Spreading and vanishing in a free boundary problem for nonlinear diffusion equations with a given forced moving boundary

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1 Introduction and Main Results

In this article, based on a recent work [16] with Kaneko, we consider the following free boundary problem of the nonlinear diffusion equation:

$$\begin{cases} u_t = u_{xx} + f(u), & t>0,\ ct<x<h(t), \\ u(t,ct) = u(t,h(t)) = 0, & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,x)=u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.1}$$

where f satisfies

$$f \in C^1, \quad f(0)=0 \ \text{ and there exists } K>0 \text{ such that } f(u)<0 \ \text{ for } u>K. \tag{1.2}$$

c, μ and h_0 are given positive constants, so $x=ct$ is a given forced moving boundary with its speed c. Moving boundary $x=h(t)$ is to be determined together with $u(t,x)$. For any given $h_0>0$ and $u_0 \in \mathscr{X}(h_0)$, we say a pair $(u(t,x),h(t))$ a classical solution of (1.1) on time a time interval $[0,T]$ for some $T>0$ if it satisfies $u \in C^{1,2}(G_T)$ and $h \in C^1([0,T])$ and all the identities in (1.1) are satisfied pointwise, where

$$G_T := \{(t,x):\ t \in (0,T],\ x \in [ct,h(t)]\}.$$

and

$$\mathscr{X}(h_0) := \left\{ \phi \in C^2[0,h_0] :\ \begin{array}{l} \phi(0)=\phi(h_0)=0, \\ \phi'(h_0)<0,\ \phi(x)>0 \text{ in } (0,h_0) \end{array} \right\}.$$

We study the dynamical behavior of solutions for (1.1) with three types of nonlinear term $f(u)$:

$$(f_{\mathrm{M}}) \ \text{ monostable case}, \quad (f_{\mathrm{B}}) \ \text{ bistable case}, \quad (f_{\mathrm{C}}) \ \text{ combustion case}.$$

In the monostable case (f_{M}), we assume that f is C^1 and it satisfies

$$f(0)=f(1)=0,\ f'(0)>0,\ f'(1)<0,\ (1-u)f(u)>0 \text{ for } u>0, u\neq 1.$$

A typical example of f which satisfies (f_{M}) is $f(u)=u(1-u)$.

In the bistable case (f_B), we assume that f is C^1 and it satisfies

$$\begin{aligned}&f(0)=f(\theta)=f(1)=0,\\&f(u)<0 \text{ in } (0,\theta),\ f(u)>0 \text{ in } (\theta,1),\ f(u)<0 \text{ in } (1,\infty)\end{aligned}$$

for some $\theta \in (0,1)$, $f'(0)<0$, $f'(1)<0$ and $\int_0^1 f(s)ds>0$. The function $f(u)=u(u-\theta)(1-u)$ with $\theta \in \left(0,\frac{1}{2}\right)$ is a typical example of f which satisfies (f_B).

In the combustion case (f_C), we assume that f is C^1 and it satisfies

$$f(u)=0 \text{ in } [0,\theta],\quad f(u)>0 \text{ in } (\theta,1),\ f'(1)<0,\ f(u)<0 \text{ in } [1,\infty)$$

for some $\theta \in (0,1)$, and there exists a small $\delta_0>0$ such that

$$f(u) \text{ is nondecreasing in } (\theta,\theta+\delta_0).$$

This model may be used to describe the spreading of a new or invasive species with population density $u(t,x)$ over one dimensional habitat $(ct,h(t))$. The free boundary $x=h(t)$ represents the spreading front. The behavior of the free boundary is determined by the Stefan-like condition which implies that the population pressure at the free boundary is driving force of the spreading front. In this model, we impose zero Dirichlet boundary condition at left moving boundary $x=ct$. This means that the left boundary of the habitat is a very hostile environment for the species and that the habitat is eroded away by the left moving boundary at constant speed c.

Recently, problem (1.1) with $c=0$ was studied in pioneer paper [6](in which Neumann boundary condition is imposed at left fixed boundary $x=0$), [14] and [15]. The authors showed that (1.1) has a unique solution which is defined for all $t>0$ and one of the following situation happens:

- (vanishing) $\lim_{t\to\infty} h(t)=h_\infty<\infty$ and $\lim_{t\to\infty}\|u(t,\cdot)\|_{C[0,h(t)]}=0$
- (spreading) $\lim_{t\to\infty} h(t)=\infty$ as $t\to\infty$ and

  $$\lim_{t\to\infty} u(t,x)=\begin{cases}1 & \text{Neumann condition case}\\ v(x) & \text{Dirichlet condition case}\end{cases}\quad \text{locally uniformly on } [0,\infty)$$

  where $v(x)$ is a unique positive solution of

  $$\begin{cases}v''+f(v)=0, & x>0,\\ v(0)=0, v(\infty)=1.\end{cases}$$

(except for a non-generic transition case when f is bistable or of combustion type). See also [7] for the double fronts free boundary problem with monostable, bistable or combustion type nonlinearity. Moreover, in the case of spreading, it is shown in [6, 7] that there exists $c^*=c^*(\mu)>0$ such that $\lim_{t\to\infty}(h(t)/t)=c^*$. In this sense, c^* is called the asymptotic spreading speed of corresponding free boundary problems. In [7], the authors showed that c^* is determined by the unique solution pair $(c,q)=(c^*,q^*)$ of the following problem

$$\begin{cases} q''+cq+f(q)=0, & z\in(-\infty,0),\\ q(0)=0,\ q(-\infty)=1,\ q'(0)=-c/\mu,\ q(z)>0 & z\in(-\infty,0). \end{cases} \tag{1.3}$$

Using a simple variation of the techniques in [6], we can see that for any $h_0>0$ and $u_0\in\mathscr{X}(h_0)$, (1.1) has a unique solution defined on some maximal time interval $(0,T_{\max})$ with maximal existence time $T_{\max}\in(0,\infty]$ (see (2.1)). The main purpose of this paper is to study the behavior of solutions to (1.1). When $T_{\max}=\infty$, the solution is global and so we can study its asymptotic behavior. On the other hand, in this problem, $T_{\max}$ may be a finite number for the reason that $h(t)-ct\to 0$ as $t\nearrow T_{\max}$, that is the habitat of the species may shrink to a single point. Such a phenomenon is observed first in free boundary problems considered by [4, 5]. We concern with the following questions:

(Q1) When the situation that $T_{\max}<\infty$ and $h(t)-ct\to 0$ as $t\nearrow T_{\max}$ occur?

(Q2) Can the situation that $T_{\max}=\infty$ and $h(t)-ct\to 0$ as $t\to\infty$ occur?

(Q3) When $T_{\max}<\infty$ and $h(t)-ct\to 0$ as $t\nearrow T_{\max}$, how about the behavior of u as $t\nearrow T_{\max}$ is ?

(Q4) When $T_{\max}=\infty$, reveal all possible long-time dynamical behavior of the solutions.

When $f(u)=u(1-u)$, this problem was considered in [17]. In [17], it was shown that for any initial data (u_0,h_0) with $u_0\in\mathscr{X}(h_0)$, exactly one of the three behaviours, called vanishing, spreading and transition, occurs.

The main purpose of this paper is to investigate the dynamical behavior of solutions to (1.1) with three basic types of nonlinearity, monostable (f_{M}), bistable (f_{B}) and combustion type (f_{C}). We remark that some approaches in [17] come from [11] and heavily rely on the special form of the logistic nonlinearity. For our purpose, we have to take a quite different approach. In particular, we use approaches employed in [9, 7], which are investigations of ω-limit set of the solution to the problem considered in a moving frame with its speed c.

Now we state our main theorems. First theorem is a trichotomy result for the case $0<c<c^*$.

