数理解析研究所講究録 第2121巻 2019年 29-40

A FREE BOUNDARY PROBLEM FOR REACTION DIFFUSION EQUATION WITH POSITIVE BISTABLE NONLINEARITY

MAHO ENDO, YUKI KANEKO AND YOSHIO YAMADA

Department of Pure and Applied Mathematics, Waseda University

1. INTRODUCTION

In this article, based on a recent work [8], we consider a free boundary problem for a reaction-diffusion equation given by:

$$\text{(FBP)}\qquad \begin{cases} u_t = du_{xx} + f(u), & t>0,\ 0<x<h(t),\\ u(t,0)=0,\ u(t,h(t))=0, & t>0,\\ h'(t) = -\mu u_x(t,h(t)), & t>0,\\ h(0)=h_0,\ u(0,x)=u_0(x), & 0\le x\le h_0, \end{cases}$$

where d, μ and h_0 are positive constants and $x = h(t)$ represents a free boundary. In (FBP), $f \in C^1([0,\infty))$ satisfies the following properties:

$$\text{(PB)}\qquad \begin{aligned} &f(u)=0 \text{ has solutions } u=0,u_1^*,u_2^*,u_3^* \ (0<u_1^*<u_2^*<u_3^*),\\ &f'(0)>0, f'(u_1^*)<0, f'(u_2^*)>0, f'(u_3^*)<0, \int_{u_1^*}^{u_3^*} f(u)du>0\\ &\text{and } f(u)\neq 0 \text{ for } u\notin\{0,u_1^*,u_2^*,u_3^*\}. \end{aligned}$$

We say that f is a function of positive bistable type when f satisfies (PB). Initial function u_0 satisfies

$$\text{(1.1)}\qquad u_0\in C^2([0,h_0]), u_0(0)=u_0(h_0)=0 \text{ and } u_0(x)>0 \text{ for } 0<x<h_0.$$

A free boundary problem like (FBP) was first proposed by Du and Lin [3] as a model of the invasion of a new species by putting homogeneous Neumann condition at $x = 0$ in place of Dirichlet condition. We denote such a free boundary problem by (FBP-N). Function $u(t,x)$ represents the population density of the species over one-dimensional habitat $(0, h(t))$. The free boundary $x = h(t)$ stands for the expanding front of the habitat and its dynamics is determined by the Stefan condition of the form $h'(t) = -\mu u_x(t, h(t))$. For the ecological meaning of this condition, see [2].

Du and Lin studied (FBP-N) with logistic nonlinearity $f(u) = u(a-bu), a, b > 0$, and established various interesting results such as spreading-vanishing dichotomy and asymptotic behaviors of solutions as $t \to \infty$ as well as the existence and uniqueness of global solutions. In particular, it was shown that any solution (u, h) of (FBP-N) satisfies either vanishing or spreading: vanishing means the case where $\lim_{t\to\infty} h(t) \le \pi/2\sqrt{d/a}$ and $\lim_{t\to\infty} \|u(t)\|_{C([0,h(t)])} = 0$, while spreading means the case where $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = a/b$ locally uniformly for $x \in [0,\infty)$. Recently, a lot of people have investigated related free boundary problems. In particular, we should refer to the work of Du and Lou [4], who have discussed a similar problem to (FBP-N) (or (FBP)) by putting free boundary conditions at both ends of the interval. As one of the most important results, it was shown that the analysis of large-time behaviors of spreading solutions is closely related to the following semi-wave problem

$$\text{(SWP)} \qquad \begin{cases} dq_{zz} - cq_z + f(q) = 0, \ q(z) > 0, & z > 0, \\ q(0) = 0, \ \mu q_z(0) = c, \ \lim\limits_{z\to\infty} q(z) = u^*, \end{cases}$$

where u^* is a stable equilibrium point of f such that $f(u^*) = 0$. When f is monostable, bistable or combustion type of nonlinearity satisfying $f(0) = f(1) = 0$ and $f(u) < 0$ for $u > 1$, it was proved in [4] that (SWP) with $u^* = 1$ admits a unique solution $(c, q) = (c^*, q^*)$. Their results show that (c^*, q^*) is available to study asymptotic behavior of any spreading solution. For its sharper asymptotic estimates, see the paper of Du-Matsuzawa-Zhou [7].

When f satisfies (PB), we recall the work of Kawai and Yamada [13] for (FBP-N). They have established the classification of solutions of (FBP-N) into four types of asymptotic behaviors: vanishing, small spreading, big spreading and transition. In particular, (FBP-N) with positive bistable nonlinearity f exhibits two types of spreading phenomena; one is the small spreading of solution (u, h) with $\lim_{t\to\infty} u(t, \cdot) = u_1^*$ locally uniformly in $[0, \infty)$ and the other is the big spreading solution (u, h) with $\lim_{t\to\infty} u(t, \cdot) = u_3^*$ locally uniformly in $[0, \infty)$. Moreover, it was also proved in [13] that under certain circumstances (SWP) does not have a solution, which is a big difference from known results for other types of nonlinearity. In this sense, positive bistable f provides us interesting and significant properties for (FBP-N). Recently, Kaneko-Matsuzawa-Yamada [12] have proved that, if (SWP) has no solutions, the corresponding spreading solution possesses a propagating terrace.

