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Minimizing movement approach without using distance function for evolving spirals by the crystalline curvature with driving force

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1. Introduction

This is a brief introduction to the numerical methods for evolving spirals having several centers by the crystalline eikonal-curvature flow.

An anisotropic curvature of a curve Γ, which is denoted by H_γ, is defined by the changing ratio of an anisotropic perimeter functional

$$E_\gamma(\Gamma) = \int_\Gamma \gamma_0(-\mathbf{n})d\sigma \tag{1.1}$$

with a density function $\gamma_0 \colon \mathbb{S}^1 \to (0,\infty)$. Here, $\mathbf{n} \in S^1$ is a continuous normal vector field of Γ denoting the orientation of Γ. It is well-known that, if $\gamma(p) = |p|\gamma_0(p/|p|)$ is smooth on $\mathbb{R}^2 \setminus \{0\}$ and strictly convex, then the Wulff shape $\mathcal{W}_\gamma$ defined by

$$\mathcal{W}_\gamma = \{p \in \mathbb{R}^2;\ \gamma^\circ(p) \le 1\} \tag{1.2}$$

has a boundary $\partial\mathcal{W}_\gamma$ with constant anisotropic curvature, where $\gamma^\circ(p) = \sup\{p\cdot q;\ \gamma(q) \le 1\}$ is the support function of γ. In fact, if the normal vector for the calculation of the curvature is oriented to the interior of $\mathcal{W}_\gamma$, then $H_\gamma = 1$ on $\partial\mathcal{W}_\gamma$; see [4] or [7] for details. We say H_γ is the crystalline curvature if $\mathcal{W}_\gamma$ is a convex polygon. Since γ° is positively homogeneous of degree 1, we here assume that $\gamma^\circ(p)$ is a convex and piecewise linear function, i.e.,

$$\gamma^\circ(p) = \sup_{0 \le j \le N_\gamma - 1} m_j \cdot p$$

for $m_j \in \mathbb{R}^2$, where $N_\gamma \ge 3$ is a number of the facets of $\mathcal{W}_\gamma$. In this case, one can find that $\gamma = (\gamma^\circ)^\circ$ is also a convex and piecewise linear function by the convex analysis provided that γ is convex. By the context of the above argument, we here impose the followings.

(A1) $\gamma \colon \mathbb{R}^2 \to [0,\infty)$ is convex,

(A2) γ is positively homogeneous of degree 1,

(A3) $\gamma > 0$ on $\mathbb{S}^1$,

(A4) γ is piecewise linear.

See also [18] or [19] for the fundamental results from convex analysis used in the above.

Under these settiings, we consider the evolving spirals Γ_t by the crystalline eikonal-curvature flow with driving force of the form

$$V_\gamma = -\rho_c H_\gamma + f, \tag{1.3}$$

where $\rho_c > 0$ is a constant, and f denotes the driving force of the evolution. Note that the normal velocity V_γ is calculated with the Finsler metric

$$d_\gamma(x, y) = \gamma^\circ(x - y)$$

so that Γ_t can be evolved as polygonal spirals having parallel facets with those of $\mathcal{W}_\gamma$.

There are some pioneering works by a discrete model by the evolution of facet length due to [3, 20], which is extended to the motion of a spiral by [8, 9]. However, the evolving spirals having several centers may merge with each other, which causes singularities on the curves. Then, it is quite natural to introduce an implicit formulation of evolving curves, which is, for example, level set method due to [17]. However, such a PDE approach with assumption (A4) includes L^1 type regularization, namely, singular and nonlocal diffusion. To overcome this difficulty, the variational approach due to [2] is one of powerful options; see [1] for the variational approach to the crystalline curvature motion. Chambolle [6] introduced an algorithm combining the variational approach and the level set method using signed distance function. Oberman, Osher, Takei and Tsai [13] proposed an iterative method to calculate Chambolle's algorithm, which is based on Bregman method [5].

The aim of this paper is to extend the numerical method due to [13] to the evolution of spirals by (1.3). The crucial difficulty of our problem lies into the fact that a spiral curve does not divide the domain into two regions. This issue seems to be the same as the level set method for spirals, which is overcome by [14]. However, even if we use the idea of the sheet structure function due to [11, 14], discontinuity of the signed distance function still remains. To overcome this issue, we introduce an idea to use a general level set function instead of the signed distance function. Note that the idea in this paper is the revised version of that introduced in [15].