Theorem A. *Suppose that f is (f_{M}), (f_{B}) or (f_{C}), $0<c<c^*$ and let (u,h) be the unique solution of (1.1) on a time interval $[0,T_{\max})$ with maximal existence time $T_{\max}$. Then exactly one of the following happens:*

(1) **Vanishing:** *$T_{\max}<\infty$, $\lim_{t\nearrow T_{\max}}(h(t)-ct)=0$ and*

$$\lim_{t\nearrow T_{\max}}\left\{\max_{x\in[ct,h(t)]}u(t,x)\right\}=0.$$

(2) **Spreading:** $T_{\max}=\infty$, $\lim_{t\to\infty}(h(t)/t)=c^*$ *and*

$$\lim_{t\to\infty} u(t,ct+z)=\phi^S(z) \quad \textit{uniformly in any compact subset of } [0,\infty),$$

where ϕ^S *is a unique solution to*

$$\begin{cases} \phi''+c\phi'+f(\phi)=0,\ \phi(z)>0,\ z\in(0,\infty), \\ \phi(0)=0,\ \phi(\infty)=1,\ z\in(0,\infty). \end{cases} \tag{1.4}$$

Moreover for any small $\varepsilon>0$

$$\lim_{t\to\infty}\left\{\max_{x\in[(c+\varepsilon)t,(c^*-\varepsilon)t]}|u(t,x)-1|\right\}=0. \tag{1.5}$$

(3) **Transition:** $T_{\max}=\infty$, $\lim_{t\to\infty}(h(t)-ct)=L_c$ *and*

$$\lim_{t\to\infty}\left\{\max_{x\in[ct,h(t)]}|u(t,x)-\mathcal{V}_c(x-h(t)+L_c)|\right\}=0,$$

where $L_c>0$ *and* $\mathcal{V}_c$ *are determined uniquely by the problem*

$$\begin{cases} \mathcal{V}''+c\mathcal{V}'+f(\mathcal{V})=0,\ \mathcal{V}(z)>0 \ \text{ for } \ z\in(0,L), \\ \mathcal{V}(0)=\mathcal{V}(L)=0,\ -\mu\mathcal{V}'(L)=c. \end{cases}$$

This theorem means that the classification for the dynamical behavior of solutions to (1.1) with three basic types of nonlinearity, (f_{M}), (f_{B}) and (f_{C}), can be expressed in a unified fashion. This result is surprising for us because when $c=0$, it was shown in [7] that the classification of asymptotic behaviors of the solutions strongly depends on the nonlinearity f.

If the initial function u_0 in (1.1) has the form $u_0=\sigma\phi$ $(\sigma>0)$ with some fixed $\phi\in\mathscr{X}(h_0)$, we can obtain the following sharp threshold result.

Theorem B. *Suppose that the initial function* u_0 *in* (1.1) *has the form* $u_0=\sigma\phi$ *with some fixed* $\phi\in\mathscr{X}(h_0)$. *Then there exists* $\overline{\sigma}\in(0,\infty]$ *such that vanishing happens when* $0<\sigma<\overline{\sigma}$, *spreading occurs when* $\sigma>\overline{\sigma}$, *and transition occurs when* $\sigma=\overline{\sigma}$.

When $c\geq c^*$, vanishing always happens.

Theorem C. *Assume that* $c^*\leq c$ *and* (u,h) *is the unique solution of* (1.1) *on a time interval* $[0,T_{\max})$ *with maximal existence time* $T_{\max}$. *Then we have* $T_{\max}<\infty$ *and* $\lim_{t\nearrow T_{\max}}(h(t)-ct)=0$ *and* $\lim_{t\nearrow T_{\max}}\max_{x\in[ct,h(t)]}u(t,x)=0$.

For the spreading case, the asymptotic profile of the solution over the whole domain will be obtained in the following theorem.

Theorem D. *Assume that* f *is* (f_{M}), (f_{B}) *or* (f_{C}), $0<c<c^*$ *and let* (u,h) *be the unique global solution to* (1.1). *If spreading happens in the sense of Theorem A, then for any* $\tilde{c}\in(c,c^*)$, *the following conclusions hold:*

(1) *There exists* $H_\infty \in \mathbb{R}$ *such that* $\lim_{t\to\infty}(h(t) - c^*t) = H_\infty$*;*

(2) $\lim_{t\to\infty} \sup_{x\in[ct,\tilde{c}t]} |u(t,x) - \phi^S(x - ct)| = 0$*;*

(3) $\lim_{t\to\infty} \sup_{x\in[\tilde{c}t,h(t)]} |u(t,x) - q^*(x - h(t))| = 0$*;*

where ϕ^S *is given in Theorem A,* q^* *and* c^* *are determined by the unique solution pair to* (1.3)*.*

2 Basic Results

In this section, I will give some basic results.

2.1 Existence and uniqueness of the solution

Proposition 2.1. *Assume that* f *satisfies* (1.2)*. For any* $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$ *and* $\alpha \in (0,1)$*, there exists* $T > 0$ *such that problem* (1.1) *admit a unique solution* (u,h) *defined on* $(0,T]$ *with*

$$u \in C^{\frac{1+\alpha}{2},1+\alpha}(\overline{D}_T) \cap C^{1+\frac{\alpha}{2},2+\alpha}(D_T),\ \ h \in C^{1+\frac{\alpha}{2}}([0,T]),$$

where $D_T := \{(t,x) \in \mathbb{R}^2 : t \in (0,T], x \in [ct, h(t)]\}$*. Moreover we have*

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}(D_T)} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C,$$

where C *and* T *depend only on* c*,* μ*,* h_0*,* α *and* $\|u_0\|_{C^2[0,h_0]}$*.*

Proposition 2.2. *Assume that* f *satisfies* (1.2) *and let* (u,h) *be any solution of* (1.1) *defined on* $[0,T_0]$ *with some* $T_0 \in (0,\infty)$*. Then the solution satisfies*

$$\begin{aligned} &0 < u(t,x) \le C_1 \quad \textit{for}\ \ 0 < t \le T_0,\ ct < x < h(t),\\ &0 < h'(t) \le \mu C_2 \quad \textit{for}\ \ 0 < t \le T_0, \end{aligned}$$

where C_1 *and* C_2 *are positive constants independent of* T_0*.*

Moreover the solution can be extended to some interval $[0,\overline{T})$ *with* $\overline{T} > T_0$ *if* $\inf_{t\in(0,T_0)]}[h(t) - ct] > 0$*.*

Now for any $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, we can define the maximal existence time $T_{\max} \in (0,\infty]$ of solution to (1.1) in the following way:

$$T_{\max} := \sup\{T > 0 : (u,h) \text{ is the solution to (1.1) on } [0,T]\}. \tag{2.1}$$

2.2 Comparison Principle

In the proof the main theorems, we will frequently construct suitable upper and lower solutions.

Lemma 2.3. *Let* ξ, $\overline{h} \in C^1([0,T])$ *and* $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T,\ \xi(t) < x < \overline{h}(t)\}$ *for* $T \in (0,\infty)$ *satisfy*

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} - f(\overline{u}) \ge 0, & 0 < t \le T,\ \xi(t) < x < \overline{h}(t), \\ \overline{u}(t, \overline{h}(t)) = 0, & 0 < t \le T, \\ \overline{h}'(t) \ge -\mu \overline{u}_x(t, \overline{h}(t)), & 0 < t \le T. \end{cases}$$

For a solution (u,h) *to* (1.1), *if*

$$ct \le \xi(t),\ u(t,\xi(t)) \le \overline{u}(t,\xi(t)) \quad \text{for} \quad 0 < t \le T,$$
$$h_0 \le \overline{h}(0),\ u_0(x) \le \overline{u}(0,x) \quad \text{for} \quad \xi(0) \le x \le h_0,$$

then

$$h(t) \le \overline{h}(t) \quad \text{for } 0 < t \le T,$$
$$u(t,x) \le \overline{u}(t,x) \quad \text{for } 0 < t \le T,\ \xi(t) < x < h(t).$$

Proof. We can prove this lemma by using a few modifications of the proof of [6, Lemma 3.5]. See the proof of [17, Lemma 2.5]. □

The function $\overline{u}$ or the pair $(\overline{u}, \overline{h})$ in Lemma 2.3 is usually called an upper solution of problem (1.1). We can define a lower solution by reversing all the inequalities in suitable places.

2.3 Zero number arguments

Our arguments in the present paper rely on the zero number argument that was originally developed by Angenent [1]. For later use, we give some basic results of the zero number argument, which is a variant of Theorem C and D in [1]. See also [8].

