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Non-expressibility of a class of finite graphs

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1 Introduction

A graph is an R-structure in which R is irreflexive and symmetric. In this article, we will use a compactness argument to examine the expressibility of a class of finite graphs. First let us explain a simple example as follows: Let $\mathcal{C}$ be the class of all finite circles, i.e., all graphs H of the form:

- $H = \{h_1, \dots, h_n\}$;
- $R^H = \{h_1h_2, h_2h_3, \dots, h_{n-1}h_n, h_nh_1\}$, where hk denotes $(h,k),(k,h)$.

Then $\mathcal{C}$ is not an elementary class, because finiteness cannot be represented by a sentence or even by a set of sentences. Furthermore, there is no R-sentence φ such that, for any finite graphs G,

$$G \models \varphi \iff G \in \mathcal{C}.$$

For showing this, we can use a compactness argument: Suppose that there were such a sentence φ. Let T be the theory consisting of the following sentences:

1. Graph axioms;
2. Every node has exactly two neighbors;
3. There is no (finite) cycles.

Since every finite part of T is satisfied by a graph in $\mathcal{C}$, by compactness, there is a countable infinite graph $G_0 \models T \cup \{\varphi\}$. By extending G_0, if necessary, we can assume G_0 is a disjoint union of countably many $\mathbb{Z}$-chains. Notice that every graph consisting of two circles does not belong to $\mathcal{C}$. So, again by compactness, there is a countable infinite graph $G_1 \models T \cup \{\neg\varphi\}$. G_1 is also assumed to be a disjoint union of countably many $\mathbb{Z}$-chains. So, we must conclude $G_0 \cong G_1$, a contradiction.

In this paper, concerning finite graphs, we consider a different type of (non-)expressibility.

2 Preliminaries

Let $L = \{R(*,*)\}$ and $L^* = L \cup \{X_i(*) : i < n\}$, where R is a binary predicate symbol, and X_i's are unary predicate symbols. Let T be a finite set of L-sentences and $\varphi(X_0, \dots, X_{n-1})$ an L^*-sentence.

Definition 1. $PC_{fin}(\varphi, T)$ is the class of L-reducts of finite models of $T \cup \{\varphi\}$. If T is the axiom for graphs, we simply write $PC_{fin}(\varphi)$ for $PC_{fin}(\varphi, T)$. PC stands for 'pseudo elementary class.'

Example 2.
1. Let $\mathcal{C}$ be the class of all non-connected finite graphs. Then, there is a sentence φ such that $PC_{fin}(\varphi) = \mathcal{C}$. Let φ be the sentence asserting that (i) both X_0 and $\neg X_0$ are non-empty, and (ii) there is no edge between X_0 and $\neg X_0$. Then clearly φ satisfies the required condition.

2. Let $\mathcal{C}$ be the class of all finite graphs with a cycle. Then Then, there is a sentence φ such that $PC_{fin}(\varphi) = \mathcal{C}$.

Now another important point will be explained below. In the structure $\mathbb{N}$, by a coding method, finite sets are represented. In other words, $\mathbb{N}$ can be considered as a model of finite set theory. So, we assume $\mathbb{N} = (\mathbb{N}, 0, 1, +, \cdot, <, \in)$, where $\in$ is the membership relation. Finite graphs are objects in $\mathbb{N}$. Let G be a finite graph with the code a_G, i.e. $G = \{g \in \mathbb{N} : \mathbb{N} \models g \in a_G\}$. The connectedness of G can be expressed by a sentence in $\mathbb{N}$ as follows: G is connected $\iff$ there is a coded function $f : [0, n] \to G$ such that (i) $\mathrm{ran}(f) = G$, and (ii) $R(f(i), f(i+1))$ $(\forall i < n)$.

Let $\mathbb{N}^* \succ \mathbb{N}$ be a recursively saturated countable model. In $\mathbb{N}^*$, a coded set $\{x \in \mathbb{N}^* : x \in a\}$, where $a \in \mathbb{N}^*$, is not necessarily finite. Let $con(x)$ be the formula expressing (in $\mathbb{N}$) that the graph coded by x is connected, and let $a \in \mathbb{N}^*$ be an element with $\mathbb{N}^* \models con(a)$. The graph coded by a is not connected in general, although it is connected in the sense of $\mathbb{N}^*$.

3 Non-expressibility

As an application of compactness argument to finite graphs, we show the following proposition, which is due to Fagin [1].

Proposition 3. *Let $\mathcal{C}$ be the class of all finite connected graphs. Then there is no L^*-sentence $\varphi = \varphi(X_0, \dots, X_{n-1})$ with $\mathcal{C} = PC_{fin}(\varphi)$.*

Sketch of Proof. A more detailed proof of a more general result be given in our forthcoming paper. Suppose that there were a sentence $\varphi(X_0, \dots, X_{n-1})$ with $\mathcal{C} = PC_{fin}(\varphi)$. Let C_n be the circle graph with the universe $[0, n-1]$ and the edges $R(i, i+1)$ $(i < n-1)$ and $R(n-1, 0)$. Now we work in $\mathbb{N}^*$. Let n^* be a nonstandard number, and let $G = C_{n^*}$. Since $C_n \models \varphi$ for all n, we have some coded sets $D_0, \dots, D_{n-1}$ such that

$$G \models \varphi(D_0, \dots, D_{n-1}). \tag{1}$$

By the recursive saturation, there are two points $a < b \in \mathbb{N}^*$ such that $\mathrm{tp}_{\mathbb{N}^*}(a/n^*, d_0, \dots, d_{n-1}) = \mathrm{tp}_{\mathbb{N}^*}(b/n^*, d_0, \dots, d_{n-1})$, where d_i is the code of D_i. We define a new graph G' by:

1. The universe of G' is the same as G, hence $|G'| = |G| = [0, n^* - 1]$;
2. $R^{G'} = R^G \setminus \{a(a+1), b(b+1)\} \cup \{a(b+1), b(a+1)\}$

Each of G and G' is a disjoint union of $\mathbb{Z}$-chains with coloring by X_i's. By our construction, G and G' are definable (and hence both are coded in $\mathbb{N}^*$). Moreover, they are isomorphic as $\{R, X_0, \dots, X_{n-1}\}$-structures, since they have the same $\mathbb{Z}$-chains (counting multiplicity) with coloring. Hence we have:

Claim A. *$G \cong_{\{R, X_0, \dots, X_{n-1}\}} G'$. This isomorphism, say σ, is not definable in $\mathbb{N}^*$. But, each D_i is σ-invariant.*

On the other hand, G' is not connected in the sense of $\mathbb{N}^*$ (in fact, it is a disjoint union of two circles), hence we have:

Claim B. *$G' \models \neg\varphi(B_0, \dots, B_{n-1})$, for all coded sets B_i's.*

The two claims above together with (1) yield a contradiction. □

References

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[4] Our forthcoming paper.