2. Proposed Algorithm

In this section we propose a minimizing movement approach for evolving spirals by (1.3).

We first review the algorithm by [6] briefly. Let $\Omega \subset \mathbb{R}^n$ be a domain, and an interface $\Gamma \subset \Omega$ be given. Let $d\colon \Omega \to \mathbb{R}$ be a signed distance function from Γ, which is positive inside of Γ. We also impose that the normal velocity of Γ is positive when Γ evolves to the region where $d < 0$. Then, consider the minimizer w^* of the functional

$$E(w) = \int_\Omega \gamma(\nabla w)dx + \frac{1}{2h}\|w - d\|_{L^2}^2.$$

Then, w^* formally satisfies

$$-\mathrm{div}[D\gamma(\nabla w^*)] + \frac{w^* - d}{h} = 0,$$

which implies

$$\Gamma_h := \{x \in \Omega;\ w^*(x) = 0\} = \{x \in \Omega;\ d(x) = -h\mathrm{div}[D\gamma(\nabla w^*(x))]\}.$$

Note that $-\mathrm{div}[D\gamma(\nabla w^*)]$ denotes an anisotropic curvature of level set of w^* with an anisotropic energy density γ (see [7]). Hence, one can regard Γ_h as the evolution of Γ by $V = -H_\gamma$ in a short time interval $[0, h]$.

When we apply this idea to the evolution of spirals, the crucial difficulty lies in the fact that spiral curve does not divide the domain into two regions, which causes not only the inconsistency between the inside and outside of the spiral, but also the discontinuity of the distance function on far away region from the spiral. To overcome these difficulty, we here propose an approach using general level set function instead of the signed distance.

2.1. Level set method for evolving spirals

We first review the level set method for evolving spirals due to [14] and its evolution equation for an anisotropic mean curvature flow with driving force.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. We denote the centers of spirals by $a_1, a_2, \ldots, a_N \in \Omega$. We choose $r > 0$ satisfying $\overline{B_r(a_i)} \cap \overline{B_r(a_j)} = \emptyset$ if $i \neq j$, and $\overline{B_r(a_j)} \subset \Omega$ so that $W := \Omega \setminus \bigcup_{j=1}^N \overline{B_r(a_j)}$ has smooth boundary, where $B_r(a) = \{x \in \mathbb{R}^2;\ |x - a| < r\}$ for $r > 0$ and $a \in \mathbb{R}^2$. Let $m_j \in \mathbb{Z}$ be a signed number of spirals associated with a_j. It means that

- $|m_j|$ curves are attached to a_j as their endpoint,
- if $m_j > 0$ (resp. $m_j < 0$), then these curves rotates around a_j with counterclockwise (resp. clockwise) rotation provided that $V > 0$.

See [14] for the details of the definition of the signed number of spirals. In this paper, we call the union of spiral curves associated with a_j for $j = 1, 2, \ldots, N$ as "spirals" or "a spiral pattern" interchangeably.

To describe the spirals with level set formulation, we introduce a pre-determined function θ due to Kobayashi [11, 12]; define

$$\theta(x) = \sum_{j=1}^{N} m_j \arg(x - a_j).$$

Let $\Gamma_t \subset \overline{W}$ be a spiral pattern at time $t \geq 0$, and $\mathbf{n} \in \mathbb{S}^1$ be a continuous unit normal vector field of Γ_t denoting the orientation of the evolution of Γ_t. Then, we describe Γ_t and $\mathbf{n}$ by

$$\Gamma_t = \{x \in \overline{W};\ u(t,x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}, \quad \mathbf{n} = -\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|}.$$