Lemma 2.4. *Let* $u : [0,T] \times [0,1] \to \mathbb{R}$ *be a bounded classical solution of*

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u, \tag{2.2}$$

with boundary conditions

$$u(t,0) = l_0(t),\ u(t,1) = l_1(t),$$

where l_0, $l_1 \in C^1[0,T]$, *and* l_0 *and* l_1 *satisfies*

$$l_i(t) \equiv 0 \quad \text{for} \quad t \in [0,T] \quad \text{or} \quad l_i(t) \ne 0 \quad \text{for any} \quad t \in [0,T]$$

for each $i = 0, 1$. *Assume that*

$$a, 1/a, a_t, a_x, a_{xx}, b, b_t, b_x, c \in L^\infty,\ a > 0, \quad \text{and} \quad u(0,\cdot) \not\equiv 0 \quad \text{when} \quad l_0 = l_1 = 0.$$

Let $z(t) = \mathcal{Z}(u(t,\cdot))$ *denote the number of zeros of* $u(t,\cdot)$ *in* $[0,1]$. *Then*

(a) *for each* $t \in (0,T]$, $z(t)$ *is finite,*

(b) $z(t)$ *is nonincreasing in* t,

(c) *if for some* $s \in (0,T)$ *the function* $u(s,\cdot)$ *has a degenerate zero* $x_0 \in [0,1]$, *that is,*

$$u(s,x_0) = u_x(s,x_0) = 0$$

holds, then $z(t_1) > z(t_2)$ *for all* $t_1 < s < t_2$.

The following variant of zero number result is useful for our problem.

Lemma 2.5. *Let* $\xi_1(t)$ *and* $\xi_2(t)$ *be continuous functions of* $t \in (t_0,t_1)$ *and assume that* $\xi_1(t) < \xi_2(t)$ *for* $t \in (t_0,t_1)$ *and that functions* a, b *and* c *satisfy same conditions in Lemma 2.4. Suppose that* $u(t,x)$ *is a continuous function of* $t \in (t_0,t_1)$ *and* $x \in [\xi_1(t),\xi_2(t)]$, *and satisfies* (2.2) *in the classical sense for* $t \in (t_0,t_1)$ *and* $x \in (\xi_1(t),\xi_2(t))$ *with*

$$u(t,\xi_1(t)) \neq 0,\ \ u(t,\xi_2(t)) \neq 0 \quad \textit{for } t \in (t_0,t_1).$$

Let $Z(t) = \mathcal{Z}(u(t,\cdot))$ *denote the number of zeros of* $u(t,\cdot)$ *in* $[\xi_1(t),\xi_2(t)]$. *Then*

(a) *for each* $t \in (t_0,t_1)$, $Z(t)$ *is finite,*

(b) $Z(t)$ *is nonincreasing in* t,

(c) *if for some* $s \in (t_0,t_1)$ *the function* $u(s,\cdot)$ *has a degenerate zero* $x_0 \in (\xi_1(s),\xi_2(s))$, *that is,*

$$u(s,x_0) = u_x(s,x_0) = 0,$$

holds, then $Z(s_1) > Z(s_2)$ *for all* s_1, s_2 *satisfying* $t_0 < s_1 < s < s_2 < t_1$.

We can find the proof of Lemma 2.5 in [5] and [8].

Remark 1. From the proof of Lemma 2.5(see [8, Lemma 2.2]), it is easily seen that Lemma 2.5 holds for the case where

$$\begin{aligned}
&\xi_1(t) \equiv 0 \ \text{ and } \ \xi_2(t) > 0 \ \text{ for } \ t \in (t_0,t_1),\\
&u(t,\xi_1(t)) = 0 \ \text{ for } \ t \in (t_0,t_1),\\
&u(t,\xi_2(t)) \neq 0 \ \ t \in (t_0,t_1).
\end{aligned}$$

We recall basic properties of $\mathcal{Z}$, number of zeros, from [9]. If $w_n(x) \to w(x)$ as $n \to \infty$ pointwisely, then we have

$$w \equiv 0 \ \text{ or } \ \mathcal{Z}(w) \leq \liminf_{n\to\infty} \mathcal{Z}(w_n).$$

Moreover, if I is a compact interval (e.g. $I = [a, b]$) and if

$$\begin{aligned}
&w_n \to w \text{ in } C^1(I) \text{ as } n \to \infty,\\
&\text{every zero of } w \text{ in } I \text{ is simple},\\
&w \neq 0 \text{ on the boundary of } I \text{ or } w_n(a) = w(a) = 0,\ w'(a) \neq 0,\ w(b) \neq 0
\end{aligned}$$

then

$$\mathcal{Z}_I(w) = \lim_{n\to\infty} \mathcal{Z}_I(w_n), \tag{2.3}$$

where $\mathcal{Z}_I$ denotes the number of zeros of a function in I.

By a simple modification of the proof of [9, Lemma 2.6], we can show the following lemma.

Lemma 2.6. *Let $I = [0, l)$ and let $\{w_n(t, x)\}$ be a sequence of functions and w be a function defined on $(t_1, t_2) \times I$ which satisfy*

- *$w_n(t, x) \to w(t, x)$ as $t \to \infty$ in $C^1((t_1, t_2) \times I)$,*
- *$w_n(t, 0) = 0$ and $w_n(t, \cdot)$ has only simple zero in I,*
- *$w(t, 0) = 0$ and w satisfies (2.2) on $(t_1, t_2) \times \mathrm{int} I$.*

If for every $t \in (t_1, t_2)$, $w(t, x) \not\equiv 0$ on I and the number of zeros of w on I is finite, then $w(t, \cdot)$ has only simple zeros on I.

2.4 Properties of vanishing solutions

In this subsection, we give some properties of the solutions for which vanishing happen. We assume that f satisfies (1.2) and let (u, h) be a unique solution to (1.1) defined on $[0, T_{\max})$ with maximal existence time $T_{\max} \in (0, \infty]$. Following proof in section 3 of [17] we can obtain the following proposition.

Proposition 2.7 ([17])**.** *If there exists $\hat{T} \in (0, \infty]$ such that $\lim_{t\nearrow\hat{T}}[h(t) - ct] = 0$, then $\hat{T} < \infty$. Moreover u satisfies $\lim_{t\nearrow\hat{T}} \|u(t, \cdot)\|_{C([ct,h(t)])} = 0$.*

Now we give a sufficient condition for vanishing.

Proposition 2.8 ([17])**.** *There exists a positive number $C = C(h_0, c)$ such that if $\|u_0\|_{C([0,h_0])} \leq C$, then $T_{\max} < \infty$ and $\lim_{t\nearrow T_{\max}}(h(t) - ct) = 0$.*

Finally, by Lemma 2.2, the proof of Proposition 2.7 and Proposition 2.8, we can show the following proposition(see [17]).

Proposition 2.9 ([17])**.** *$T_{\max} < \infty$ if and only if $\lim_{t\nearrow T_{\max}}(h(t) - ct) = 0$.*

2.5 Stationary solutions

Define $v(t,z) := u(t,z+ct)$ and $H_c(t) := h(t) - ct$. It is clear that (v, H_c) satisfies

$$\begin{cases} v_t = v_{zz} + cv_z + f(v), & t>0,\ 0<z<H_c(t), \\ v(t,0) = v(t,H_c(t)) = 0, & t>0, \\ H_c'(t) = -\mu v_z(t,H_c(t)) - c, & t>0, \\ H_c(0) = h_0, v(0,z) = u_0(z), & 0 \le z \le h_0. \end{cases} \tag{2.4}$$

The stationary problem corresponding to (2.4) is the following problem

$$\begin{cases} v_{zz} + cv_z + f(v) = 0, \quad z>0, \\ v(0) = 0. \end{cases}$$

We recall the phase plane analysis in [13, 12] for the following equation:

$$q''(z) + \gamma q'(z) + f(q) = 0, \quad z > 0. \tag{2.5}$$

The above equation is equivalent to the following first order differential system:

$$\begin{cases} q'(z) = p \\ p'(z) = -\gamma p - f(q). \end{cases} \tag{2.6}$$

A solution $(q(z), p(z))$ of the above system generates the trajectory in the q–p phase plane. Such a trajectory has a slope

$$\frac{dp}{dq} = -\gamma - \frac{f(q)}{p}$$

at any point where $p \neq 0$.

For (f_{M}), (f_{B}) and (f_{C}), we are only interested in the case $\gamma \in (0, c_0)$, where c_0 is the minimal speed of the traveling wave when f is (f_{M}) or it is the unique speed of the traveling wave when f is (f_{B}) or (f_{C}).

When f is (f_{M}), $(0,0)$ and $(1,0)$ are two singular points on the phase plane. For such a $\gamma \in (0, c_0)$, the eigenvalues of the corresponding linearizations at the singular points are

$$\lambda_{0,\gamma}^{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4f'(0)}}{2} \quad \text{at } (0,0),$$
$$\lambda_{1,\gamma}^{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4f'(1)}}{2} \quad \text{at } (1,0),$$

Since $f'(0) > 0$ and $f'(1) < 0$, $(1,0)$ is a saddle point, $(0,0)$ is a center when $\gamma = 0$, or a focus when $0 < \gamma < c_0$. By phase plane analysis (see [2, 3, 7, 13]), one can easily obtain all kinds of bounded, nonnegative solution of (2.5).

