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# ON THE AUTOMORPHISM GROUP OF A HRUSHOVSKI'S PSEUDOPLANE ASSOCIATED TO 5/8

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ABSTRACT. The automorphism group of Hrushovski's pseudoplane associated to 5/8 is a simple group.

Hrushovski's construction, automorphism group, simple group
03C10, 03C13, 03C25, 03C30

## 1. INTRODUCTION

D. Evans, Z. Ghadernezhad, and K. Tent have shown that the automorphisms groups of certain countable structures obtained using the Hrushovski amalgamation method are simple groups. Among them, there are generic structure of $\mathbf{K}_f$ for certain $f$ with coefficient $1/2$ for the predimension function. They conjectured that the automorphism group of the generic structure of $\mathbf{K}_f$ is a simple group if the coefficient of the predimension function for $\mathbf{K}_f$ is rational.

In this paper, we show that the automorphism group of Hrushovski's original pseudoplane associated to a predimension function with coefficient 5/8 is a simple group. Actually, we prove a sufficient condition given by Evans, Ghadernezhad, and Tent. We are going to treat the general rational cases in another paper [14].

We essentially use notation and terminology from Baldwin-Shi [3] and Wagner [15]. We also use some terminology from graph theory [4].

For a set $X$, $[X]^n$ denotes the set of all subsets of $X$ of size $n$, and $|X|$ the cardinality of $X$.

We recall some of the basic notions in graph theory we use in this paper. These appear in [4]. Let $G$ be a graph. $V(G)$ denotes the set of vertices of $G$ and $E(G)$ the set of edges of $G$. $E(G)$ is a subset of $[V(G)]^2$. $|G|$ denotes $|V(G)|$. The *degree* of a vertex $v$ is the number of edges at $v$. A vertex of degree 1 is a *leaf*. $G$ is a *path* $x_0x_1\dots x_k$ if $V(G)=\{x_0,x_1,\dots,x_k\}$ and $E(G)=\{x_0x_1,x_1x_2,\dots,x_{k-1}x_k\}$ where the $x_i$ are all distinct. $x_0$ and $x_k$ are *ends* of $G$. The number of edges of a path is its *length*. A path of length 0 is a single vertex. $G$ is a *cycle* $x_0x_1\dots x_{k-1}x_0$ if $k\geq 3$, $V(G)=\{x_0,x_1,\dots,x_{k-1}\}$ and $E(G)=\{x_0x_1,x_1x_2,\dots,x_{k-2}x_{k-1},x_{k-1}x_0\}$ where the $x_i$ are all distinct. The number of edges of a cycle is its *length*. A non-empty graph $G$ is *connected* if any two of its vertices are linked by a path in $G$. A *connected component* of a graph $G$ is a maximal connected subgraph of $G$. A *forest* is a graph not containing any cycles. A *tree* is a connected forest.

To see a graph $G$ as a structure in the model theoretic sense, it is a structure in language $\{E\}$ where $E$ is a binary relation symbol. $V(G)$ will be the universe, and $E(G)$ will be the interpretation of $E$. The language $\{E\}$ will be called *the graph language*.

Suppose $A$ is a graph. If $X\subseteq V(A)$, $A|X$ denotes the substructure $B$ of $A$ such that $V(B)=X$. If there is no ambiguity, $X$ denotes $A|X$. We usually follow this convention. $B\subseteq A$ means that $B$ is a substructure of $A$. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is the same as $A[X]$ in Diestel's book [4].

We say that $X$ is *connected* in $A$ if $X$ is a connected graph in the graph theoretical sense [4]. A maximal connected substructure of $A$ is a *connected component* of $A$.

Let $A$, $B$, $C$ be graphs such that $A\subseteq C$ and $B\subseteq C$. $AB$ denotes $C|(V(A)\cup V(B))$, $A\cap B$ denotes $C|(V(A)\cap V(B))$, and $A-B$ denotes $C|(V(A)-V(B))$. If $A\cap B=\emptyset$, $E(A,B)$ denotes the set of edges $xy$ such that $x\in A$ and $y\in B$. We put $e(A,B)=|E(A,B)|$. $E(A,B)$ and $e(A,B)$ depend on the graph in which we are working.