The main purpose of this article is to give precise information on asymptotic behaviors of solutions for (FBP) when f satisfies (PB). As our first main result (Theorem 4.1), we will show that any solution (u, h) of (FBP) satisfies one of the following properties:

(I) vanishing: $\lim\limits_{t\to\infty} h(t) \le \pi\sqrt{d/f'(0)}$ and $\lim\limits_{t\to\infty} \|u(t)\|_{C([0,h(t)])} = 0$;

(II) small spreading: $\lim\limits_{t\to\infty} h(t) = \infty$ and $\lim\limits_{t\to\infty} u(t, \cdot) = v_1$ locally uniformly in $[0, \infty)$;

(III) big spreading: $\lim\limits_{t\to\infty} h(t) = \infty$ and $\lim\limits_{t\to\infty} u(t, \cdot) = v_3$ locally uniformly in $[0, \infty)$.

Here v_1 and v_3 are bounded solutions of the following problem

$$\text{(SP)} \qquad \begin{cases} dv_{xx} + f(v) = 0, \ v(x) > 0 \quad \text{for } 0 \le x < \infty, \\ v(0) = 0 \end{cases}$$

with $\lim_{x\to\infty} v_1(x) = u_1^*$ and $\lim_{x\to\infty} v_3(x) = u_3^*$, respectively. Note that (SP) has no bounded solutions other than v_1 and v_3 (refer to Section 3). In Section 4 we will also give sufficient conditions for each behavior. Moreover, in order to get better understanding on the above asymptotic behaviors, we will introduce parameter $\sigma > 0$. Let any (u_0, h_0) satisfying (1.1) be fixed and consider (FBP) with (u_0, h_0) replaced by $(\sigma u_0, h_0)$. We denote such a free boundary problem by $(\text{FBP})_\sigma$. Let $(u(t, x; \sigma), h(t; \sigma))$ be the solution of $(\text{FBP})_\sigma$. Then it is possible to show the existence of two threshold numbers σ_1^* and σ_2^* ($\sigma_1^* < \sigma_2^*$) such that the vanishing of $(u(\sigma), h(\sigma))$ occurs for $0 \le \sigma \le \sigma_1^*$, the small spreading of $(u(\sigma), h(\sigma))$

occurs for $\sigma_1^* < \sigma \le \sigma_2^*$ and the big spreading of $(u(\sigma), h(\sigma))$ occurs for $\sigma_2^* < \sigma$ (Theorems 4.10 and 4.11).

As the second step, in Section 5 we will derive asymptotic estimates for two types of spreading solutions. Let (u, h) be any big spreading solution of (FBP) and let (SWP) with $u^* = u_3^*$ admit a unique solution (c_B, q_B). (For the existence and nonexistence of such a solution, see [13]). Then we will give the result that (u, h) satisfies

$$\lim_{t\to\infty} h'(t) = c_B \quad \text{and} \quad \lim_{t\to\infty}(h(t) - c_B t) = H_B \tag{1.2}$$

with some $H_B \in \mathbb{R}$ and

$$\lim_{t\to\infty} \sup_{h(t)/2 \le x \le h(t)} |u(t,x) - q_B(h(t) - x)| = 0 \tag{1.3}$$

(Theorem 5.4). In this sense, (c_B, q_B) gives a good approximation of (u, h) near the free boundary $x = h(t)$ for large t. Moreover, we can also show that for any $c \in (0, c_B)$

$$\lim_{t\to\infty} \sup_{0 \le x \le ct} |u(t,x) - v_3(x)| = 0 \tag{1.4}$$

(Theorem 5.5). For a small spreading solution (u, h), it will be seen that analogous estimates as (1.2)-(1.4) are valid provided that $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} < u_2^*$. Here we should remark that there exists a small spreading solution which does not satisfy this condition. For example, when we take $(u(t.x;\sigma_2^*), h(t;\sigma_2^*))$ which is a borderline solution between the small spreading and the big spreading for $(\text{FBP})_\sigma$, this solution satisfies $\lim_{t\to\infty} \|u(t;\sigma_2^*)\|_{C([0,h(t;\sigma_2^*)])} \ge u_2^* > u_1^*$. We have not obtained satisfactory asymptotic estimates for such small spreading solution.

2. Preliminaries

First of all, in this section we will introduce basic properties of the solutions for (FBP). We begin with the global existence result for (FBP).

Theorem 2.1 (Existence and uniqueness of bounded global solution)**.** *Let f satisfy* (PB) *and let u_0 satisfy* (1.1). *Then* (FBP) *has a unique solution* (u, h) *satisfying*

$$(u, h) \in \{C^{\frac{1+\alpha}{2}, 1+\alpha}(\bar{\Omega}) \cup C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega)\} \times C^{1+\frac{\alpha}{2}}([0,\infty)),$$

for any $\alpha \in (0,1)$ with $\Omega = \{(t,x) \in \mathbb{R}^2 : t > 0, 0 \le x \le h(t)\}$. Moreover, it holds that

$$u_x(t,x) < 0 \quad \text{for } t > 0, \max\Big\{h_0, \frac{h(t)}{2}\Big\} \le x \le h(t)$$

and there exist positive constants $C_1 = C_1(\|u_0\|_{C([0,h_0])}, h_0)$ and $C_2 = C_2(\|u_0\|_{C^1([0,h_0])}, h_0)$ such that

$$\begin{aligned} &0 < u(t,x) \le C_1, && \text{for } t > 0,\ 0 < x < h(t),\\ &0 < h'(t) \le \mu C_2, && \text{for } t > 0. \end{aligned}$$

Theorem 2.1 has been shown by Du-Lin [3, Theorems 2.1, 2.3 and Lemma 2.2] and Kaneko-Yamada [10, Theorem 2.7] and [11, LemmaA.1]. In particular $h'(t) > 0$ means the existence of $\lim_{t\to\infty} h(t) \in (0, \infty]$.