Consider the following initial value problem for $\gamma \in (0, c_0)$ and $\alpha \ge 0$:

$$\begin{cases} \phi'' + \gamma\phi' + f(\phi) = 0, \ z > 0, \\ \phi(0) = 0, \ \phi'(0) = \alpha. \end{cases} \tag{2.7}$$

By phase plane analysis by [13], we can obtain all kinds of solution ϕ to (2.7).

Proposition 2.10. *Suppose that f is (f_{M}) and $\gamma \in (0, c_0)$, then there exists $\alpha_\gamma \in (0, \infty)$ such that the solutions ϕ to (2.7) can be classified as follows:*

(i) *When $\alpha = 0$, $\phi = \phi_0^{\mathrm{M}}(z) \equiv 0$ for $z > 0$.*

(ii) *For $\alpha \in (0, \alpha_\gamma)$ there exists a unique $l(\alpha) > 0$, ϕ is a solution with compact support $(0, l(\alpha))$ (we call such ϕ type ϕ_1^{M}), that is*

$$\phi_1^{\mathrm{M}}(0) = 0,\ \phi_1^{\mathrm{M}}(l(\alpha)) = 0,\ \phi_1^{\mathrm{M}}(z) > 0 \ \text{ for } \ z \in (0, l(\alpha)).$$

(iii) *When $\alpha = \alpha_\gamma$, ϕ is a strictly increasing solution (we call such ϕ type ϕ_2^{M}), that is*

$$\phi_2^{\mathrm{M}}(0) = 0,\ \phi_2^{\mathrm{M}}(\infty) = 1,\ \phi_2^{\mathrm{M}}(z) > 0,\ (\phi_2^{\mathrm{M}})'(z) > 0 \text{ for } z > 0.$$

(iv) *For $\alpha > \alpha_\gamma$, there exists $z_0 > 0$ which depends on α such that $\phi(z_0) > 1$ (we call such ϕ as ϕ_3^{M}).*

When f is (f_{B}), $(0,0)$, $(\theta, 0)$ and $(1,0)$ are three singular points on the phase plane. The eigenvalues of the corresponding linearizations at singular point $(\theta, 0)$ is

$$\lambda_{\theta,\gamma}^{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4f'(\theta)}}{2} \quad \text{at } (\theta, 0)$$

Since $f'(0) < 0$, $f'(\theta) > 0$ and $f'(1) < 0$, $(0,0)$ and $(1,0)$ are saddle points, $(\theta, 0)$ is a center when $\gamma = 0$, or a focus when $0 < \gamma < 2\sqrt{f'(\theta)}$, or node $\gamma \geq 2\sqrt{f'(\theta)}$. It is easily seen that $c_0 < 2\sqrt{f'(\theta)}$. So for $c \in (0, c_0)$, singular point $(\theta, 0)$ is always focus. Hence one can obtain all kinds of bounded, nonnegative solution of (2.5).

Consider initial value problem (2.7) for $\gamma \in (0, c_0)$ and $\alpha \geq 0$. By phase plane analysis by [12], we can obtain all kinds of solution ϕ to (2.7).

Proposition 2.11. *Suppose that f is (f_{B}) and $\gamma \in (0, c_0)$, then there exist α_γ, $\beta_\gamma \in (0, \infty)$ with $\beta_\gamma < \alpha_\gamma$ such that the solutions ϕ to (2.7) can be classified as follows:*

(i) *When $\alpha = 0$, $\phi = \phi_0^{\mathrm{B}}(z) \equiv 0$ for $z \geq 0$.*

(ii) *For $\alpha \in (0, \beta_\gamma)$ ϕ is a solution with θ-tail (we call such ϕ type ϕ_1^{B}), that is*

$$\phi_1^{\mathrm{B}}(0) = 0,\ \phi_1^{\mathrm{B}}(\infty) = \theta,\ \phi_1^{\mathrm{B}}(z) > 0 \ \textit{ for } \ z > 0$$

(iii) *When $\alpha = \beta_\gamma$, ϕ is a positive solution on the half-line (we call such ϕ type ϕ_2^{B}), that is*

$$\phi_2^{\mathrm{B}}(0) = 0,\ \phi_2^{\mathrm{B}}(\infty) = 0,\ \phi_2^{\mathrm{B}}(z) > 0 \textit{ for } z > 0.$$

(iv) *For $\alpha \in (\beta_\gamma, \alpha_\gamma)$, there exists $l(\alpha) > 0$ such that, ϕ is a solution with compact support $(0, l(\alpha))$ (we call such ϕ as ϕ_3^{B}), that is*

$$\phi_3^{\mathrm{B}}(0) = \phi_3^{\mathrm{B}}(l(\alpha)) = 0,\ \phi_3^{\mathrm{B}}(z) > 0 \textit{ for } z \in (0, l(\alpha)).$$

(v) *When* $\alpha = \alpha_\gamma$, ϕ *is a strictly increasing solution (we call such* ϕ *type* ϕ_4^{B}*), that is*

$$\phi_4^{\mathrm{B}}(0) = 0,\ \phi_4^{\mathrm{B}}(\infty) = 1,\ \phi_4^{\mathrm{B}}(z) > 0,\ (\phi_4^{\mathrm{B}})'(z) > 0 \textit{ for } z > 0.$$

(vi) *For* $\alpha > \alpha_\gamma$, *there exists* z_0 *which depends on* α *such that* $\phi(z_0) > 1$ *(we call such* ϕ *type* ϕ_5^{B}*).*

Next consider problem (2.7) for f satisfying (f_{C}).

Proposition 2.12 (Proposition 2.8 in [16])**.** *Suppose that* f *is* (f_{C}) *and* $\gamma \in (0, c_0)$, *then there exist* α_γ, $\beta_\gamma \in (0, \infty)$ *with* $\beta_\gamma < \alpha_\gamma$ *such that the solutions* ϕ *to* (2.7) *can be classified as follows:*

(i) *When* $\alpha = 0$, $\phi \equiv 0$ *for* $z \geq 0$.

(ii) *For* $\alpha \in (0, \gamma\theta]$, *solution* ϕ *satisfies*

$$\phi(0) = 0,\ \phi(\infty) \in (0, \theta],\ \phi(z) > 0,\ \phi'(z) > 0 \textit{ for } \ z > 0$$

We call this type solution type ϕ_1^{C}.

(iii) *For* $\alpha \in (\gamma\theta, \beta_\gamma)$, *there exists* $m(\alpha) > 0$ *such that* ϕ *satisfies*

$$\begin{aligned}&\phi(0) = 0,\ \phi(\infty) \in (0, \theta),\ \phi(z) > 0 \textit{ for } z > 0 \textit{ and}\\ &\phi'(z) > 0 \text{ for } z \in (0, m(\alpha)),\quad \phi'(m(\alpha)) = 0,\quad \phi'(z) < 0 \textit{ for } z \in (m(\alpha), \infty).\end{aligned}$$

We call this type solution type ϕ_2^{C}.

(iv) *When* $\alpha = \beta_\gamma$, ϕ *satisfies*

$$\phi(0) = 0,\ \phi(\infty) = 0,\ \phi(z) > 0 \textit{ for } z > 0.$$

We call this type solution type ϕ_3^{C}.

(v) *For* $\alpha \in (\beta_\gamma, \alpha_\gamma)$ *there exists* $l(\alpha) > 0$ *such that,* ϕ *is a solution with compact support* $(0, l(\alpha))$, *that is*

$$\phi(0) = \phi(l(\alpha)) = 0,\ \phi(z) > 0 \text{ for } z \in (0, l(\alpha)).$$

We call such ϕ *type* ϕ_4^{C}.

(vi) *When* $\alpha = \alpha_\gamma$, ϕ *is a strictly increasing solution, that is*

$$\phi(0) = 0,\ \phi(\infty) = 1,\ \phi(z) > 0,\ \phi'(z) > 0 \textit{ for } z > 0.$$

We call such ϕ *type* ϕ_5^{C}.

(vii) *When* $\alpha > \alpha_\gamma$, *there exists* $z_0 > 0$ *which depends on* α *such that* $\phi(z_0) > 1$. *We call such* ϕ *type* ϕ_6^{C}.