Let $D$ be a graph and $A$, $B$, and $C$ substructures of $D$. We write $D=B\otimes_A C$ if $D=BC$, $B\cap C=A$, and $E(D)=E(B)\cup E(C)$. $E(D)=E(B)\cup E(C)$ means that there are no edges between $B-A$ and $C-A$. $D$ is called a *free amalgam of $B$ and $C$ over $A$*. If $A$ is empty, we write $D=B\otimes C$, and $D$ is also called a *free amalgam of $B$ and $C$*.

**Definition 1.1.** Let $\alpha$ be a real number such that $0<\alpha<1$.

(1) For a finite graph $A$, we define a predimension function $\delta$ by $\delta(A)=|A|-\alpha|E(A)|$.
(2) Let $A$ and $B$ be substructures of a common graph. Put $\delta(A/B)=\delta(AB)-\delta(B)$.

**Definition 1.2.** Let $A$ and $B$ be graphs with $A\subseteq B$, and suppose $A$ is finite.

$A \le B$ if whenever $A \subseteq X \subseteq B$ with $X$ finite then $\delta(A) \le \delta(X)$.
$A < B$ if whenever $A \subsetneq X \subseteq B$ with $X$ finite then $\delta(A) < \delta(X)$.
We say that $A$ is *closed* in $B$ if $A < B$.
$A <^{-} B$ if whenever $A \subsetneq X \subsetneq B$ with $X$ finite then $\delta(A) < \delta(X)$.

Let $\mathbf{K}_\alpha$ be the class of all finite graphs $A$ such that $\emptyset < A$.
Some facts about $<$ appear in [3, 15, 16]. Some proofs are given in [12].

**Fact 1.3.** *If $A < B \subseteq D$ and $C \subseteq D$ then $A \cap C < B \cap C$.*

**Fact 1.4.** *Let $D = B \otimes_A C$.*
(1) $\delta(D/A) = \delta(B/A) + \delta(C/A)$.
(2) *If $A < C$ then $B < D$.*
(3) *If $A < B$ and $A < C$ then $A < D$.*

**Fact 1.5.** (1) *Let $A$, $B$, $C$ and $D$ be graphs with $D = B \otimes C$ and $A \subseteq D$. Then $\delta(D/A) = \delta(B/A \cap B) + \delta(C/A \cap C)$.*
(2) *Let $D$ be a graph and $A$ a substructure of $D$. Let $\{D_1, D_2, \dots, D_k\}$ be the set of all connected components of $D$ where the $D_i$ are all distinct. Then*

$$\delta(D/A) = \sum_{i=1}^{k} \delta(D_i/A \cap D_i).$$

Let $B$, $C$ be graphs and $g : B \to C$ a graph embedding. $g$ is a *closed embedding* of $B$ into $C$ if $g(B) < C$. Let $A$ be a graph with $A \subseteq B$ and $A \subseteq C$. $g$ is a *closed embedding over* $A$ if $g$ is a closed embedding and $g(x) = x$ for any $x \in A$.

In the rest of the paper, $\mathbf{K}$ denotes a class of finite graphs closed under isomorphisms.

**Definition 1.6.** Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *amalgamation property* if for any finite graphs $A, B, C \in \mathbf{K}$, whenever $g_1 : A \to B$ and $g_2 : A \to C$ are closed embeddings then there is a graph $D \in \mathbf{K}$ and closed embeddings $h_1 : B \to D$ and $g_2 : C \to D$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs $A, B$, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

A countable graph $M$ is a *generic structure* of $(\mathbf{K}, <)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and $A$ is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B < M$.
(2) If $A \subseteq M$ then $A \in \mathbf{K}$.
(3) For any $A$, $B \in \mathbf{K}$, if $A < M$ and $A < B$ then there is a closed embedding of $B$ into $M$ over $A$.

Let $A$ be a finite structure of $M$. There is a smallest $B$ satisfying $A \subseteq B < M$, written $\mathrm{cl}(A)$. The set $\mathrm{cl}(A)$ is called the *closure* of $A$ in $M$.