We define spreading and vanishing of solutions under general situations.

Definition 2.2. Let (u,h) be any global solution of (FBP). Then (u,h) is a vanishing solution if

$$\lim_{t\to\infty} \|u(t)\|_{C([0,h(t)])} = 0,$$

and (u,h) is a spreading solution if

$$\lim_{t\to\infty} h(t) = \infty \text{ and } \liminf_{t\to\infty} \|u(t)\|_{C([0,h(t)])} > 0.$$

We next give a comparison theorem for (FBP) proved by Du and Lin [3, Lemma 3.5].

Theorem 2.3 (Comparison theorem)**.** *Suppose that* $\overline{h} \in C^1([0,T])$ *and* $\overline{u} \in C^{1,2}(\bar{\Omega})$ *with* $T > 0$ *and* $\Omega = \{(t,x) \in \mathbb{R}^2 \mid 0 < t \le T, 0 \le x \le \overline{h}(t)\}$ *satisfy*

$$\begin{cases} \overline{u}_t \ge d\overline{u}_{xx} + f(\overline{u}), & (t,x) \in \bar{\Omega}, \\ \overline{u}(t,0) \ge 0,\ \overline{u}(t,\overline{h}(t)) = 0, & t \in (0,T], \\ \overline{h}'(t) \ge -\mu\overline{u}_x(t,\overline{h}(t)), & t \in (0,T]. \end{cases}$$

If $\overline{h}(0) \ge h_0$, $\overline{u}(0,x) \ge u_0(x)$ *in* $[0,h_0]$, *then the solution* (u,h) *of* (FBP) *satisfies*

$$\overline{h}(t) \ge h(t) \text{ in } [0,T] \text{ and } \overline{u}(t,x) \ge u(t,x) \text{ in } (t,x) \in [0,T] \times [0,h(t)].$$

The pair $(\overline{u},\overline{h})$ is called an upper solution of (FBP) when it satisfies the assumptions of Theorem 2.3. Similarly a lower solution is defined by reversing all inequality signs in the assumptions of Theorem 2.3.

3. Analysis of stationary problem

We introduce the following stationary problem defined in the finite interval $[0,\ell]$:

$$(\mathrm{SP}-\ell) \qquad \begin{cases} d\phi_{xx} + f(\phi) = 0, \quad \phi(x) > 0 \quad \text{for } 0 < x < \ell, \\ \phi(0) = \phi(\ell) = 0 \end{cases}$$

with positive number ℓ. In this section we will study two stationary problems; (SP) and (SP-ℓ) for (FBP) with nonlinearity f satisfying (PB) by making use of the phase plane analysis (see for instance [15] and Figure 1). We first give the existence of bounded nonnegative solutions of (SP).

Proposition 3.1 (Existence of bounded solutions of (SP))**.** *Under assumption* (PB), (SP) has three bounded solutions $v \equiv 0, v_1(x)$ and $v_3(x)$, where $v_1 = v_1(x)$ (resp. $v_3 = v_3(x)$) is an increasing function satisfying $\lim_{x\to\infty} v_1(x) = u_1^*$ (resp. $\lim_{x\to\infty} v_3(x) = u_3^*$) and $v_1(x) < v_3(x)$ for $x > 0$.

In order to find a solution of (SP-ℓ) we consider the following initial value problem

$$(3.1) \qquad \begin{cases} d\phi'' + f(\phi) = 0, \\ \phi(0) = 0,\ \phi'(0) = P > 0. \end{cases}$$

Let $\phi = \phi(x;P)$ be a solution of (3.1) and define $\ell = \ell(P)$ by

$$\ell(P) := \inf\{x > 0 : \phi(x;P) = 0\}.$$

We also define

$$F(u) := \int_0^u f(s)ds.$$

For f satisfying (PB), we can choose $\hat{u} \in (u_2^*, u_3^*)$ such that

$$(3.2) \qquad F(\hat{u}) = F(u_1^*).$$

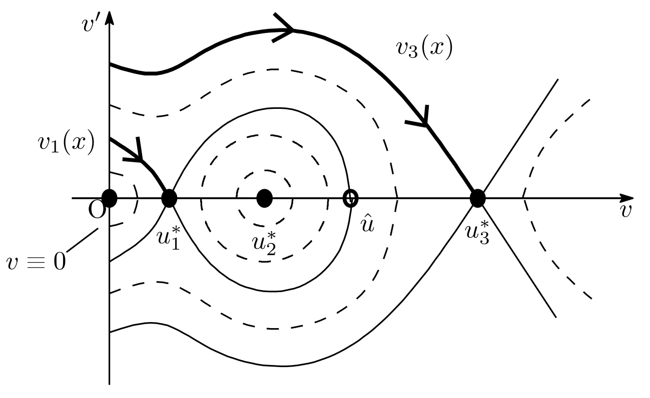

FIGURE 1. The phase plane of (SP)

For functions v_1 and v_3 in Proposition 3.1 set $v_1'(0) =: \omega_1$ and $v_3'(0) =: \omega_3$, then $\ell(P)$ can be represented by

$$\ell(P) = \sqrt{2d} \int_0^{\phi_P} \frac{d\phi}{\sqrt{F(\phi_P) - F(\phi)}} \quad \text{for } P \in (0, \omega_1) \cup (\omega_1, \omega_3) \tag{3.3}$$

where $\phi_P = \phi(\ell(P)/2; P)$. Note that if one can find P^* satisfying $\ell(P^*) = \ell$, then $\phi(x; P^*)$ becomes a solution of (SP-ℓ). The following result gives an elementary property of $\ell(P)$.