In what follows, we use notation $\phi_j^i(z)$ $(i = \mathrm{M},\ \mathrm{B},\ \mathrm{C},\ j \in \mathbb{N})$ to express one of the type ϕ_j^i solutions of (2.7).

To obtain (1.5) in Theorem A, we will construct a lower solution. For the construction of the lower solution we prepare the following lemma which is easily obtained by a simple phase plane argument.

Lemma 2.13. *Suppose that f is (f_{M}), (f_{B}) or (f_{C}) and $\gamma \in (0, c_0)$.*

(1) *When f is $(\mathrm{f_M})$, for any $Q \in (0,1)$, there exist a unique $\alpha \in (0, \alpha_\gamma)$, a unique $z(\gamma, Q) > 0$ and a solution to (2.7) of type ϕ_1^{M} such that*

$$\phi_1^{\mathrm{M}}(z(\gamma, Q)) = Q,\ \ (\phi_1^{\mathrm{M}})'(z(\gamma, Q)) = 0$$

(2) *When f is (f_{B}) or (f_{C}), there exists $Q_0 \in (0,1)$ such that the following holds. For each $Q \in (Q_0, 1)$, there exist $\alpha \in (\beta_\gamma, \alpha_\gamma)$, a unique positive number $z(\gamma, Q) > 0$ and the solution to (2.7) of type ϕ_3^{B} or type ϕ_4^{C}, respectively, such that*

$$\begin{aligned} &\phi_3^{\mathrm{B}}(z(\gamma, Q)) = Q,\ \ (\phi_3^{\mathrm{B}})'(z(\gamma, Q)) = 0 \ \ \textit{when } f \textit{ is } (f_{\mathrm{B}}), \\ &\phi_4^{\mathrm{C}}(z(\gamma, Q)) = Q,\ \ (\phi_4^{\mathrm{C}})'(z(\gamma, Q)) = 0 \ \ \textit{when } f \textit{ is } (f_{\mathrm{C}}). \end{aligned}$$

Now we introduce the following notation:

$$\phi^S(z) = \phi^S(z\,;\,\gamma) := \begin{cases} \phi_2^{\mathrm{M}}(z) & (\text{when } f \text{ is } (f_{\mathrm{M}})), \\ \phi_4^{\mathrm{B}}(z) & (\text{when } f \text{ is } (f_{\mathrm{B}})), \\ \phi_5^{\mathrm{C}}(z) & (\text{when } f \text{ is } (f_{\mathrm{C}})), \end{cases} \tag{2.8}$$

$$q(z\,;\,c, Q) = \begin{cases} \phi_1^{\mathrm{M}}(z) & (\text{when } f \text{ is } (f_{\mathrm{M}})), \\ \phi_3^{\mathrm{B}}(z) & (\text{when } f \text{ is } (f_{\mathrm{B}})), \\ \phi_4^{\mathrm{C}}(z) & (\text{when } f \text{ is } (f_{\mathrm{C}})) \end{cases} \tag{2.9}$$

where ϕ_1^{M}, ϕ_3^{B} and ϕ_4^{C} above are functions which are uniquely determined for each Q in Lemma 2.13.

2.6 Traveling waves

To investigate global solutions we introduce semi-wave and compact supported traveling wave.

First we consider the following problem

$$\begin{cases} q'' + \gamma q' + f(q) = 0 \text{ in } (-\infty, 0), \\ q(0) = 0,\ q(-\infty) = 1,\ q(z) > 0 \text{ in } (-\infty, 0). \end{cases} \tag{2.10}$$

Proposition 2.14 (Proposition 1.8 and Theorem 6.2 of [7])**.** *Suppose that f is (f_{M}), (f_{B}) or (f_{C}). For any $\mu > 0$ there exists a unique $c^* = c^*(\mu) > 0$ such that (2.10) with $\gamma = c^*$ admits a unique solution q^* satisfying $(q^*)'(0) = -c^*/\mu$.*

We remark that this function q^* is shown in [7] to satisfy $(q^*)'(z) < 0$ for $z \le 0$.

We call q^* a *semi-wave with speed* c^*, since the function $w(t,x) := q^*(x - c^*t)$ satisfies

$$\begin{cases} w_t = w_{xx} + f(w) \quad \text{for} \quad t \in \mathbb{R}^1, \ x < c^*t, \\ w(t, c^*t) = 0, \ w_x(t, c^*t) = -c^*/\mu, \ w(t, -\infty) = 1, \quad t \in \mathbb{R}^1. \end{cases}$$

We give a result for a wave of finite length which is needed to construct a lower solution to obtain (1.5) in Theorem A. Consider the following problem:

$$\begin{cases} q'' + \gamma q' + f(q) = 0 \text{ for } z < 0, \\ q(0) = 0, \ q'(0) = -c^*/\mu. \end{cases} \tag{2.11}$$

Lemma 2.15 ([7])**.** *Suppose that f is (f_{M}), (f_{B}) or (f_{C}). For any $\gamma \in (0, c^*)$ and $\mu > 0$, there exists a unique $z^*(\gamma) > 0$ such that solution q_γ to* (2.11) *satisfies*

$$q_\gamma'(-z^*(\gamma)) = 0, \ q'(z) < 0 \quad \text{for } z \in (-z^*(\gamma), 0].$$

Moreover $\lim_{\gamma \nearrow c^*} z^*(\gamma) = \infty$, $\lim_{\gamma \nearrow c^*} q_\gamma(-z^*(\gamma)) = 1$.

We set

$$Q_\gamma = q_\gamma(-z^*(\gamma)). \tag{2.12}$$

Finally we introduce the compact supported traveling wave. We consider the following problem:

$$\begin{cases} \phi'' + \gamma \phi' + f(\phi) = 0, \quad z < 0, \\ \phi(0) = 0, \ \phi'(0) = -\alpha. \end{cases} \tag{2.13}$$

By virtue of a phase-plane analysis, for any $\gamma \in (0, c_0)$ there exists $P(\gamma) > 0$ such that

- For $\alpha \in (0, P(\gamma))$, there exists $L_\gamma(\alpha) > 0$ such that the solution ϕ to (2.13) satisfies

$$\phi(-L_\gamma(\alpha)) = \phi(0) = 0, \ \phi(z) > 0 \text{ for } z \in (-L_\gamma(\alpha), 0).$$

- For $\alpha = P(\gamma)$, the solution ϕ to (2.13) satisfies

$$\phi(-\infty) = 1, \ \phi(0) = 0, \ \phi(z) > 0, \ \phi'(z) < 0 \quad \text{for } z \in (-\infty, 0).$$

By [7, 13, 12], it was shown that $P(\gamma)$ is a continuous, monotone decreasing function of γ. We note that c^* is a unique solution to $P(\gamma) = \gamma/\mu$. Thus for any $c \in (0, c^*)$ we have $0 < c/\mu < P(c)$.

So there exists unique solution ϕ to the following problem:

$$\begin{cases} \phi'' + c\phi' + f(\phi) = 0, \quad -L_c(c/\mu) < z < 0. \\ \phi(-L_c(c/\mu)) = \phi(0) = 0, \ \phi'(0) = -c/\mu, \ \phi(z) > 0 \text{ for } z \in (-L_c(c/\mu), 0). \end{cases}$$

Set $L_c = L_c(c/\mu)$ and $\mathcal{V}_c(z) = \phi(z - L_c)$. Now we obtain the following lemma.

Lemma 2.16 ([13, 12]). *Suppose that f is $(f_{\rm M})$, $(f_{\rm B})$ or $(f_{\rm C})$. For any $\mu > 0$ and $c \in (0, c^*)$, there exist a unique constant $L_c > 0$ and a unique $\mathcal{V}_c \in C^2([0, L_c])$ such that*

$$\begin{cases} \mathcal{V}_c'' + c\mathcal{V}_c' + f(\mathcal{V}_c) = 0, \ \mathcal{V}_c > 0 \ \text{ for } \ z \in (0, L_c), \\ \mathcal{V}_c(0) = \mathcal{V}_c(L_c) = 0, \ -\mu\mathcal{V}'(L_c) = c. \end{cases}$$

Moreover when f is $(f_{\rm B})$ or $(f_{\rm C})$, $\mathcal{V}_c$ satisfies $\max_{z\in[0,L_c]} \mathcal{V}_c(z) > \theta$.