Here and hereafter, we write the above shortly as $\Gamma_t = \{u(t) - \theta \equiv 0\}$. Note that θ should be a multi-valued function to describe the spirals completely. However, $\nabla\theta$ is defined as a single-valued function. Therefore, we observe that Γ_t is locally given as a usual level set formulation by $u-\theta$, and then we find

$$H_\gamma = -\mathrm{div} D\gamma(\nabla(u-\theta)).$$

On the normal velocity of Γ_t, we impose that the normal velocity of Γ_t should be calculated with the metric derived by γ, so that Γ_t should be evolved with the polygonal spiral curve having parallel facets between those of $\mathcal{W}_\gamma$. Then, we have

$$V_\gamma = \frac{u_t}{\gamma(\nabla(u-\theta))}.$$

Hence, we obtain the level set equation of (1.3) as

$$u_t - \gamma(\nabla(u-\theta))\left\{\rho_c \mathrm{div}[D\gamma(\nabla(u-\theta))] + f\right\} = 0 \quad \text{in } (0,T)\times W. \tag{2.1}$$

2.2. Without distance function scheme

We propose a numerical method to calculate (2.1) due to the minimizing movement approach with a general level set function.

Consider the situation that a spiral pattern denoted by $\Gamma \subset \overline{W}$ is given by the level set formulation

$$\Gamma = \{x \in \overline{W};\ u(x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}$$

with $u \in C(\overline{W})$. Note that u possibly is not the signed distance function. Let us consider the minimizer w^* of

$$E(w;u) = \int_W \rho_c \gamma(\nabla(w-\theta))dx - \int_W fwdx + \frac{1}{2h}\left\| \frac{w-u}{\gamma(\nabla(u-\theta))}\right\|_{L^2}^2.$$

Then, w^* should satisfy

$$w^* = u + h\gamma(\nabla(u-\theta))\left\{\rho_c \mathrm{div}[D\gamma(\nabla(w^*-\theta))] + f\right\}$$

with a suitable boundary condition. We now focus on the analogy between the above and the implicit finite difference scheme of (2.1), i.e.,

$$u(t+h) \approx u(t) + h\gamma(\nabla(u-\theta))\left\{\rho_c \mathrm{div}[D\gamma(\nabla(u-\theta))] + f\right\}.$$

Accordingly, we regard

$$S_h(\Gamma) := \{x \in \overline{W};\ w^* - \theta \equiv 0 \mod 2\pi\mathbb{Z}\}$$

as the result of the evolution of Γ by (1.3) in a short time span $h > 0$. Hence, for every given time $t > 0$ and the initial spiral patterm Γ_0, we obtain the solution Γ_t of the evolution of spirals by (1.3) by $\Gamma_t = S_h^{[t/h]}(\Gamma_0)$, where $[a]$ is the Gaussian bracket of $a \in \mathbb{R}$.

3. Split Bregman iteration

Our proposed method includes the problem finding the minimizer w^* of an energy functional of the form

$$E(w;g) = \int_W \rho_c\gamma(\nabla(w-\theta))dx - \int_W fwdx + \frac{1}{2h}\left\|\frac{w-g}{\psi}\right\|_{L^2}^2$$

for $f, g \in L^2(W)$ and $\psi\colon \overline{W} \to \mathbb{R}$ satisfying $\varphi/\psi \in L^2(W)$ for $\varphi \in L^2(W)$. In this section, we give a brief introduction to the split Bregman iteration to solve the above problem, which is due to [13].

3.1. Review: Bregman method

We first describe the split Bregman method due to [13] briefly.

The key idea to find w^* is to interpret the problem to the following constraint minimization problem: find a minimizer (w^*, d^*) of

$$\begin{aligned} F(w,d;g) &:= \int_W \rho_c\gamma(d-\nabla\theta)dx - \int_W fwdx + \frac{1}{2h}\left\|\frac{w-g}{\psi}\right\|_{L^2}^2 \\ &\text{subject to } d = \nabla w. \end{aligned}$$

For this problem, we introduce a functional with penalty term:

$$\begin{aligned} F_\mu(w,d;g) :=& F(w,d;g) + \frac{\mu}{2}\|d-\nabla w\|_{L^2}^2 \\ =& \int_W \rho_c\gamma(d-\nabla\theta)dx - \int_W fwdx + \frac{1}{2h}\left\|\frac{w-g}{\psi}\right\|_{L^2}^2 + \frac{\mu}{2}\|d-\nabla w\|_{L^2}^2 \end{aligned}$$

for $(w,d) \in H^1(W) \times L^2(W;\mathbb{R}^2)$. We now apply Bregman iteration to obtain (w^*, d^*).

