopf algebra* and $\mathcal{A}$ is called an *antipode*. For a Hopf algebra H, we denote by $\mathrm{Ch}(H)$ the set of all nonzero algebra homomorphisms $f : H\to\mathbb{R}$. We define the product on $\mathrm{Ch}(H)$ by

$$f * g = (f\otimes g)\Delta,\quad f,g\in\mathrm{Ch}(H).$$

By the Hopf algebra structure, we can see that $\mathbf{1}^*$ is a unit of $\mathrm{Ch}(H)$ and $f\mathcal{A}$ is an inverse of f. Hence $(\mathrm{Ch}(H),*)$ is a group.

An example is the word Hopf algebra, which is used in the geometric rough path theory. Another example is the Connes-Kreimer Hopf algebra, which is used in the branched rough path theory. See [8] for example. Our Hopf algebra below is a kind of generalization of the word Hopf algebra for the multidimensional space and for higher regularities.

3.2. **Construction of a Hopf algebra associated with paraproducts.** We start from the word Hopf algebra.

Definition 3.1. *Let S be a finite set. Let $W(S)=\bigcup_{n=0}^{\infty}S^n$ be the set of all "words" generated by "alphabets" S, and let $\mathcal{W}(S)$ be the algebra generated by $W(S)$. We define the coproduct $\mathring{\Delta}:\mathcal{W}(S)\to\mathcal{W}(S)\otimes\mathcal{W}(S)$ by extending the formula*

$$\mathring{\Delta}(\mathbb{F}^1,\dots,\mathbb{F}^n)=\sum_{i=0}^{n}(\mathbb{F}^{i+1},\dots,\mathbb{F}^n)\otimes(\mathbb{F}^1,\dots,\mathbb{F}^i),\quad \mathbb{F}^1,\dots,\mathbb{F}^n\in S,$$

where we interpret "$(\mathbb{F}^{n+1},\dots,\mathbb{F}^n)$" and "$(\mathbb{F}^1,\dots,\mathbb{F}^0)$" as an empty word $\mathbf{1}=\emptyset$.

Remark 3.1. *We do not assume any expansion formula for the product, like the shuffle product*

$$(\mathbb{F}^i,\mathbb{F}^j)\sqcup\!\sqcup(\mathbb{F}^k)=(\mathbb{F}^i,\mathbb{F}^j,\mathbb{F}^k)+(\mathbb{F}^i,\mathbb{F}^k,\mathbb{F}^j)+(\mathbb{F}^k,\mathbb{F}^i,\mathbb{F}^j).$$

Such relation is used in the geometric rough path theory. This is why our Hopf algebra is "a kind of" generalization of the word Hopf algebra.

Proposition 3.2. *$(\mathcal{W}(S),\mathring{\Delta})$ is a Hopf algebra, i.e., there exists an antipode $\mathring{\mathcal{A}}$.*

Now we define the homogeneity for all words $\tau\in W(S)$. We define $|\mathbf{1}|=0$. We assume that each $\mathbb{F}\in S$ is given a homogeneity $|\mathbb{F}|>0$. Then we define

$$|(\mathbb{F}^1,\dots,\mathbb{F}^n)|:=\sum_{i=1}^{n}|\mathbb{F}^i|.$$

We add the notions of "derivatives" and "polynomials" to $\mathcal{W}(S)$. Such structures did not appear in the rough path theory, because there we need not consider a regularity greater than 1.

Definition 3.2. *Fix $d\in\mathbb{N}$. We define*

$$\tilde{W}(S)=\{\tau_m:=(\tau,m)\in W(S)\times\mathbb{N}^d;|\tau|>|m|\}.$$

Moreover, we introduce the Taylor polynomials $T=\{\mathbb{X}_1,\dots,\mathbb{X}_d\}$. We define the homogeneity by

$$|\tau_m|:=|\tau|-|m|,\quad |\mathbb{X}_j|=1.$$

Let $\mathcal{H}(S)$ be the algebra generated by $\tilde{W}(S)\cup T$.