If we define function $w(t, x) := \mathcal{V}_c(x - ct)$, w satisfies

$$\begin{cases} w_t = w_{xx} + f(w) \ \text{ for } \ t \in \mathbb{R}^1, \ ct < x < ct + L_c, \\ w(t, ct) = w(t, ct + L_c) = 0, \ \ t \in \mathbb{R}^1, \\ -\mu w_x(t, ct + L_c) = c, \ t \in \mathbb{R}^1. \end{cases}$$

and w resemble a traveling wave with a compact support moving to the right at constant speed c.

3 Proof of Theorem A

In this paper we only give a sketch of proof of Theorem A. For the proof of Theorems B, C and D, please see [16].

Throughout this subsection, we assume that f is $(f_{\rm M})$, $(f_{\rm B})$ or $(f_{\rm C})$, $c \in (0, c^*)$ and (u, h) is a unique solution to (1.1) defined for all $t > 0$.

We begin with an upper estimate which is easily given by a simple comparison argument.

Lemma 3.1. *Suppose that (u, h) be a solution to (1.1) defined for any $t > 0$. Then for any $\delta \in (0, -f'(1))$ there exists $M_0 > 0$ such that $u(t, x) \le 1 + M_0 e^{-\delta t}$ for $t > 0$ and $x \in [ct, h(t)]$.*

Proof. See the proof of [17, Lemma 2.3] or [7, Lemma 6.5(iii)]. □

We next give an important lemma to show Theorem A.

Lemma 3.2. *If $h(t) - ct$ is unbounded function, then $\lim_{t\to\infty}(h(t) - ct) = \infty$. Moreover for any $l > 0$, there exists $T_l > 0$ such that*

$$u(t, x) \ge \mathcal{V}_c(x - ct - l) \ \text{ for } \ t \ge T_l, \ ct + l \le x \le ct + l + L_c.$$

where $\mathcal{V}_c$ is the function determined by Lemma 2.16.

Proof. See [11, Lemma 3.2] or [17, Lemma 4.2]. □

Note that if we define

$$z = x - ct, \ v(t, z) = u(t, z + ct), \ H_c(t) = h(t) - ct.$$

(v, H_c) satisfies (2.4). We first assume that $H_c(t)$ is unbounded. By Lemma 3.2, we have $\lim_{t\to\infty} H_c(t) = \infty$.

We will introduce a notion of ω-limit set as in [9] and [7]. Denote by $\omega(v)$ the ω-limit set of $v(t,\,\cdot\,)$ in the topology of $L^\infty_{\mathrm{loc}}([0,\infty))$. Then a function $w(z)$ belongs to $\omega(v)$ if and only if there exists a sequence $\{t_n\}$ with $0 < t_1 < t_2 < \cdots < t_n < \cdots \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} v(t_n, z) = w(z) \ \text{ locally uniformly in } [0,\infty). \tag{3.1}$$

It is easy to see that, when f is of (f_{M}), (f_{B}) or (f_{C}), $\omega(v)$ is not empty. By local parabolic estimates, we see that the convergence (3.1) implies that in the $C^2_{\mathrm{loc}}([0,\infty))$ topology. Thus the definition of $\omega(v)$ remains unchanged if the topology of $L^\infty_{\mathrm{loc}}([0,\infty))$ is replaced by that of $C^2_{\mathrm{loc}}([0,\infty))$.

It is well-known that $\omega(v)$ is compact and connected, and it is an invariant set. This means that for any $w \in \omega(v)$, there exists an entire orbit(namely a solution of $W_t = W_{zz} + cW_z + f(W)$ defined for all $t \in \mathbb{R}$ and $z \in [0,\infty)$) passing through w.

Choosing a suitable sequence $\{t_n\}$ with $0 < t_1 < t_2 < \cdots < t_n < \cdots \to \infty$, we can find such an entire solution $W(t,z)$ as follows:

$$v(t + t_n, z) \to W(t,z) \ \text{ as } \ n \to \infty. \tag{3.2}$$

Here the convergence is understood in the $L^\infty_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))$ sense, but by parabolic regularity, it is taken place in $C^{1,2}_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))$.

By Lemma 3.1 and (3.1), we find that for $w \in \omega(v)$,

$$0 \le w(z) \le 1 \ \text{ for } \ z \in [0,\infty). \tag{3.3}$$

Moreover, since $v \ge 0$ and W satisfies

$$\begin{cases} W_t = W_{zz} + cW_z + f(W), & t \in \mathbb{R}, z > 0, \\ W(t,0) = 0, & t \in \mathbb{R}, \\ W(0,z) = w(z), & z > 0 \end{cases}$$

and $W \ge 0$, by the strong maximum principle, we deduce either $W(t,z) > 0$ for all $t \in \mathbb{R}^1$ and $z \in (0,\infty)$ or $W \equiv 0$. In the former case, $w > 0$ for $z \in (0,\infty)$, while in the latter case $w \equiv 0$.

Proposition 3.3. *If* $\lim_{t\to\infty} H_c(t) = \infty$, *then* $\omega(v) = \{\phi^S\}$, *where* ϕ^S *is defined in* (2.8).

Proof. Take any $w \in \omega(w)$. Then there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $v(t_n,\,\cdot\,) \to w$ as $n \to \infty$ in $L^\infty_{\mathrm{loc}}(\mathbb{R})$. Consider the following problem

$$\begin{cases} \phi'' + c\phi' + f(\phi) = 0, & z > 0, \\ \phi(0) = 0, \ \phi'(0) = w'(0) \ge 0. \end{cases} \tag{3.4}$$

We only give the proof for the case where f is (f_{M}). From (3.3), it is easily seen that type ϕ^{M}_3 solution of (3.4) does belong to $\omega(v)$. We also obtain that type ϕ^{M}_1 solution is not an element of $\omega(v)$ since $w(z) > 0$ for $z \in (0,\infty)$ for any nonzero $w \in \omega(v)$.

(i) We will prove $w'(0) \le \alpha_c$. Otherwise $w'(0) > \alpha_c$. Then, from Proposition 2.10, solution ϕ of (3.4) is type $\phi_3^{\rm M}$. Take $l > 0$ satisfying $\phi_3^{\rm M}(l) > 1$ and set $I = [0, l)$. Since

$$\begin{aligned} &v(t,0) = \phi_3^{\rm M}(0) = 0 \ \text{ for } t > 0,\\ &v(t,l) - \phi_3^{\rm M}(l) \neq 0 \ \text{ for sufficiently large } t. \end{aligned}$$

We can apply Lemma 2.5 and Remark 1 to obtain

$$\mathcal{Z}_I(v(t,\,\cdot\,) - \phi_3^{\rm M}) < \infty \ \text{ for sufficiently large } t.$$

We may assume that all zeros on I of $v(t,\,\cdot\,) - \phi_3^{\rm M}$ are nondegenerate since degenerate zeros appear at finitely many time moment by Lemma 2.5 in this case, and so

$$\mathcal{Z}_I(v(t_n,\,\cdot\,) - \phi_3^{\rm M}) < \infty$$

for sufficiently large n and all zeros are nondegenerate. By Lemma 2.6 we can conclude that

$$w \equiv \phi_3^{\rm M} \ \text{ on } \ I, \ \text{ or } \ w - \phi_3^{\rm M} \ \text{ has only simple zeros on } I.$$

However $w - \phi_3^{\rm M}$ has degenerate zero at $z = 0$. Therefore $w \equiv \phi_3^{\rm M}$, contradicting $\phi_3^{\rm M} \notin \omega(v)$. Thus $w'(0) \le \alpha_c$.

(ii) We next suppose that $0 < w'(0) < \alpha_c$. Then, from Proposition 2.10, solution ϕ of (3.4) is type $\phi_1^{\rm M}$ defined on $[0, l(w'(0))]$. Set $I = [0, l(w'(0))]$. By a similar argument to (i), we can conclude that

$$w \equiv \phi_1^{\rm M} \ \text{ on } \ I, \ \text{ or } \ w - \phi_1^{\rm M} \ \text{ has only simple zeros on } I.$$

However $w - \phi_1^{\rm M}$ has a degenerate zero at $z = 0$. Therefore $w \equiv \phi_1^{\rm M}$, contradicting $\phi_1^{\rm M} \notin \omega(v)$. Thus $w'(0) \notin (0, \alpha_c)$.

(iii) We next assume that $w'(0) = 0$. Then solution of (3.4) is identical with 0. If $w \not\equiv 0$, from the arguments above Proposition 3.3, we have $W(t,z) > 0$ for $t \in \mathbb{R}^1$, $z \in (0,\infty)$. Then by the Hopf Lemma, we can see that $W_z(t,0) > 0$ and then $w'(0) > 0$. This is a contradiction. Hence $w \equiv 0$.

