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An approximation of Keisler measure by using Morley sequences

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Abstract

We discuss on the Vapnik-Chervonenkis inequality in a stable structure following [1] and [2]. Some proofs may be modified but the results are essentially the same.

1 Introduction and Preliminaries

In this note we are interested in the Vapnik-Chervonenkis inequality with model theoretic approach, especialy with assuming stability of structures or formulas. The mathematical result is already known and pointed out in [1], and its essential part is given in [2] and [4]. Here we follow them however some proofs may be easier to understand.

Let X be a non-empty set and let $\mathcal{B} \subset \mathcal{P}(X)$ be a σ-algebra (boolean algebra, respectively) with $X \in \mathcal{B}$. A (finitely additive, resp.) probability space on $\mathcal{B}$ is a tuple $(X, \mathcal{B}, \mu)$ such that $\mu : \mathcal{B} \to [0, 1]$ is a σ-additive (finitely additive, resp.) function with $\mu(X) = 1$.

For a given sequence $\bar{a} = a_0, ..., a_{n-1} \in X$ and $B \in \mathcal{B}$, we put $\mathrm{Av}(B, \bar{a}) = |\{k < n \mid a_k \in B\}|/n$. We consider $\mathrm{Av}(B, \bar{a})$ as an approximation of $\mu(B)$ by $\bar{a}$.

For $\mathcal{C} \subset \mathcal{B}$ and $\epsilon > 0$, we say $\bar{a} \in X$ an ϵ-approximation of μ on $\mathcal{C}$ if $|\mathrm{Av}(C, \bar{a}) - \mu(C)| < \epsilon$ for all $C \in \mathcal{C}$. In general, there is no ϵ-approximation of μ on $\mathcal{C}$ (for example consider the case $\mathcal{C} = \mathcal{P}(X)$), however it is well known that if X is a probability space (that is, not finitely additive but σ-additive), $\mathcal{C}$ is dependent (equivalently, $\mathcal{C}$ has finite Vapnick-Chervonenkis(VC)-dimension) and well-behaved (see Section 2 for the definition), then for all $\epsilon, \delta > 0$ there is $n \in \omega$ and $B \subset X^n$ with $\mu(B) > 1 - \delta$ such that for all $\bar{a} = (a_0, ..., a_{n-1}) \in B$, μ is ϵ-approximated by $\bar{a}$ on $\mathcal{C}$. This result is called the Vapnick-Chervonenkis inequality. (See [2] for a detail.)

It is also known that there are counter examples for not well-behaved case.

Example 1. Let $X = \omega_1$ and $\mathcal{C} = \{[a, \infty) \subset \omega_1 \mid a \in \omega_1\}$. Let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{C}$. Consider $\mu : \mathcal{B} \to \{0, 1\}$ such that $\mu(A) = 1$ if and only if $A \subset \omega_1$ is unbounded. It is easy to check μ is σ-additive and $\mathcal{C}$ has finite VC-dimension. However, every finite $\bar{a} \in X$ cannot $\frac{1}{2}$-approximate since there is $C \in \mathcal{C}$ with $\bar{a} \cap C = \emptyset$. In this case, $\mathcal{C}$ is not well-behaved.

Now we introduce model theoretic setting. Let T be an L-theory and let $M \models T$. Let $\varphi(x, y)$ be an L-formula with $|x| = |y| = 1$. (Here we assume x and y are singleton, but it is easy generalizing them to tuples.)

Definition 2.
1. A Keisler measure μ on M is a finitely additive probability measure on the set $\mathrm{Def}_1(M)$ of M-definable subsets of M.
2. Let μ be a Keisler measure on M. We say μ is ϵ-approximated by $\bar{a} \in M$ with respect to φ over $B \subset M$ if $|\mathrm{Av}(\varphi(M, b), \bar{a}) - \mu(\varphi(M, b))| < \epsilon$ for all $b \in B$.