We define the differentiation maps ∂^m on $\mathcal{H}(S)$.

Definition 3.3. *We define the map $\partial_i:\mathcal{H}(S)\to\mathcal{H}(S)$ for $i=1,\dots,d$ by*

$$\partial_i\tau_m=\tau_{m+e_i},\quad e_i=(0,\dots,0,\overset{i}{1},0,\dots,0),$$

for $\tau\in W(S)$ and $m\in\mathbb{N}^d$,

$$\partial_i\mathbb{X}_j=0$$

for $\mathbb{X}_j\in T$, and the Leibniz rule

$$\partial_i(\tau\sigma)=(\partial_i\tau)\sigma+\tau(\partial_i\sigma).$$

For each mutiindex $m=(m_1,\dots,m_d)\in\mathbb{N}^d$, we write $\partial^m=\partial_1^{m_1}\cdots\partial_d^{m_d}$.

Proposition 3.3. *We define the coproduct $\mathring{\Delta}$ on $\mathcal{H}(S)$ as an extension of the coproduct $\mathring{\Delta}$ on $\mathcal{W}(S)$, by imposing*

$$\mathring{\Delta}\partial_i=(\partial_i\otimes\mathrm{Id}+\mathrm{Id}\otimes\partial_i)\mathring{\Delta},$$
$$\mathring{\Delta}\mathbb{X}_i=\mathbb{X}_i\otimes\mathbf{1}+\mathbf{1}\otimes\mathbb{X}_i,$$

$$\mathring{\Delta}(\tau\sigma) = (\mathring{\Delta}\tau)(\mathring{\Delta}\sigma).$$

Then $(\mathcal{H}(S), \mathring{\Delta})$ *is a Hopf algebra. We denote by* $\mathring{\mathcal{A}}$ *again the antipode of* $\mathcal{H}(S)$.

To obtain the algebraic structure associated with paraproducts, we define the "twisted" coproduct.

Definition 3.4. *We define the new coproduct* Δ *by*

$$\Delta = \exp(\mathbb{X} \otimes \partial)\mathring{\Delta} := \exp(\sum_i \mathbb{X}_i \otimes \partial_i)\mathring{\Delta},$$

where we use the notation $\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}$. *Since* $\partial^m \tau = 0$ *for given* $\tau \in \mathcal{H}(S)$ *and a large* $m \in \mathbb{N}^d$, $\Delta\tau$ *is actually a finite series.*

Proposition 3.4. $(\mathcal{H}(S), \Delta)$ *is a Hopf algebra. Its antipode* $\mathcal{A}$ *is given by*

$$\mathcal{A} = \exp(-\mathbb{X}\partial)\mathring{\mathcal{A}} := \exp(-\sum_i \mathbb{X}_i \partial_i)\mathring{\mathcal{A}}.$$

3.3. **Models and modelled distributions on $\mathcal{H}(S)$.** We interpret the result in Section 2 in the algebraic language as above. We fix the functions

$$f_1 \in \mathcal{C}^{\alpha_1}, f_2 \in \mathcal{C}^{\alpha_2}, \dots, f_n \in \mathcal{C}^{\alpha_n}$$

for $\alpha_i > 0$ such that,

$$\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k} \notin \mathbb{N}$$

for any $i_1 < i_2 < \cdots < i_k$. Then we consider the set $S = \{\mathbb{F}^1, \dots, \mathbb{F}^n\}$ and homogeneities $|\mathbb{F}^i| = \alpha_i$.