(iv) Finally we assume that $w'(0) = \alpha_c$. Then solution ϕ of (3.4) is type $\phi_2^{\rm M}$. Set $I_t = [0, H_c(t)]$. Since

$$\begin{aligned} &v(t,0) - \phi_2^{\rm M}(0) = 0 \ \text{ for } t > 0,\\ &v(t,H_c(t)) - \phi_2^{\rm M}(H_c(t)) \neq 0 \ \text{ for } t > 0, \end{aligned}$$

we can apply Lemma 2.5 and Remark 1 to obtain that

$$\mathcal{Z}_{I_t}(v(t,\,\cdot\,) - \phi_2^{\rm M}) < \infty \ \text{ for sufficiently large } t.$$

and all zeros are nondegenerate, and then

$$\begin{aligned} &\mathcal{Z}_{I_{t_n}}(v(t_n,\,\cdot\,) - \phi_2^{\rm M}) < \infty \ \text{ for sufficiently large } n,\\ &\text{all zeros of } v(t_n,\,\cdot\,) - \phi_2^{\rm M} \text{ are nondegenerate.} \end{aligned} \tag{3.5}$$

Since $\lim_{t\to\infty} H(t) = \infty$, for any $L > 0$, (3.5) is valid if we replace interval I_{t_n} by $I = [0, L)$. By Lemma 2.6 we can conclude that

$$w \equiv \phi_2^{\mathrm{M}} \ \text{ on } \ I, \ \text{ or } \ w - \phi_2^{\mathrm{M}} \ \text{ has only simple zeros on } I.$$

However $w - \phi_2^{\mathrm{M}}$ has degenerate zero at $z = 0$. Therefore $w \equiv \phi_2^{\mathrm{M}}$.

By summarizing arguments in (i)–(iv) we can conclude that $w(v) = \{\phi_2^{\mathrm{M}}\} = \{\phi^S\}$, or $\omega(v) = \{0\}$ because $\omega(v)$ must be connected in C^2_{loc} topology. However by Lemma 3.2, we can show that 0 can not be in $\omega(v)$ since for any $l > 0$ there exists $T_l > 0$ such that $v(t, z) \geq \mathcal{V}_c(z - l)$ for $t \geq T_l$ and $z \in [l, l + L_c]$. Therefore $\omega(v) = \{\phi_2^{\mathrm{M}}\} = \{\phi^S\}$. □

Proposition 3.4. *If* $\lim_{t\to\infty} H_c(t) = \infty$, *then* $\lim_{t\to\infty} v(t, z) = \phi^S(z)$ *uniformly on any compact subset of* $[0, \infty)$.

Proof. Take any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and define $H_{c,n}(t) = H_c(t+t_n)$, $v_n(t, z) = v(t + t_n, z)$. Then v_n and $H_{c,n}$ satisfy

$$\begin{cases} \dfrac{\partial v_n}{\partial t} = \dfrac{\partial^2 v_n}{\partial z^2} + c\dfrac{\partial v_n}{\partial z} + f(v_n), & t > -t_n, \ 0 < z < H_{c,n}(t), \\ v_n(t, 0) = v_n(t, H_{c,n}(t)) = 0, & t > -t_n. \end{cases} \tag{3.6}$$

Since $\|v_n\|_{L^\infty}$ are uniformly bounded in n, we can apply standard parabolic L^p estimate, Sobolev embedding and Schauder estimate to (3.6) to find that for any $\alpha \in (0, 1)$, there exists a subsequence $\{\tilde{t}_n\}$ of $\{t_n\}$ and a function $\hat{v} \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\mathbb{R} \times [0, \infty))$ such that

$$v_n \to \hat{v} \text{ as } n \to \infty \text{ in } C^{1+\frac{\alpha}{2}, 2+\alpha}_{\mathrm{loc}}(\mathbb{R} \times [0, \infty))$$

along the subsequence. Then $\hat{v}$ satisfies

$$\hat{v}_t = \hat{v}_{zz} + c\hat{v}_z + f(\hat{v}), \ t \in \mathbb{R}, \ z \in [0, \infty).$$

In particular,

$$v_n(0, \cdot\,) = v(\tilde{t}_n, \cdot\,) \to \hat{v}(0, \cdot\,) \ \text{ in } \ C^2_{\mathrm{loc}}([0, \infty)) \ \text{ as } n \to \infty.$$

By the definition of $\omega(v)$, we obtain $\hat{v}(0, z) = \phi^S(z)$ for all $z \in [0, \infty)$.

Now we have shown that for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{\tilde{t}_n\}$ of $\{t_n\}$ such that $v(\tilde{t}_n, \cdot\,) \to \phi^S$ in $C^2_{\mathrm{loc}}([0, \infty))$. Since the limit ϕ^S does not depend on the subsequence, we can conclude the desired convergence as in the statement of the proposition. □

We next investigate the asymptotic spreading speed of $h(t)$ for the global-in-time solutions. By constructing an upper solution of the form

$$\begin{aligned} \overline{h}(t) &:= c^* t + M(e^{-\delta T} - e^{-\delta t}) + H, \\ \overline{u}(t, x) &:= (1 + Me^{-\delta t}) q^*(x - \overline{h}(t)), \end{aligned}$$

with suitable M, δ, H and $T > 0$ as in [10, Lemma 3.2] we can obtain an upper estimate of $h(t)$.

Lemma 3.5. *Let* (u,h) *be a global solution to* (1.1). *Then there exists* $C_0 > 0$ *such that* $h(t) - c^*t < C_0$ *for* $t > 0$.

Lemma 3.6. *If* $H_c(t)$ *is unbounded,* $\lim_{t\to\infty}(h(t)/t) = c^*$.

Proof. By Lemma 3.5, we have

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le c^*.$$

Hence we will show

$$\liminf_{t\to\infty} \frac{h(t)}{t} \ge c^*.$$

Take $\varepsilon > 0$ so small that $c < c^* - \varepsilon$ holds. Then by Lemma 2.16, there exists a unique positive number $L_{c^*-\varepsilon}$ such that the following problem has a unique positive solution $\mathcal{V} = \mathcal{V}_{c^*-\varepsilon}$:

$$\begin{cases} \mathcal{V}'' + (c^* - \varepsilon)\mathcal{V}' + f(\mathcal{V}) = 0, \ z \in (0, L_{c^*-\varepsilon}), \\ \mathcal{V}(0) = \mathcal{V}(L_{c^*-\varepsilon}) = 0, \\ -\mu\mathcal{V}'(L_{c^*-\varepsilon}) = c^* - \varepsilon. \end{cases}$$

Take $\delta > 0$ small so that

$$0 < \delta < 1 - \max_{z\in[0,L_{c^*-\varepsilon}]} \mathcal{V}_{c^*-\varepsilon}(z)$$

holds. Then there exists a positive number $K_{\varepsilon,\delta} > 0$ such that

$$\begin{aligned} &\phi^S(K_{\varepsilon,\delta}) = \max_{z\in[0,L_{c^*-\varepsilon}]} \mathcal{V}_{c^*-\varepsilon}(z) + \delta, \\ &\phi^S(z) > \max_{z\in[0,L_{c^*-\varepsilon}]} \mathcal{V}_{c^*-\varepsilon}(z) + \delta \ \text{ for } \ z > K_{\varepsilon,\delta}. \end{aligned}$$

Since $\lim_{t\to\infty} H_c(t) = \infty$, by Proposition 3.4, we can choose $T = T_{\varepsilon,\delta} > 0$ such that $h(T) \ge cT + K_{\varepsilon,\delta} + L_{c^*-\varepsilon}$ and

$$\phi^S(z) - \delta \le v(T,z) \le \phi^S(z) + \delta \ \text{ for } z \in [K_{\varepsilon,\delta}, K_{\varepsilon,\delta} + L_{c^*-\varepsilon}]. \tag{3.7}$$

From (3.7) we have

$$\mathcal{V}_{c^*-\varepsilon}(z) \le v(T,z) \ \text{ for } \ z \in [K_{\varepsilon,\delta}, K_{\varepsilon,\delta} + L_{c^*-\varepsilon}],$$

that is,

$$\mathcal{V}_{c^*-\varepsilon}(x - cT) \le u(T,x) \ \text{ for } \ x \in [K_{\varepsilon,\delta} + cT, K_{\varepsilon,\delta} + L_{c^*-\varepsilon} + cT].$$