We shall give a brief review of the Bregman method with an abstract functional. See [16] for details. Let X be a reflexive Banach space. Assume that $F, H\colon X \to [0,\infty]$ be convex functionals, and H is smooth. Let us consider the following constrained minimization problem:

$$J(\mathbf{u}) = F(\mathbf{u}) + \mu H(\mathbf{u}) \quad \text{subject to } H(\mathbf{u}) = 0.$$

We introduce the Bregman distance $D_F^{\mathbf{p}}(\mathbf{u};\mathbf{v})$ for F of the form

$$D_F^{\mathbf{p}}(\mathbf{u};\mathbf{v}) = F(\mathbf{u}) - F(\mathbf{v}) - \langle \mathbf{p}, \mathbf{u}-\mathbf{v}\rangle,$$

where $\mathbf{p} \in \partial F(\mathbf{v})$ and $\partial F(\mathbf{v}) = \{\mathbf{q} \in X^*;\ F(\mathbf{u}) \geq F(\mathbf{v}) + \langle \mathbf{q}, \mathbf{u}-\mathbf{v}\rangle\}$ is a subdifferential of F at $\mathbf{v} \in X$. Then, one can find that the iteration

$$\mathbf{u}^{k+1} = \arg\min_{\mathbf{u}\in X}(D_F^{\mathbf{p}^k}(\mathbf{u};\mathbf{u}^k) + \mu H(\mathbf{u})), \quad \mathbf{u}^0 = 0,\ \mathbf{p}^0 = 0,\ \mathbf{p}^k \in \partial F(\mathbf{u}^k)$$

yields the desired minimizer $\mathbf{u}^* \in X$ by $\mathbf{u}^* = \lim_{k\to\infty}\mathbf{u}^k$ under some suitable assumptions. Moreover, it is important to find that, if $H(\mathbf{u}) = \|A\mathbf{u} - \zeta\|_Y^2/2$ with a linear operator

$A: X \to Y$ onto a Hilbert space Y and $\zeta \in Y$, then the minimizer $\mathbf{u}^{k+1}$ of the above is equivalent to the following iteration:

$$\begin{aligned} \mathbf{u}^{k+1} &= \arg\min_{\mathbf{u}\in X} \left(J(\mathbf{u}) + \frac{\mu}{2}\|A\mathbf{u} - \zeta - b^k\|_Y^2 \right), \\ b^{k+1} &= b^k + \zeta - A\mathbf{u}^{k+1}, \quad b^0 = 0. \end{aligned}$$

By applying the above sketch to our problem, we consider the itaration

$$\begin{aligned} (u^{k+1}, d^{k+1}) = \arg\min_{(w,d)} \Bigg[&\int_W \rho_c \gamma(d - \nabla\theta) dx - \int_W f w dx \\ &+ \frac{1}{2h} \left\| \frac{w-g}{\psi} \right\|_{L^2}^2 + \frac{\mu}{2}\|d - \nabla w - b^k\|_{L^2}^2 \Bigg], \end{aligned} \tag{3.1}$$

$$b^{k+1} = b^k + \nabla u^{k+1} - d^{k+1}, \quad u^0 = g, \ d^0 = b^0 = 0 \tag{3.2}$$

with $\psi = \gamma(\nabla(g - \theta))$ to obtain the minimizer (u^*, d^*) of $F(w, d; g)$. To solve this problem, we introduce the following alternate iteration;

$$u^{k,j+1} = \arg\min_w \left[-\int_W f w dx + \frac{1}{2h} \left\| \frac{w-g}{\psi} \right\|_{L^2}^2 + \frac{\mu}{2}\|d^{k,j} - \nabla w - b^k\|_{L^2}^2 \right], \tag{3.3}$$

$$d^{k,j+1} = \arg\min_d \left[\int_W \rho_c \gamma(d - \nabla\theta) dx + \frac{\mu}{2}\|d - \nabla u^{k,j+1} - b^k\|_{L^2}^2 \right]. \tag{3.4}$$

Then, we observe that $(u^{k+1}, d^{k+1}) = \lim_{j\to\infty}(u^{k,j}, d^{k,j})$.