**Fact 1.7** ([3, 15, 16])**.** *Let* $(\mathbf{K}, <)$ *be an amalgamation class. Then there is a generic structure of* $(\mathbf{K}, <)$*. Let M be a generic structure of* $(\mathbf{K}, <)$*. Then any isomorphism between finite closed substructures of M can be extended to an automorphism of M.*

**Definition 1.8.** Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *free amalgamation property* if whenever $D = B \otimes_A C$ with $B, C \in \mathbf{K}$, $A < B$ and $A < C$ then $D \in \mathbf{K}$.

By Fact 1.4 (2), we have the following.

**Fact 1.9.** *Let* $\mathbf{K}$ *be a subclass of* $\mathbf{K}_\alpha$*. If* $(\mathbf{K}, <)$ *has the free amalgamation property then it has the amalgamation property.*

**Definition 1.10.** Let $\mathbb{R}^+$ be the set of non-negative real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave (convex upward) unbounded function. Assume that $f(0) = 0$, and $f(1) \leq 1$. We assume that $f$ is piecewise smooth. $f'_+(x)$ denotes the right-hand derivative at $x$. We have $f(x+h) \leq f(x) + f'_+(x)h$ for $h > 0$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta(B) \geq f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic structure of $(\mathbf{K}_f, <)$ has a countably categorical theory [16].

A graph $X$ is *normal to* $f$ if $\delta(X) \geq f(|X|)$. A graph $A$ belongs to $\mathbf{K}_f$ if and only if $U$ is normal to $f$ for any substructure $U$ of $A$.

## 2. Theorems by Evans, Ghadernezhad, and Tent

In this section, we fix a generic structure $M$ of $\mathbf{K}_f$. Many of the following definitions and facts are by Evans, Ghadernezhad, and Tent[5].

**Definition 2.1.** Let $A \subseteq M$. $\mathrm{Aut}(M/A)$ denotes the set of automorphisms of $M$ fixing $A$ pointwise. Let $b \in M$. $\mathrm{orb}(b/A)$ denotes the $\mathrm{Aut}(M/A)$-orbit of $b$. So, $\mathrm{orb}(b/A) = \{\sigma(b) \mid \sigma \in \mathrm{Aut}(M/A)\}$.

**Definition 2.2.** Let $A \subseteq M$ be finite. The *dimension* $d(A)$ of $A$ is defined by $d(A) = \delta(\mathrm{cl}(A))$. Let $B \subseteq M$ be also finite. The *relative dimension* $d(A/B)$ is defined by $d(A/B) = d(AB) - d(B)$.

**Definition 2.3.** Suppose $b \in M$ and $A < M$ with $A$ finite. We say that $b$ is *basic* over $A$ if $b \notin A$ and whenever $A \subseteq C < M$ and $d(b/C) < d(b/A)$ then $b \in C$. In this case, $\mathrm{orb}(b/A)$ is called a *basic orbit* over $A$.

**Definition 2.4.** We say that $M$ is *monodimensional* if for every finite $A < M$ and basic orbit $D$ over $A$ there is a finite $B < M$ with $M = \mathrm{cl}(BD)$ and $A \subseteq B$.

**Definition 2.5.** Suppose $A < M$ and $b \in M$ a single element. $b \perp A$ if $\mathrm{cl}(bA) = bA$ and $d(b/A) = d(b)$.

**Fact 2.6.** *Suppose $A < M$ and $b_1, b_2 \in M$ be single elements. If $b_1 \perp A$ and $b_2 \perp A$ then $b_1$ and $b_2$ are conjugate over $A$ in $M$.*

*Proof.* Suppose $b_1 \perp A$ and $b_2 \perp A$. We have $\mathrm{cl}(b_1A) = b_1A$ by the definition. So, $\delta(b_1/A) = \delta(b_1) = 1$. This means that there are no edges between $b_1$ and $A$. By the same argument, there are no edges between $b_2$ and $A$. Hence, $b_1A$ and $b_2A$ are isomorphic over $A$ and also $b_1A < M$ and $b_2A < M$. Therefore, the partial isomorphism between $b_1A$ and $b_2A$ over $A$ can be extended to an automorphism of $M$ by Fact 1.7. $\square$

**Fact 2.7.** *If $M = \mathrm{cl}(AD)$ for some finite $A < M$ and a basic orbit $D$ over $A$ then $M$ is monodimensional.*