Let

$$\begin{aligned} &\xi(t) := K_{\varepsilon,\delta} + cT + (c^* - \varepsilon)(t - T), \\ &\underline{u}(t,x) := \mathcal{V}_{c^*-\varepsilon}(x - K_{\varepsilon,\delta} - cT - (c^* - \varepsilon)(t - T)) = \mathcal{V}_{c^*-\varepsilon}(x - \xi(t)), \\ &\underline{h}(t) := K_{\varepsilon,\delta} + L_{c^*-\varepsilon} + cT + (c^* - \varepsilon)(t - T) = \xi(t) + L_{c^*-\varepsilon}. \end{aligned}$$

It is easy to check that

$$\begin{cases} \underline{u}_t = \underline{u}_{xx} + f(\underline{u}), & t > T,\ \xi(t) < x < \underline{h}(t), \\ \underline{u}(t, \underline{h}(t)) = 0, & t > T, \\ \underline{h}'(t) = -\mu \underline{u}_x(t, \underline{h}(t)), & t > T, \end{cases}$$

and

$$\begin{aligned} &\underline{h}(T) \le h(T), \\ &\underline{u}(T, x) \le u(T, x),\ \xi(T) \le x \le \underline{h}(T), \\ &ct < \xi(t),\ t > T, \\ &\underline{u}(t, \xi(t)) = 0 \le u(t, \xi(t)),\ t > T, \end{aligned}$$

that is, $(\underline{u}, \underline{h})$ is a lower solution to (1.1). Hence Lemma 2.3 gives

$$\begin{aligned} &\underline{u}(t, x) \le u(t, x),\ t > T,\ \xi(t) < x < h(t), \\ &\underline{h}(t) \le h(t), \quad t > T. \end{aligned}$$

Therefore we obtain

$$c^* - \varepsilon = \lim_{t\to\infty} \frac{\underline{h}(t)}{t} \le \liminf_{t\to\infty} \frac{h(t)}{t}.$$

Letting $\varepsilon \to 0$ we have

$$c^* \le \liminf_{t\to\infty} \frac{h(t)}{t}.$$

The proof has been completed. □

Proposition 3.7. *If* $\lim_{t\to\infty} H_c(t) = \infty$*, then for any* $\varepsilon > 0$*, there exists* $M > 0$*,* $\delta_0 \in (0, -f'(1))$ *and* $\hat{T} > 0$ *such that*

$$\sup_{x\in[(c+\varepsilon)t, (c^*-\varepsilon)t]} |u(t, x) - 1| \le M e^{-\delta_0 t} \quad \text{for} \quad t > \hat{T}.$$

In particular, for $\varepsilon > 0$*,*

$$\lim_{t\to\infty} \sup_{x\in[(c+\varepsilon)t, (c^*-\varepsilon)t]} |u(t, x) - 1| = 0.$$

Proof. See the proof of Proposition 3.10 in [16]. □

We next consider the case where $\sup_{t>0} H_c(t) < \infty$ for global solution (u, h). Following the proof of Lemma 4.5, Proposition 4.6 and Theorem 4.10 in [17], we can obtain the following proposition.

Proposition 3.8. *If $H_c(t)$ is bounded, then* $\lim_{t\to\infty} H_c(t) = L_c$ *and*

$$\lim_{t\to\infty}\left\{\sup_{x\in[ct,h(t)]}|u(t,x)-\mathcal{V}_c(x-h(t)+L_c)|\right\}=0. \tag{3.8}$$

Now I will give the proof of Theorem A.

Proof of Theorem A. Suppose that $T_{\max}<\infty$. From Proposition 2.9 and Lemma 2.7 we have

$$\lim_{t\nearrow T_{\max}}(h(t)-ct)=0,\quad \lim_{t\nearrow T_{\max}}\|u(t,\,\cdot\,)\|_{C([ct,h(t)])}=0.$$

This means that when $T_{\max}<\infty$, vanishing happens in Theorem A.

We next suppose that $T_{\max}=\infty$. If $H_c(t):=h(t)-ct$ is unbounded function on $[0,\infty)$, then by Propositions 3.4 and 3.6 we have

$$\begin{aligned}&\lim_{t\to\infty}u(t,z+ct)=\lim_{t\to\infty}v(t,z)=\phi^S(z)\ \text{ locally uniformly on } [0,\infty),\\&\lim_{t\to\infty}\frac{h(t)}{t}=c^*.\end{aligned}$$

Moreover by Proposition 3.7 we have for any small $\varepsilon>0$

$$\lim_{t\to\infty}\sup_{x\in[(c+\varepsilon)t,(c^*-\varepsilon)t]}|u(t,x)-1|=0.$$

This means that if $T_{\max}=\infty$ and $H_c(t)$ is unbounded function, then spreading happens in Theorem A.

Finally we assume that $T_{\max}=\infty$ and $H_c(t)$ is bounded function on $[0,\infty)$. By Proposition 3.8 we have that

$$\begin{aligned}&\lim_{t\to\infty}(h(t)-ct)=\lim_{t\to\infty}H_c(t)=L_c,\\&\lim_{t\to\infty}\left\{\sup_{x\in[ct,h(t)]}|u(t,x)-\mathcal{V}_c(x-h(t)+L_c)|\right\}=0,\end{aligned}$$

where $L_c>0$ and $\mathcal{V}_c$ are determined in Lemma 2.16.

This means that if $T_{\max}=\infty$ and $H_c(t)$ is unbounded function, then transition happens in Theorem A.

Therefore we have shown that for any initial data (u_0,h_0) exactly one of the situations, vanishing, spreading and transition, happens for the unique solution (u,h) to (1.1).

The proof of Theorem A has completed. $\square$

References

[1] S. B. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math., 390 (1988), 79–96.

[2] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in Partial Differential Equations and Related Topics,

[3] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math., 30 (1978), 33–76.

[4] J. Cai, *Asymptotic behavior of solutions of Fisher-KPP equation with free boundary conditions*, Nonlinear Anal., 16 (2014), 170–177.

[5] J. Cai, B. Lou and M. Zhou, *Asymptotic behavior of solutions of a reaction diffusion equation with free boundary conditions*, J. Dynam. Differential Equations, 26(2014), 1007–1028.

[6] Y. Du and Z. Lin, *Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM J. Math. Anal., 42 (2010), 377–405.

[7] Y. Du and B. Lou, *Spreading and vanishing in nonlinear diffusion problems with free boundaries*, J. Eur. Math. Soc., 17 (2015) 2673–2724.

[8] Y. Du, B. Lou and M. Zhou, *Nonlinear diffusion problems with free boundaries: convergence, transition and zero number arguments*, SIAM J. Math. Anal. 47 (2015) 3555–3584.

[9] Y. Du and H. Matano, *Convergence and sharp thresholds for propagation in nonlinear diffusion problems*, J. Eur. Math. Soc., 12(2012) 279–312

[10] Y. Du, H. Matsuzawa and M. Zhou, *Sharp estimate of the spreading speed determined by nonlinear free boundary problems*, SIAM J. Math. Anal., 46 (2014), 375–396.

[11] Y. Du, L. Wei and L. Zhou, *Spreading in a shifting environment modeled by the diffusive logistic equation with a free boundary*, preprint.

[12] H. Gu and B. Lou, *Spreading in advective environment modeled by a reaction diffusion equation with free boundaries*, J. Differential Equations, 260(2016) 3991–4015.

[13] H. Gu, B. Lou and M. Zhou, *Long time behavior of solutions of Fisher-KPP equation with advection and free boundaries*, J. Funct. Anal., 269 (2015) 1714–1768.

[14] Y. Kaneko, K. Oeda and Y. Yamada, *Remarks on spreading and vanishing for free boundary problems of some reaction-diffusion equations*, Funkcial. Ekvac., 57 (2014), 449–465.

[15] Y. Kaneko and Y. Yamada, *A free boundary problem for a reaction-diffusion equation appearing in ecology*, Adv. Math. Sci. Appl., 21 (2011), 467–492.

[16] Y. Kaneko and H. Matsuzawa, *Spreading and vanishing in a free boundary problem for nonlinear diffusion equations with a forced given moving boundary*, Journal of Differential Equations, 265(2018), 1000–1043.

[17] H. Matsuzawa, *A free boundary problem for the Fisher-KPP equation with a given moving boundary*, Communications on pure and applied analysis, 17(2018) 1821–1852.

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