The first minimization (3.3) is established by just solving the following boundary value problem of an elliptic equation;

$$w - h\psi\Delta w = g + h\psi(f - \operatorname{div}(d^{k,j} - b^k)) \qquad \text{in } W, \tag{3.5}$$

$$\frac{\partial w}{\partial \nu} = d^{k,j} - b^k \qquad \text{on } \partial W. \tag{3.6}$$

The second minimization (3.4) is established by calculating the minimizer of integrating function $x \mapsto \rho_c \gamma(x - z) + (\mu/2)|x - y|^2$, which is considered in the next subsection.

3.2. Shrinkage function

According to [13], we calculate the polyhedral shrinkage function for the second minimizing problem of the split Bregman iteration. Moreover, our integrating function includes the translation term $\nabla\theta$ in γ. Thus, we present the calculation of polyhedral shrinkage as in [13] with the translation briefly.

We first consider the minimizing problem of the integrating function in (3.4) with general settings. Remark that $\gamma(x)$ is represented as

$$\gamma(x) = \sup_{p \in \mathcal{W}_\gamma} p \cdot x$$

when γ is convex, where $\mathcal{W}_\gamma = \{p \in \mathbb{R}^2;\ \gamma^\circ(p) \leq 1\}$, and $\gamma^\circ(p) = \sup\{p \cdot q;\ \gamma(q) \leq 1\}$. Let us define the projection map $P_{\mathcal{W}_\gamma} : \mathbb{R}^2 \to \mathcal{W}_\gamma$ onto $\mathcal{W}_\gamma$, which is defined by

$$P_{\mathcal{W}_\gamma}(x) = \arg\min_{y \in \mathcal{W}_\gamma} |y - x|^2.$$

Let us consider the minimization problem

$$x^* = x^*(y, z) = \arg\min_{x \in \mathbb{R}^2} \left\{ \frac{1}{c}\gamma(x - z) + \frac{1}{2}|x - y|^2 \right\} \tag{3.7}$$

with a constant $c > 0$ with the above notations.

We first derive the necessary condition for the minimizer.

Lemma 3.1 *Let $x^* = x^*(y, z)$ be the minimizer defined by* (3.7). *Then,*

$$x^* = y - \frac{1}{c} P_{\mathcal{W}_\gamma}(c(y - z)).$$

Proof. Note that, in this proof, we denote the part which has no influence to the minimizer x^* by C, which may change by step-by-step calculation.

Set $\varphi(x, p) = (1/c)p \cdot (x - z) + (1/2)|x - y|^2$ and interpret the minimizing problem (3.7) to finding x^* which attains the following minimum:

$$M = \min_{x \in \mathbb{R}^2} \max_{p \in \mathcal{W}_\gamma} \varphi(x, p) = \min_{x \in \mathbb{R}^2} \max_{p \in \mathcal{W}_\gamma} \left(\frac{1}{c} p \cdot (x - z) + \frac{1}{2}|x - y|^2 \right).$$

Note that $x \mapsto \varphi(x, p)$ is convex, $\lim_{|x| \to \infty} \varphi(x, p) = \infty$ uniformly for $p \in \mathcal{W}_\gamma$, and $p \mapsto \varphi(x, p)$ is affine. Then, by the minimax theorem due to [21] we observe that

$$M = \max_{p \in \mathcal{W}_\gamma} \min_{x \in \mathbb{R}^2} \left(\frac{1}{c} p \cdot (x - z) + \frac{1}{2}|x - y|^2 \right).$$

By stragithforward calculation we have

$$\varphi(x, p) = \frac{1}{2}|x|^2 - \left(y - \frac{1}{c}p \right) \cdot x + C = \frac{1}{2} \left| x - \left(y - \frac{1}{c}p \right) \right|^2 + C.$$

Hence, we obtain

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \varphi(x, p) &= \varphi\left(y - \frac{1}{c}p, p \right) = \frac{1}{c}p \cdot \left(y - z - \frac{1}{c}p \right) + \frac{1}{2c^2}|p|^2 \\ &= -\frac{1}{2c^2}|p|^2 + \frac{1}{c}p \cdot (y - z) = -\frac{1}{2c^2}\left| p - c(y - z) \right|^2 + C. \end{aligned}$$

Consequently, the minimum M is attained at

$$x^*(y, z) = y - \frac{1}{c}p^*, \quad p^* = P_{\mathcal{W}_\gamma}(c(y - z)). \ \square$$

The following corollary directly follows from Lemma 3.1, so that we omit its proof.