**Fact 2.8.** *If $M$ is monodimensional then the automorphism group of $M$ is a simple group.*

## 3. Hrushovski's Boundary Functions

**Definition 3.1** ([7])**.** Let $\alpha$ be a positive real number. We define $x_n$, $e_n$, $k_n$, $d_n$ for integers $n \geq 1$ by induction as follows: Put $x_1 = 2$ and $e_1 = 1$. Assume that $x_n$ and $e_n$ are defined. Let $r_n$ be a smallest rational number $r$ such that $r = k/d > \alpha$ with $d \leq e_n$ where $k$ and $d$ are positive integers. Let $k_n$ and $d_n$ be coprime positive integers with $k_n/d_n = r_n$. Finally, let $x_{n+1} = x_n + k_n$, and $e_{n+1} = e_n + d_n$.

Let $a_0 = (0,0)$, and $a_n = (x_n, x_n - e_n\alpha)$ for $n \geq 1$. Let $f$ be a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ whose graph on interval $[x_n, x_{n+1}]$ with $n \geq 0$ is a line segment connecting $a_n$ and $a_{n+1}$. We call $f$ a *Hrushovski's boundary function associated to $\alpha$*.

**Fact 3.2** ([7]). *Let $D = B \otimes_A C$. Suppose $B$ is normal to $f$, $x_n \leq |B| < x_{n+1}$ and $\delta(C/A) = x - e\alpha > 0$ with positive integers $x$, $e$. If $x/e \geq k_n/d_n$ then $D$ is normal to $f$.*

**Fact 3.3** ([7]). *Let $D = B \otimes_A C$. If $\delta(A) < \delta(B)$, $\delta(A) < \delta(C)$, and $A$, $B$, $C$ are normal to $f$ then $D$ is normal to $f$.*

**Fact 3.4** ([7]). *Let $f$ be a Hrushovski's boundary function associated to $\alpha$. Then $f$ is strictly increasing and concave, and $(\mathbf{K}_f, <)$ has the free amalgamation property. Therefore, there is a generic structure of $(\mathbf{K}_f, <)$. Any one point structure is closed in any structure in $\mathbf{K}_f$. If $\alpha$ is rational then $f$ is unbounded.*

In the rest of the paper, we fix $\alpha = 5/8$.
Note that $2 \cdot 8 - 3 \cdot 5 = 1$. Therefore, $2 - 3\alpha = 1/8$.

**Proposition 3.5.**
(1) *Let $k \geq 0$ be an integer. Whenever $0 < x < 3 + 8(k+1)$ and $y/x > 5/8$ then $y/x \geq (2+5k)/(3+8k)$.*
(2) *We refer to Definition 3.1. Suppose $e_n \geq 3$. Let $l$ be a largest integer $l'$ with $3 + 8l' \leq e_n$. Then $k_n = 2 + 5l$ and $d_n = 3 + 8l$.*

*Proof.* (2) follows from (1). So, we show (1).

First, note that for any integers $u$, $v$, $8v - 5u = 1$ if and only if $u = 3 + 8k$ and $v = 2 + 5k$ with an integer $k$. Also, $(2+5k)/(3+8k)$ is decreasing on $k$.

By inspection, whenever $0 < x < 3 + 8 = 11$ and $y/x > 5/8$ then $y/x \geq 2/3$.

Suppose $u = 3 + 8k$ and $v = 2 + 5k$ with $k \geq 1$. We have

$$\frac{v}{u} - \frac{5}{8} = \frac{1}{8u}.$$

Assume $u < x < u + 8$ and $y/x > 5/8$. Note that $x < 2u$ because $8 < u$. We have

$$\frac{y}{x} - \frac{5}{8} = \frac{8y - 5x}{8x} \geq \frac{2}{8x} > \frac{1}{8u}.$$

Therefore,

$$\frac{y}{x} > \frac{v}{u}.$$

We have (1). □

By this proposition, we have a following chart:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 2 | 3 | 4 | 6 | 8 | 10 | 17 | 29 | 51 |
| $e_n$ | 1 | 2 | 3 | 6 | 9 | 12 | 23 | 42 | 77 |
| $k_n$ | 1 | 1 | 2 | 2 | 2 | 7 | 12 | 22 | 47 |
| $d_n$ | 1 | 1 | 3 | 3 | 3 | 11 | 19 | 35 | 75 |

Also, for $n \geq 3$, we have

$$f(x_{n+1}) = f(x_n) + \frac{1}{8}.$$

The following are easy.