Lemma 3.2. *Define $\ell(P)$ by* (3.3)*. Then $\ell(P)$ is a continuous function of $P \in (0, \omega_1) \cup (\omega_1, \omega_3)$ and satisfies*

$$\lim_{P \to 0} \ell(P) = \pi \sqrt{\frac{d}{f'(0)}}, \quad \lim_{P \to \omega_1 - 0} \ell(P) = \lim_{P \to \omega_1 + 0} \ell(P) = \lim_{P \to \omega_3} \ell(P) = \infty.$$

For the proof of this lemma, see [14].

We are led to the existence of a minimum of $\ell(P)$ in (ω_1, ω_3) by virtue of Lemma 3.2, namely

$$\ell^* := \min_{\omega_1 < P < \omega_3} \ell(P). \tag{3.4}$$

Lemma 3.2 ensures the following result on the structure of solutions of (SP-ℓ).

Proposition 3.3 (The structure of solutions for (SP-ℓ))**.** *Assume* (PB) *and define ℓ^* by* (3.4)*. Then the following properties hold true:*

(i) *For each $\ell \in (\pi\sqrt{d/f'(0)}, \infty)$,* (SP-$\ell$) *has a positive solution $\phi_1 = \phi_1(x; \ell)$ satisfying $\|\phi_1\|_{C([0,\ell])} < u_1^*$. Moreover, $\lim_{\ell \to \pi\sqrt{d/f'(0)}} \|\phi_1\|_{C([0,\ell])} = 0$ and $\lim_{\ell \to \infty} \|\phi_1\|_{C([0,\ell])} = u_1^*$.*

(ii) *For each $\ell \in [\ell^*, \infty)$,* (SP-$\ell$) *has two positive solutions $\phi_2 = \phi_2(x; \ell)$ and $\phi_3 = \phi_3(x; \ell)$ satisfying*

$$\hat{u} < \|\phi_2\|_{C([0,\ell])} \le \|\phi_3\|_{C([0,\ell])} < u_3^*,$$
$$\lim_{\ell \to \infty} \|\phi_2\|_{C([0,\ell])} = \hat{u} \text{ and } \lim_{\ell \to \infty} \|\phi_3\|_{C([0,\ell])} = u_3^*.$$

Here $\hat{u} \in (u_2^*, u_3^*)$ is a constant defined in (3.2)*. Moreover, $\phi_1(x; \ell) < \phi_2(x; \ell) < \phi_3(x; \ell)$ for $0 < x < \ell$ when they exist.*

4. Asymptotic behaviors of solutions

In this section, we will study asymptotic behaviors of solutions of (FBP).

4.1. **Classification of asymptotic behaviors.** We first show the first main result about the classification of solutions of (FBP) in terms of their asymptotic behaviors.

Theorem 4.1. *Assume* (PB). *Then any solution* (u,h) *of* (FBP) *satisfies one of the following properties:*

- (I) *Vanishing:* $\lim_{t\to\infty} h(t) \le \pi\sqrt{d/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C([0,h(t)])} = 0$*;*
- (II) *Small spreading:* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_1(x)$ *uniformly in* $x \in [0,R]$ *for any* $R>0$*;*
- (III) *Big spreading:* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_3(x)$ *uniformly in* $x\in[0,R]$ *for any* $R>0$*,*

where v_1 *and* v_3 *are bounded increasing functions in Proposition 3.1.*

In order to prove Theorem 4.1, we will make use of the zero number arguments developed by Angenent [1]. Denote by $\mathcal{Z}_I(w)$ the number of zero points of a continuous function w in an interval $I \subset \mathbb{R}$. Du-Lou-Zhou [5, Lemma 2.2] and Du-Matano [6, Lemma 2.6] have extended Angenent's result as follows:

Lemma 4.2. *Let* $\xi(t) \ge 0$ *be a continuous function for* $t\in(t_1,t_2)$ *and set* $I(t) := [-\xi(t),\xi(t)]$*. Assume that* $w(t,x)$ *is a continuous function defined in* $(t_1,t_2)\times I(t)$ *and that it satisfies*

$$w_t = dw_{xx} + c(t,x)w, \quad (t,x)\in(t_1,t_2)\times(-\xi(t),\xi(t)), \tag{4.1}$$

where c *is bounded in* $[t_1,t_2]\times I(t)$*. If* $w(t,-\xi(t)) \ne 0$ *and* $w(t,\xi(t))\ne 0$ *for* $t\in(t_1,t_2)$*, then the following properties hold true:*

- (i) $\mathcal{Z}_{I(t)}(w(t)) < \infty$ *for any* $t\in(t_1,t_2)$ *and it is non-increasing in* t*;*
- (ii) *If* $w(s,x)$ *has a degenerate zero* $x_0 \in (-\xi(s),\xi(s))$ *at some* $s\in(t_1,t_2)$ *, then*

$$\mathcal{Z}_{I(s_1)}(w(s_1)) > \mathcal{Z}_{I(s_2)}(w(s_2)) \text{ for any } s_1\in(t_1,s) \text{ and } s_2\in(s,t_2).$$