Remark 3. Let μ be a $\{0, 1\}$-valued Keisler measure on M. Then $p_\mu(x) = \{\varphi(x, a) \mid \mu(\varphi(x, a)) = 1\}$ is a complete type over M. Conversely, if $p(x) \in S_1(M)$ then by defining $\mu_p(\varphi(x, a)) = 1$ if $\varphi(x, a) \in p$ we can see μ_p is a Keisler measure on M. Hence, Keisler measures on M can be considered as a generalization of types.

2 Approximating Keisler measures by types

Let $(X, \mathcal{B}, \mu)$ be a probability space. For $A \in \mathcal{B}$, let $g_{A,k} : X^k \to [0, 1]$ be defined by $g_{A,k}(\bar{a}) = |\mathrm{Av}(A, \bar{a}) - \mu(A)|$ and let $h_{A,k} : X^{2k} \to [0, 1]$ be defined by $h_{A,k}(\bar{a}, \bar{b}) = |\mathrm{Av}(A, \bar{a}) - \mathrm{Av}(A, \bar{b})|$.

Definition 4. A class $\mathcal{C} \subset \mathcal{B}$ is said to be well-behaved if $\sup_{A \in \mathcal{C}} g_{A,k}$ and $\sup_{A \in \mathcal{C}} h_{A,k}$ are μ-measurable.

By the definition, if $\mathcal{C}$ is countable then it is well-behaved.

Example 5. Consider $\mathcal{C}$ in Example 1. Then $\mathcal{C}$ is not well-behaved. Indeed, if $\sup_{A\in\mathcal{C}} g_{A,k}$ and $\sup_{A\in\mathcal{C}} h_{A,k}$ are μ-measurable, then so is $D = \{(a,b) \in \omega_1^2 \mid a < b\}$. However, for all $a \in \omega_1$ $D_a = \{b \in \omega \mid (a,b) \in D\}$ has measure 1, so by Fubini's theorem $\mu(D)$ must be 1. On the other hand, for all $b \in \omega_1$ $D_b = \{a \in \omega_1 \mid (a,b) \in D\}$ has measure 0. Again, by Fubini's theorem, $\mu(D)$ must be 0. This is a contradiction.

In the VC-inequality, we have to assume $\mathcal{C} \subset \mathcal{B}$ is well-behaved. However, if we consider Keisler measures in a saturated structure, then the assumption is not needed. In fact, we can prove the following.

Fact 6. [2, Lemma 4.8] Let $M^* \succ M$ be an $|M|$-saturated elementary extension of M. Take a dependent L-formula $\varphi(x,y)$ and put $\mathcal{C} = \{\varphi(M^*, a) \mid a \in M\}$. Suppose that μ is a Keisler measure on M^* and let $\epsilon, \delta > 0$. Then there is $k \in \omega$ and μ-measurable $B \subset (M^*)^k$ with $\mu(B) > 1 - \delta$ such that every $\bar{b} = (b_0, ..., b_{k-1}) \in B$ ϵ-approximates μ on $\mathcal{C}$.

On the other hands, every Keisler measure on M can be naturally extended to a σ-additive probability measure on M^* (see [3] for example). Therefore we conclude that:

Corollary 7. Let M be an L-structure and $\varphi(x,y)$ be a dependent L-formula. Let $\mathcal{C} = \{\varphi(M,a) \mid a \in M\}$. Then for any Keisler measure μ on M and $\epsilon > 0$ there are types $p_0(x), ..., p_{k-1}(x) \in S(M)$ such that $|\mu(C) - (\mu_{p_0}(C) + ... + \mu_{p_{k-1}}(C))/k| < \epsilon$ for all $C \in \mathcal{C}$.

Hence we can say that every Keisler measure can be approximated by $\{0,1\}$-valued Keisler measures.