**Lemma 3.6.** (1) *Let $C = A \otimes_p B$ where $p$ is a single vertex and $A, B \in \mathbf{K}_f$. Then $C \in \mathbf{K}_f$.*
(2) *Any finite forest belongs to $\mathbf{K}_f$.*
(3) *Any cycle of length 6 or more belongs to $\mathbf{K}_f$.*

**Lemma 3.7.** *Let $B = A \otimes_{\{x,y\}} P$ where $P = x \cdots y$ is a path of length 3 or more. If the distance of $x$ and $y$ is 3 or more in $A$ and $A \in \mathbf{K}_f$ then $B \in \mathbf{K}_f$.*

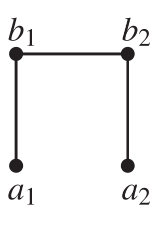
**Lemma 3.8.** *Let $B = A \otimes_{\{x,y,z\}} P$ where*

$$V(P) = \{x, y, z, x', y', z', w\} \text{ and } E(P) = \{xx', x'w, yy', y'w, zz', z'w\}.$$

*If the pairwise distances among $x$, $y$, $z$ in $A$ are 2 or more and $A \in \mathbf{K}_f$ then $B \in \mathbf{K}_f$.*

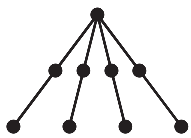
## 4. Special Structures

Let $B$ be a graph with $V(B) = \{a_1, a_2, b_1, b_2\}$ and $E(B) = \{a_1b_1, a_2b_2, b_1b_2\}$ and let $A = \{a_1, a_2\}$. Then $A < B$ and $\delta(B/A) = 1/8$.



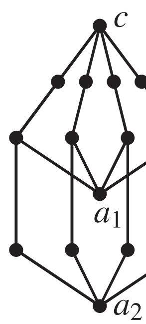

Suppose that $B$ is a closed subset of $M$. Then $b_1$ and $b_2$ are basic over $A$ because $1/8$ is the smallest positive possible dimension.

Let $W_1$ be the following graph:



Let $F_1$ be the set of leaves of $W_1$. Then $F_1 <^- W_1$ and $\delta(W_1/F_1) = 0$.

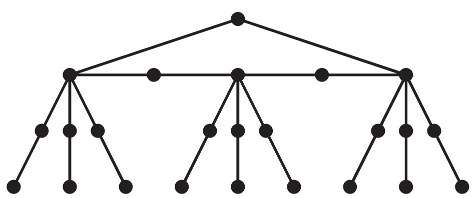
Make a free amalgam of 4 copies of $B$ over $A$ and attaching $W_1$ to it, we get:




The point $c$ belongs to the closure of basic points over $A$.

Unfortunately, This structure does not belong to $\mathbf{K}_f$. But it turns out that any proper substructures belong to $\mathbf{K}_f$.

We can make a "wreath" $W_3$ with 3 copies of $W_1$:

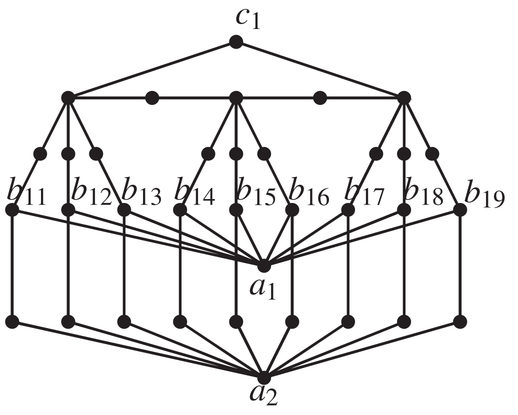


Let $F$ be the leaves of $W_3$. Then $F <^- W_3$ and $\delta(W_3/F) = 0$. A cycle with length 6 belongs to $\mathbf{K}_f$. Therefore, $W_3$ belong to $\mathbf{K}_f$.