Lemma 4.3. *Let* $I\subset\mathbb{R}$ *be an open interval and let* $\{w_n(t,x)\}_{n=1}^{\infty}$ *be a sequence of functions which converges to* $w(t,x)$ *in* $C^1((t_1,t_2)\times I)$*. Assume that for every* $t\in(t_1,t_2)$ *and* $n\in\mathbb{N}$*, the function* $x\mapsto w_n(t,x)$ *has only simple zeros in* I *and that* $w(t,x)$ *satisfies an equation of the form* (4.1) *in* $(t_1,t_2)\times I$*. Then for every* $t\in(t_1,t_2)$*, either* $w(t,x)\equiv 0$ *in* I*, or* $w(t,x)$ *has only simple zeros in* I*.*

We will prove the following convergence property of the solutions of (FBP) by using Lemmas 4.2, 4.3 and the elementary properties of the structure of ω-limit set.

Proposition 4.4. *Let* (u,h) *be the solution of* (FBP)*. If* $\lim_{t\to\infty} h(t)=\infty$*, then*

$$\lim_{t\to\infty} u(t,\cdot) = v^* \text{ uniformly in } [0,R] \text{ for any } R>0,$$

where v^* *is a bounded positive solution of* (SP).

Sketch of Proof of Proposition 4.4. Let $\omega(u)$ be an ω-limit set of $u(t,\cdot)$ in the topology of $L^\infty_{loc}([0,\infty))$, that is, for every $w\in\omega(u)$ there exists a sequence $0<t_1<t_2<\cdots<t_n<t_{n+1}<\cdots\to\infty$ such that

$$\lim_{n\to\infty} u(t_n,x) = w(x) \text{ uniformly in } x\in[0,R] \text{ for any } R>0. \tag{4.2}$$

By local parabolic regularity estimates, we can replace the topology of $L^\infty_{loc}([0,\infty))$ by that of $C^2_{loc}([0,\infty))$. Since $\omega(u)$ is a compact, connected and invariant set, for any $w \in \omega(u)$ there exists an entire orbit $\{W(t,x)\}_{t\in\mathbb{R}}$ with $W(0,x) = w(x)$. This fact implies that for every $w \in \omega(u)$ there exists $W(t,x)$ satisfying

$$\begin{cases} W_t = dW_{xx} + f(W), & t \in \mathbb{R}, \quad x > 0, \\ W(t,0) = 0, & t \in \mathbb{R}, \\ W(0,x) = w(x) \in \omega(u), & x > 0, \end{cases}$$

and

$$\lim_{n\to\infty} u(t+t_n, x) = W(t,x) \text{ in } L^\infty_{loc}(\mathbb{R}\times[0,\infty)). \tag{4.3}$$

This convergence can be also replaced by the topology of $C^{1,2}_{loc}(\mathbb{R}\times[0,\infty))$ on account of parabolic regularity.

Let $v = v(x)$ be a unique solution of

$$\begin{cases} dv'' + f(v) = 0, & x > 0, \\ v(0) = 0, \\ v'(0) = w'(0). \end{cases}$$

We will investigate intersection points between $W(t,x)$ and $v(x)$. Let v_1 and v_3 be functions given in Proposition 3.1; then

$$v_1(x) \le w(x) \le v_3(x) \text{ for } x \ge 0. \tag{4.4}$$

Since $v_1(0) = w(0) = v_3(0) \equiv 0$, we have $0 < v_1'(0) \le w'(0) = v'(0) \le v_3'(0)$. Therefore, by the phase plane analysis (see Figure 1), it is seen that either

(i) $v(x) > 0$ for $x > 0$, or

(ii) there exists a positive number R such that $v(R) = 0$ and $v(x) > 0$ for $x \in (0,R)$.

We first consider the case (i). Let $\hat{u}(t,x), \hat{W}(t,x)$ and $\hat{v}(x)$ be odd extensions of $u(t,x), W(t,x)$ and $v(x)$, respectively. It follows from Lemmas 4.2 and 4.3 that, for every $t \in \mathbb{R}$, either $\hat{W}(t,x) - \hat{v}(x) \equiv 0$ in $\mathbb{R}$, or $\hat{W}(t,x) - \hat{v}(x)$ has only simple zeros in $\mathbb{R}$. However we see that the latter case never occurs because $\hat{W}(t,0) - v(0) = \hat{W}_x(t,0) - \hat{v}_x(0) = 0$ at $t = 0$. Therefore, $\hat{W}(t,x) \equiv \hat{v}(x)$ in $\mathbb{R}$, that is,

$$W(t,x) \equiv W(0,x) = w(x) \equiv v(x) \text{ for } x \ge 0.$$

Thus any $w \in \omega(u)$ is equal to v which is a bounded positive solution of (SP).

Similarly we can prove that the case (ii) never occurs, so the proof is compete. □

Proposition 4.4 enables us to prove Theorem 4.1. For the proof, see [8] in more detail.

4.2. **Sufficient conditions for asymptotic behavior.** In this subsection we will give some sufficient conditions for (I)-(III) of Theorem 4.1. We first introduce a sufficient condition for the vanishing which can be proved in the same way as [9, Theorem 2.2].