3 Approximating $\{0,1\}$-valued Keisler measures by Morley sequences

In this section we fix a model $M \models T$, a stable L-formula $\varphi(x,y)$ and a complete type $p(x)$ in $S_\varphi(M)$. Here $S_\varphi(X)$ is the class of sets $p(x)$ of formulas of the form $\varphi(x,a)$ or $\neg\varphi(x,a)$ with $a \in M$ such that $p(x)$ is consistent and maximal. It is not essential but for simplicity we assume M is countable. If we consider $p(x) \in S(M)$ instead of $S_\varphi(M)$, it may be necessary that M is ω-saturated.

In what follows we construct an infinite sequence J which approximates μ with respect to φ over M. In addition, we can construct J as a Morley sequence (with respect to φ), though we don't show it, however one can easily notice how to do that from the proof. The idea of this construction is essentially in [4, Lemma 2.2].

Let $A_0 \subset_{\text{fin}} A_1 \subset_{\text{fin}} \cdots \subset M$ be an increasing sequence of finite sets with $\bigcup_n A_n = M$. Let $c_n \in M$ be any realization of $p|A_n$ and put $I = \{c_n \mid n \in \omega\}$. (Since $p|A_n$ is finite, M contains such solutions.)

Proposition 8. For any infinite subsequence $I' \subset I$ there is an infinite subsequence $J \subset I'$ such that for any $\epsilon > 0$ there is $m \in \omega$ such that for any subsequence $\bar{c}$ of J with $|\bar{c}| = m$ ϵ-approximates μ_p for φ over M.

Proof. It is easy I' satisfies the same condition assumed for I. Hence without loss of generality we assume $I' = I$. We first define the subsequence $J = \{d_k \mid k \in \omega\} \subset I'$. Take c^* in a big model M^* such that $c^* \models p(x)$ and put $d_{-1} = c^*$. Let $d_0 = c_0$ and suppose that $d_i = c_{n_i}$ is defined for all $i < k$. Let n_k be the minimum natural number n such that for all $b \in M$ there are $a, a' \in A_n$ such that $M^* \models \varphi(d_i, b) \leftrightarrow \varphi(d_i, a)$ for all $-1 \leq i < k$. Since the number of φ-types over $d_{-1} \cdots d_{k-1}$ is finite, we can find such A_n. Then put $n_k = n$ and $d_k = c_{n_k}$.

Now we prove J satisfies the required condition. Since $\varphi(x, y)$ is stable, there is $l \in \omega$ such that there is no a_i, b_j $(i, j < l)$ such that $\varphi(a_i, b_j)$ if and only if $i < j$ for all $i, j < l$. Let $m > l/\epsilon$ and $\bar{d} \subset J$ with $|\bar{d}| = m$. Since the general cases are similar, we assume that $\bar{d} = d_0 ... d_{m-1}$. We show that $|\mathrm{Av}(\varphi(M, a), \bar{d}) - \mu(\varphi(M, a))| < \epsilon$ for all $a \in M$. First we prove that:

Claim A. If $|\{i < m \mid d_i \in \varphi(M, a)\}| \geq l$ then $M^* \models \varphi(c^*, a)$.

Let $d_{i_0}, ..., d_{i_l} \in \{i < m \mid d_i \in \varphi(M, a)\}$ and suppose $M^* \models \neg\varphi(c^*, a)$. Then for each $l' < l$ there is $a_{l'} \in A_{i_{l'}}$ such that $\varphi(d_{i_s}, a_t)$ if and only if $s < t$ for all $s, t < l$, by the definition of J. This contradicts to the assumption of stability of φ.

Similarly, we can prove:

Claim B. If $|\{i < m \mid d_i \in \neg\varphi(M, a)\}| \geq l$ then $M^* \models \neg\varphi(c^*, a)$.

By using the above claims, we know that:

1. If $\varphi(x,a) \in p(x)$ then $\mathrm{Av}(\varphi(M,a),\bar{d}) \geq 1 - l/m$.
2. If $\varphi(x,a) \notin p(x)$ then $\mathrm{Av}(\varphi(M,a),\bar{d}) \leq l/m$.

Since $m > l/\epsilon$, we conclude that $l/m < \epsilon$. □

References

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