**Lemma 4.1.** *Let*

$$C_1 = (B_1 \otimes_A B_2 \otimes_A \cdots \otimes_A B_9) \otimes_F W_3$$

*where each $B_i$ is isomorphic to $B$ over $A$, $F = \{b_{11}, b_{12}, \ldots, b_{19}\}$, with $b_{1i} \in B_i$ the isomorphic images of $b_1$, and $F$ is also the set of leaves of $W_3$. Then $C_1 \in \mathbf{K}_f$. The following is a picture of $C_1$:*




*Proof.* Let $S$ be the cycle of length 6 at the top including $c_1$ in $C_1$. We can represent $C_1$ as

$$C_1 = H_1 \otimes_{AS} H_2 \otimes_{AS} H_3 \otimes_{AS} H_4 \otimes_{AS} H_5 \otimes_{AS} H_6 \otimes_{AS} H_7 \otimes_{AS} H_8 \otimes_{AS} H_9$$

where each $H_i$ is $B_i p_i S$, $p_i$ is adjacent to $b_{1i}$ and to a single vertex in $S$. We have $H_i \in \mathbf{K}_f$ and $AS <^- H_i$ for each $i$.

We have to show that if $U \subseteq C_1$ then $U$ is normal to $f$.

Case $U \cap A \neq A$. In this case, $U \cap AS < U \cap H_i$ for each $i$. Therefore, $U \in \mathbf{K}_f$ by the free amalgamation property.

Case $A \subseteq U$.

Subcase 1: $U = B_I \otimes_{F_I} U'$ where $I$ is a subset of $\{1, 2, \ldots, 9\}$, $B_I = \bigotimes_A \{B_i\}_{i \in I}$, $F_I = \{b_{1i}\}_{i \in I}$, and $U'$ is a substructure of $W_3$.

Let $x$ be the number of points in $U' - F_I$ and $e$ the number of edges in $U'$. To show that $U$ is normal to $f$, we can assume that $e \leq 3|I|$ and $x < 2|I|$ by Lemmas 3.6, 3.7, and 3.8.

Suppose $|I| \leq 4$. Then $|B_I| = x_{2+|I|}$, and $e \leq 3|I| = e_{2+|I|}$. Since $x - e\alpha = \delta(U'/F_I) > 0$, we have $x/e > \alpha$. Hence, $x/e \geq k_{2+|I|}/d_{2+|I|}$. Therefore, $U'$ is normal to $f$ by Fact 3.3.

Suppose $|I| > 4$. We have $\delta(B_I) = 2 + |I|/8$, and $|B_I| = 2 + 2|I|$. $|B_I|$ is 12, 14, 16, 18, 20 for $|I| = 5, 6, 7, 8, 9$, respectively.

We also have $f(10) = 2 + 4/8$, $f(17) = 2 + 5/8$, $f(29) = 2 + 6/8$, and $f(51) = 2 + 7/8$.

Suppose $|I| = 5$. $|U| \leq 12 + 10 = 22$, and $\delta(U) \geq 2 + 5/8 + 1/8 = 2 + 6/8$. Since $|U| \leq 22 < 29$, $U$ is normal to $f$.

Suppose $|I| = 6$. $|U| \leq 14 + 12 = 28$, and $\delta(U) \geq 2 + 6/8 + 1/8 = 2 + 7/8 > f(29)$. Since $|U| \leq 28 < 29$, $U$ is normal to $f$.

Suppose $|I| = 7$. $|U| \leq 16+14 = 30$, and $\delta(U) \geq 2+7/8+1/8 = 2+1$. Since $|U| \leq 30 < 51$, and $f(51) = 2+7/8 < \delta(U)$, $U$ is normal to $f$.

Suppose $|I| = 8$. $|U| \leq 18+16 = 34$, and $\delta(U) \geq 2+8/8+1/8 = 2+9/8$. Since $|U| \leq 30 < 51$, and $f(51) = 2+7/8 < \delta(U)$, $U$ is normal to $f$.

Finally, suppose $|I| = 9$. $|U| \leq 20+18 = 38$, and $\delta(U) \geq 2+9/8$. Since $|U| \leq 38 < 51$, and $f(51) = 2+7/8 < \delta(U)$, $U$ is normal to $f$.

Now, consider the general case. We can assume that $A \subseteq U$, and $U$ is smooth over $AS \cap U$. Let $I$ be the set of $i$ such that $B_i \subseteq U \cap H_i$.

Let $U_I = \bigotimes_{AS\cap U}\{H_i \cap U\}_{i\in I}$, and $j \notin I$.