Theorem 4.5. *Assume $h_0 < \pi\sqrt{d/f'(0)}$. Then there exists a positive function V^* such that, if $u_0(x) \le V^*(x)$ in $[0,h_0]$, then the solution (u,h) of* (FBP) *satisfies the vanishing.*

The following result gives a sufficient condition for the spreading when $h_0 < \pi\sqrt{d/f'(0)}$:

Theorem 4.6. *Assume $h_0 < \pi\sqrt{d/f'(0)}$. If*

$$\int_0^{h_0} x u_0(x)dx > \frac{d}{2\mu}\Big(\pi^2\frac{d}{f'(0)} - h_0^2\Big)\max\left\{1, \frac{\|u_0\|_{C([0,h_0])}}{u_1^*}\right\},$$

then the solution of (FBP) *satisfies the spreading.*

The following two results are sufficient conditions for the small and big spreading of solutions.

Theorem 4.7. *If $h_0 \geq \pi\sqrt{d/f'(0)}$ and $\|u_0\|_{C([0,h_0])} < u_2^*$, then the solution of* (FBP) *satisfies* (II) *of Theorem 4.1.*

Theorem 4.8. *Let ℓ^* be a positive number defined by* (3.4) *and assume $h_0 \geq \ell^*$. If there exists a positive constant $\ell \in [\ell^*, h_0]$ such that $u_0(x) \geq \phi_2(x;\ell)$ in $[0,\ell]$, then the solution of* (FBP) *satisfies* (III) *of Theorem 4.1, where $\phi_2(x;\ell)$ is the solution of* (SP-ℓ) *given in Proposition 3.3.*

We can prove Theorem 4.7 in the same way as the proof of [Proposition 4.8][10]. One can prove Theorem 4.8 in a similar way to [13, Theorem 3.6].

4.3. **Sharp threshold numbers.** In this subsection we give a more detailed description on the asymptotic behavior of the solution of (FBP). We introduce a parameter $\sigma \geq 0$ and consider (FBP)$_\sigma$ with initial data $(u_0, h_0) = (\sigma\phi, h_0)$ for any fixed (ϕ, h_0) satisfying (1.1). Denote by $(u(t,x;\sigma), h(t;\sigma))$ the solution of (FBP)$_\sigma$ with initial data $(\sigma\phi, h_0)$. It is clear from Theorems 2.1 and 2.3 that, if $\sigma_1 > \sigma_2$, then

$$h(t;\sigma_1) > h(t;\sigma_2) \text{ and } u(t,x;\sigma_1) > u(t,x;\sigma_2) \text{ for } t \geq 0,\ x \in (0, h(t;\sigma_2)]. \tag{4.5}$$

We define two numbers σ_1^* and σ_2^* by

$$\sigma_1^* := \sup\{\sigma;\ \text{the vanishing occurs for } (u(t,x;\sigma), h(t;\sigma))\} \tag{4.6}$$

and

$$\sigma_2^* := \inf\{\sigma;\ \text{the big spreading occurs for } (u(t,x;\sigma), h(t;\sigma))\}. \tag{4.7}$$

Note that $\sigma_1^* \leq \sigma_2^*$ by the comparison theorem. We begin with the following lemma which gives a condition for $\sigma_1^* < \infty$:

Lemma 4.9. *Assume that (ϕ, h_0) satisfies* (1.1) *and*

$$\pi^2\frac{d}{f'(0)} - \frac{2\mu u_1^* \int_0^{h_0} x\phi(x)dx}{d\|\phi\|_{C([0,h_0])}} \leq h_0^2 < \pi^2\frac{d}{f'(0)}. \tag{4.8}$$

Then there exists a positive number $\overline{\sigma}$ such that $(u(t,x;\sigma), h(t;\sigma))$ satisfies the spreading for every $\sigma \geq \overline{\sigma}$.

Using Theorem 4.5, Lemma 4.9 and (4.5), one can see that σ_1^* given in (4.6) is the threshold number which separates the vanishing and the spreading:

Theorem 4.10. *Let $(u(t,x;\sigma), h(t;\sigma))$ be the solution of* (FBP)$_\sigma$ *with initial data $(\sigma\phi, h_0)$ for $\sigma > 0$. Then $(u(t,x;\sigma), h(t;\sigma))$ satisfies the vanishing for every $\sigma \leq \sigma_1^*$ and the spreading for every $\sigma > \sigma_1^*$. Moreover, $\sigma_1^* \in (0,\infty]$ if $h_0 < \pi\sqrt{d/f'(0)}$, $\sigma_1^* = 0$ if $h_0 \geq \pi\sqrt{d/f'(0)}$, and $\sigma_1^* \in (0,\infty)$ if (ϕ, h_0) satisfies* (4.8).

It is possible to prove Theorem 4.10 in a similar manner to [4, Theorem 5.2] or [13, Theorem 3.7].

Next we will show that σ_2^* defined as (4.7) is another threshold number which separates the small spreading and the big spreading:

Theorem 4.11. *Let $(u(t,x;\sigma), h(t;\sigma))$ be the solution of* (FBP)$_\sigma$ *with initial data $(\sigma\phi, h_0)$ for $\sigma > 0$. Then $(u(t,x;\sigma), h(t;\sigma))$ satisfies the small spreading for every $\sigma \in (\sigma_1^*, \sigma_2^*]$ and the big spreading for every $\sigma \in (\sigma_2^*, \infty)$. Moreover, $\sigma_2^* \in (\sigma_1^*, \infty)$ if $h_0 > \ell^*$, where ℓ^* is a positive constant given in Proposition 3.3.*

The proof of this theorem is similar to that of [13, Theorem 3.8].