Proposition 3.5. *Define the family* $\{\mathsf{g}_x\}_{x\in\mathbb{R}^d} \subset \mathrm{Ch}(\mathcal{H}(S))$ *by* $\mathsf{g}_x(\mathbb{X}_j) = x_j$ *and*

$$\mathsf{g}_x((\mathbb{F}^{i_1}, \dots, \mathbb{F}^{i_k})_m) = \partial_*^m(f_{i_1} \dots, f_{i_k})(x),$$

where $\partial_*^m(f_{i_1} \dots, f_{i_k})$ *is a function continuously depending on* $f_{i_1} \dots, f_{i_k}$ *defined in Section 2. Then we have*

$$|(\mathsf{g}_y * \mathsf{g}_x^{-1})(\tau)| \lesssim |y - x|^{|\tau|}$$

for any $\tau \in \tilde{W}(S) \cup T$. *In other words,* g *is a* model *on* $\mathcal{H}$ *(see Section 1).*

Theorem 3.6. *For given* $g \in \mathcal{C}^\beta$ *with* $\beta > 0$ *such that* $\beta + \alpha_1 + \cdots + \alpha_n \notin \mathbb{N}$, *we define the function* $\mathbb{G} : \mathbb{R}^d \to \mathcal{H}(S)$ *by*

$$\mathbb{G} = \sum \frac{1}{k!}\partial_*^k(g, f_1, \dots, f_n)\mathbb{X}^k + \sum \frac{1}{k_1!}\partial_*^{k_1}(g, f_1, \dots, f_{n-1})\mathbb{X}^{k_1}\mathbb{F}^n$$
$$+ \cdots + \sum \frac{1}{k_n!}(\partial^{k_n} g)\mathbb{X}^{k_n}(\mathbb{F}^1, \dots, \mathbb{F}^n).$$

Then $\mathbb{G}$ *is a* $(\beta + \alpha_1 + \cdots + \alpha_n)$*-class* modelled distribution, *i.e.,*

$$\|\mathbb{G}_y - (\mathsf{g}_y * \mathsf{g}_x^{-1})\mathbb{G}_x\|_\theta \lesssim |y - x|^{\beta+\alpha_1+\cdots+\alpha_n-\theta}$$

for any $\theta < \beta+\alpha_1+\cdots+\alpha_n$, *where* $\|\cdot\|_\theta$ *denotes the (equivalent) norm on the finite dimensional space* $\mathcal{H}(S)_\theta := \langle \tau \in \tilde{W}(S) \cup T; |\tau| = \theta \rangle$.

The above theorem implies the equivalence between the modelled distribution and the paracontrolled distribution in the space $\mathcal{H}(S)$.

Theorem 3.7. *If a function* $\mathbb{G} : \mathbb{R}^d \to \mathcal{H}(S)$ *of the form*

$$\mathbb{G} = \sum_{|k|<\beta+\alpha_1+\cdots+\alpha_n} g_k \mathbb{X}^k + \sum_{|k_1|<\beta+\alpha_1+\cdots+\alpha_{n-1}} g_{k_1,n} \mathbb{X}^{k_1} \mathbb{F}^n \\ + \cdots + \sum_{|k_n|<\beta} g_{k_n,1,\dots,n} \mathbb{X}^{k_n} (\mathbb{F}^1, \dots, \mathbb{F}^n)$$

is a $(\beta + \alpha_1 + \cdots + \alpha_n)$*-class modelled distribution, then there exist functions*

$$\begin{aligned} g^{\#}_{1\dots n} &\in \mathcal{C}^{\beta}, \\ g^{\#}_{2\dots n} &\in \mathcal{C}^{\beta+\alpha_1}, \\ &\cdots \\ g^{\#}_{n} &\in \mathcal{C}^{\beta+\alpha_1+\cdots+\alpha_{n-1}}, \\ g^{\#} &\in \mathcal{C}^{\beta+\alpha_1+\cdots+\alpha_n}, \end{aligned}$$

such that, we have the formula

$$g_{k,j\dots n} = \frac{1}{k!} \{ \partial^k_* (g^{\#}_{1\dots n}, f_1, \dots, f_{j-1}) + \partial^k_* (g^{\#}_{2\dots n}, f_2, \dots, f_{j-1}) \\ + \cdots + \partial^k_* (g^{\#}_{(j-1)\dots n}, f_{j-1}) + \partial^k g^{\#}_{j\dots n} \}.$$

Namely, the modelled distribution $\mathbb{G}$ *is determined by only* $(n+1)$ *independent functions.*

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