Put $U''_j = U_I \otimes_{AS\cap U} H_j$. Then $U_I$ and $U''_j$ are normal to $f$ by Subcase 1. Put $U_j = U_I \otimes_{AS\cap U} (H_j \cap U)$. Then $|U_j| < |U''_j|$ and $\delta(U_j) \geq \delta(U''_j)$ because $\delta(H_j \cap U/AS \cap H_j \cap U) \geq \delta(H_j/AS \cap H_j)$. Hence, $U_j$ is normal to $f$. Also, we have $U_I < U_j$. Therefore,

$$U = \bigotimes_{U_I}\{U_j\}_{j\notin I}$$

is normal to $f$ by Fact 3.2. □

**Lemma 4.2.** *Let $C_2 = C_1 \otimes_{c_1} c_1pc$ where $c_1pc$ is a path of length 2. Then $C_2 \in \mathbf{K}_f$ and $Ac < C_2$.*

*Proof.* $C_2 \in \mathbf{K}_f$ because $C_1$ and $c_1pc$ belong to $\mathbf{K}_f$ and one point is always closed in any structure.

By inspection, for any $U \subseteq C_1$ with $A \subseteq U$ and $c_1 \in U$, we have $\delta(U/A) \geq 1$. Therefore, for any $U \subseteq C_2$ with $A \subseteq U$ and $c \in U$, we have $\delta(U/A) > 1$. Therefore, $\delta(U/Ac) = \delta(U/A) - 1 > 0$. Hence, $Ac < C_2$. □

**Lemma 4.3.** *Let $C_3 = \bigotimes_{Ac}\{C_{11}, C_{12}, C_{13}, C_{14}\}$ where $C_{1j}$ are isomorphic to $C_1$ over $Ac$. Then $C_3 \in \mathbf{K}_f$, $A < C_3$, $Ac < C_3$.*

## 5. Monodimensionality

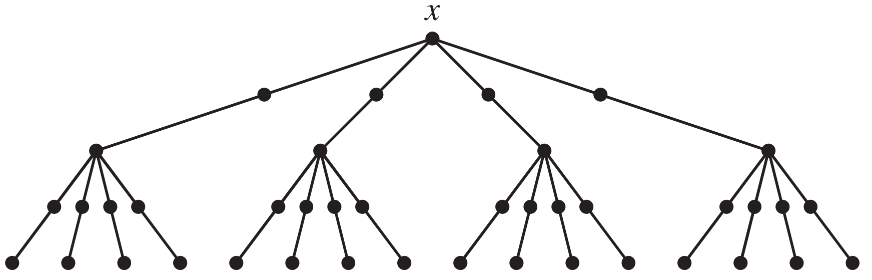
We can prove the following theorem as in [5].

**Theorem 5.1.** *The generic structure $M$ of $\mathbf{K}_f$ is monodimensional. Therefore, the automorphism group of $M$ is a simple group.*

*Proof.* Consider $C_3$ from 4.3. We can assume $a_1$ and $a_2$ are not connected and $A = \{a_1, a_2\} < M$. So, we can embed $C_3$ as a closed substructure of $M$. We can assume that $C_3 < M$. Isomorphic images of $B$ over $A$ in $C_3$ are also closed in $M$. Therefore, isomorphic images of $b_1$ are in a same basic orbit over $A$, say $D$. Hence, $c \in \mathrm{cl}(A,D)$. Since $cA < C_3$, $cA$ is closed in $M$ and thus $c \perp A$.

By Fact 2.6, we have $\{e \in M \mid e \perp A\} \subseteq \mathrm{cl}(A,D)$.

Now, let $x \in M - A$. Let $X = \mathrm{cl}(xA)$. Consider the following structure $U$:




Then $X < X \otimes_x U$. Embed this structure over $X$ as a closed structure of $M$. So, we can assume $XU < M$. For each leaf $y$ of $U$, $xy < U$. Therefore, $Ay < M$. Hence, $y \perp A$. On the other hand, by the structure of $U$, $x$ is in the closure of the leaves of $U$. Thus, $x \in \mathrm{cl}(\{e \in M \mid e \perp A\})$. Therefore, $x \in \mathrm{cl}(A,D)$. $\square$

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