Remark 1. If we consider (FBP-N), then the solution satisfies the transition at $\sigma = \sigma_2^*$ when σ_1^* and σ_2^* are defined by (4.6) and (4.7) (see, [13, Theorem 3.8]). This fact implies that the transition is a borderline behavior between the small spreading and big spreading in the case of zero Neumann boundary condition at $x = 0$.

We next state the result concerned with the small spreading for $\sigma \in (\sigma_1^*, \sigma_2^*]$.

Theorem 4.12. *Let $(u, h; \sigma) = (u(t,x;\sigma), h(t;\sigma))$ be the solution of* (FBP)$_\sigma$ *with initial data $(\sigma\phi, h_0)$ for $\sigma > 0$. Then there exists a positive number $\sigma_{1,2}^* \in (\sigma_1^*, \sigma_2^*]$ such that $(u, h; \sigma)$ satisfies the small spreading and $\liminf_{t\to\infty} \|u(t;\sigma)\|_{C([0,h(t;\sigma)])} < u_2^*$ for every $\sigma \in (\sigma_1^*, \sigma_{1,2}^*)$, while $(u, h; \sigma)$ satisfies the small spreading and $\liminf_{t\to\infty} \|u(t;\sigma)\|_{C([0,h(t;\sigma)])} \geq u_2^*$ for every $\sigma \in [\sigma_{1,2}^*, \sigma_2^*]$.*

Remark 2. The notion of small spreading in Theorem 4.1 is defined by $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = v_1(x)$ in $[0, R]$ for any $R > 0$. It may be classified into two sub-cases; (i) $\liminf_{t\to\infty} \|u(t)\|_{C([0,h(t)])} < u_2^*$, (ii) $\liminf_{t\to\infty} \|u(t)\|_{C([0,h(t)])} \geq u_2^*$. In particular, case (ii) implies that $u(t,x)$ has a peak at $x = x^*(t)$ satisfying $u(t, x^*(t)) \geq u_2^*$ for sufficiently large t. This is an interesting phenomenon, but we have no further information on this kind of small spreading. The phenomenon of case (ii) may correspond to the "transition", which is a borderline solution between small spreading and big spreading for solutions of (FBP-N).

5. Spreading speed and profiles of solutions

In this section we will discuss an asymptotic spreading speed of the free boundary and an asymptotic profile of any spreading solution of (FBP). It was shown by Du and Lou [4] that the analysis of asymptotic spreading speed and profile of the solution for (FBP) is closely related with the semi-wave problem:

$$\text{(SWP)} \qquad \begin{cases} dq_{zz} - cq_z + f(q) = 0, \ q(z) > 0, & z > 0, \\ q(0) = 0, \ \mu q_z(0) = c, \ \lim\limits_{z\to\infty} q(z) = u^* \end{cases}$$

with $u^* = u_1^*$ (resp. $u^* = u_3^*$). We first recall the following existence and uniqueness of the solution of (SWP) for f satisfying (PB) which was proved by Kawai and Yamada [13] by applying the phase plane method (see [13, Theorem 4.1]).

Theorem 5.1. *The following properties hold true.*

(i) *For $u^* = u_1^*$,* (SWP) *has a unique solution $(c, q) = (c_S, q_S)$ for each $\mu > 0$.*
(ii) *For $u^* = u_3^*$, either Case A or Case B holds true;*
Case A: (SWP) *has a unique solution $(c, q) = (c_B, q_B)$ for each $\mu > 0$,*
Case B: *there exists a positive number μ^* such that* (SWP) *has a unique solution $(c, q) = (c_B, q_B)$ for each $\mu \in (0, \mu^*)$, whereas* (SWP) *has no solution for $\mu \geq \mu^*$.*

5.1. **Asymptotic spreading speed.** In this subsection, we discuss the asymptotic speed of the free boundary for the spreading solution of (FBP).

Theorem 5.2. *Let* (u, h) *be the solution of* (FBP) *and let* c_S, c_B *and* μ^* *be positive constants given in Theorem 5.1.*

(i) *If* (u, h) *is a small spreading solution and* $\liminf_{t\to\infty} \|u(t)\|_{C([0,h(t)])} < u_2^*$, *then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S.$$

(ii) *If* (u, h) *is a big spreading solution and* (SWP) *has a unique solution, then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_B,$$

and, if (SWP) *has no solution, then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S.$$

We can prove this theorem in the same way as [11, Theorem 2]. See also [13, Theorem 4.2].

5.2. **Asymptotic profiles of solutions.** We will show sharp estimates of spreading speed and profile to each spreading solution for $h(t)/2 \le x \le h(t)$. (See [11, Theorems 3, 5 and 6].)

Theorem 5.3. *Let* (u, h) *be any small spreading solution of* (FBP) *satisfying* $\liminf_{t\to\infty} \|u(t)\|_{C([0,h(t)])} < u_2^*$. *Then there exists a constant* $H_S \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} (h(t) - c_S t) = H_S \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_S.$$

Moreover, it holds that

$$\lim_{t\to\infty} \sup_{h(t)/2\le x\le h(t)} |u(t, x) - q_S(h(t) - x)| = 0.$$

Here (c_S, q_S) *is a unique solution of* (SWP) *with* $u^* = u_1^*$.

Theorem 5.4. *Let* (u, h) *be any big spreading solution of* (FBP) *and assume that* (SWP) *with* $u^* = u_3^*$ *has a unique solution* (c_B, q_B). *Then there exists a constant* $H_B \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} (h(t) - c_B t) = H_B \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_B.$$

Moreover, it holds that

$$\lim_{t\to\infty} \sup_{h(t)/2\le x\le h(t)} |u(t, x) - q_B(h(t) - x)| = 0.$$

As to an asymptotic estimate in another interval containing $x = 0$, we get the convergence to the stationary solutions.

Theorem 5.5. *Let* (u, h) *be any small spreading solution of* (FBP) *satisfying* $\liminf_{t\to\infty} \|u(t)\|_{C([0,h(t)])} < u_2^*$. *Then for any* $c \in (0, c_S)$, *it holds that*

$$\lim_{t\to\infty} \sup_{0\le x\le ct} |u(t, x) - v_1(x)| = 0,$$

where v_1 is the solution of (SP) *given in Proposition 3.1. Similarly, let* (u, h) *be any big spreading solution of* (FBP) *and assume that* (SWP) *with* $u^* = u_3^*$ *has a unique solution* (c_B, q_B)*. Then for any* $c \in (0, c_B)$*, it holds that*

$$\lim_{t\to\infty} \sup_{0\le x\le ct} |u(t,x) - v_3(x)| = 0,$$

where v_3 is the solution of (SP) *given in Proposition 3.1.*

Remark 3. Consider any big spreading solution (u, h) of (FBP) when (SWP) with $u^* = u_3^*$ has no solutions. In this case, the result of [12, Proposition 4.2] implies that

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le c_S,$$

where c_S is a semi-wave speed of solution (q_S, c_S) of (SWP) with $u^* = u_1^*$. On the other hand, Theorem 2.3 implies

$$\liminf_{t\to\infty} \frac{h(t)}{t} \ge c_S.$$

Therefore, it holds that

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S.$$

When we discuss a big spreading solution of (FBP-N) in the case (SWP) with $u^* = u_3^*$ has no solutions, we already know from [12] that it possesses a propagating terrace. This is composed of a semi-wave corresponding to a small spreading solution and a traveling wave connecting u_1^* and u_3^*. Therefore we infer that any big spreading solution of (FBP) also has a similar propagating terrace.

Acknowledgments

Maho Endo would like to thank Prof. Ken Shirakawa for giving her the opportunity to present the study in RIMS workshop. Yuki Kaneko has been Partially supported by Waseda University Grant for Special Research Project (2018K-203, 2018B-103). Yoshio Yamada has been Partially supported by Grant-in-Aid for Scientific Research (C) 16K-05244.

References

[1] S. Angenent, The zero set of a solution of a parabolic equation, *J. Reine Angew. Math.*, **390** (1988), 79–96.

[2] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media*, **7** (2012), 583–603.

[3] Y. Du and Z. G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, **42** (2010), 377–405, *SIAM J. Math. Anal.*, **45** (2013), 1995–1996 (Erratum).

[4] Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J. Eur. Math. Soc.*, **17** (2015), 2673–2724.

[5] Y. Du, B. Lou and M. Zhou, Nonlinear diffusion problems with free boundaries: convergence, transition speed, and zero number arguments, *SIAM J. Math. Anal.*, **47** (2015), 3555–3584.

[6] Y. Du and H. Matano, Convergence and sharp thresholds for propagation in nonlinear diffusion problems, *J. Eur. Math. Soc.*, **12** (2010), 279–312.

[7] Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, *SIAM J. Math. Anal.*, **46** (2014), 375–396.

[8] M. Endo, Y. Kaneko and Y. Yamada, Free boundary problem for a reaction-diffusion equation with positive bistable nonlinearity, submitted.

[9] Y. Kaneko, K. Oeda and Y. Yamada, Remarks on spreading and vanishing for free boundary problems of some reaction-diffusion equations, *Funkcial. Ekvac.*, **57** (2014), 449–465.

[10] Y. Kaneko and Y. Yamada, A free boundary problem for a reaction-diffusion equation appearing in ecology, *Adv. Math. Sci. Appl.*, **21** (2011), 467–492.
[11] Y. Kaneko and Y. Yamada, Spreading speed and profiles of solutions to a free boundary problem with Dirichlet boundary conditions, *J. Math. Anal. Appl.*, **465** (2018), 1159–1175.
[12] Y. Kaneko, H. Matsuzawa and Y. Yamada, Asymptotic profiles of solutions and propagating terrace for a free boundary problem of reaction-diffusion equation with positive bistable nonlinearity, preprint.
[13] Y. Kawai and Y. Yamada, Multiple spreading phenomena for a free boundary problem of a reaction-diffusion equation with a certain class of bistable nonlinearity, *J. Differential Equations*, **261** (2016), 538–572.
[14] D. Ludwig, D.G. Aronson and H.F. Weinberger, Spatial patterning of the spruce budworm, *J. Math. Biol.*, **8** (1979), 217–258.
[15] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations*, **39** (1981), 269–290.

Department of Pure and Applied Mathematics
Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555,
JAPAN
E-mail address: e-maho@toki.waseda.jp
E-mail address: y.kaneko@aoni.waseda.jp
E-mail address: yamada@waseda.jp

早稲田大学大学院 基幹理工学研究科　遠藤 真帆
早稲田大学 基幹理工学部 数学科　兼子 裕大
早稲田大学 理工学術院　山田 義雄