Corollary 3.2 *It holds that $c(y-z) \in \mathcal{W}_\gamma$ if and only if $x^*(y,z) = z$.*

Let us consider the case when $c(y-z) \notin \mathcal{W}_\gamma$. From here on, let γ be a convex and piecewise linear function, i.e.,

$$\gamma(p) = \max_{0 \le j \le N_\gamma - 1} n_j \cdot p,$$

where $n_j = r_j(\cos\theta_j, \sin\theta_j)$ ($r_i > 0$ is a constant) satisfies

(W1) $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{N_\gamma-1} < \theta_0 + 2\pi$,

(W2) $\theta_j < \theta_{j+1} < \theta_j + \pi$ for $j = 0, 1, \dots, N_\gamma - 1$.

Note that, the index number j of n_j has been extended to $\mathbb{Z}$ by regarding $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$, in other words, $n_{j+kN_\gamma} = n_j$ for $k \in \mathbb{Z}$. For the calculation of the subdifferential $\partial\gamma$ of γ, we partition $\mathbb{R}^2 \setminus \{0\}$ into the regions

$$\begin{aligned} Q_i &= \{p \neq 0;\ n_i \cdot p > n_j \cdot p \quad \text{for } j \neq i\}, \\ R_{i,i+1} &= \{p \neq 0;\ n_i \cdot p = n_{i+1} \cdot p > 0\} \end{aligned}$$

for $i \in \mathbb{Z}$. Assume the following.

(W3) $Q_i = \Xi_{i,i-1} \cap \Xi_{i,i+1} \neq \emptyset$, where $\Xi_{j,k} = \{p \neq 0;\ n_j \cdot p > n_k \cdot p\}$.

Note that (W3) implies $\partial Q_i = R_{i-1,i} \cup R_{i,i+1} \cup \{0\}$. Moreover, γ is smooth on Q_i and has singularities on each $R_{i,i+1}$. By using the above notations, we now characterize x^* when $c(y-z) \notin \mathcal{W}_\gamma$.

Lemma 3.3 *Let $\gamma(p) = \max_{0 \le j \le N_\gamma - 1} n_j \cdot p$ with $n_j = r_j(\cos\theta_j, \sin\theta_j)$ for $r_j > 0$ and $\theta_j \in \mathbb{R}$, and assume that (W1)–(W3) holds. Let $c > 0$ and $y, z \in \mathbb{R}^2$ satisfy $c(y-z) \notin \mathcal{W}_\gamma$. Set*

$$\lambda_i = \frac{c(y-z) \cdot (n_i - n_{i+1}) - n_i \cdot n_{i+1} + |n_{i+1}|^2}{|n_i - n_{i+1}|^2}, \tag{3.8}$$

$$\xi_i = \lambda_i n_i + (1 - \lambda_i) n_{i+1}. \tag{3.9}$$

For the minimizer x^* defined by (3.7), the followings hold.

(i) *If $x^* - z \in R_{i,i+1}$, then $\lambda_i \in [0,1]$ and $(c(y-z) - \xi_i) \cdot \xi_i \geq 0$.*

(ii) *If $x^* - z \in Q_i$, then $\lambda_i > 1$ and $\lambda_{i-1} < 0$.*

Proof. By (3.7) we have

$$\frac{1}{c}\gamma(x-z) + \frac{1}{2}|x-y|^2 \geq \frac{1}{c}\gamma(x^*-z) + \frac{1}{2}|x^*-y|^2.$$

This implies that

$$\gamma(x-z) \geq \gamma(x^*-z) - c(x^*-y) \cdot (x-x^*) - \frac{c}{2}|x-x^*|^2,$$

and thus $-c(x^*-y)\in\partial\gamma(x^*-z)$. The subdifferential $\partial\gamma$ can be explicitly calculated as

$$\partial\gamma(q)=\begin{cases}\{n_i\} & \text{if } q\in Q_i \\ & \text{for some } i\in\{0,1,\dots,N_\gamma-1\}, \\ \{\lambda n_i+(1-\lambda)n_{i+1};\ \lambda\in[0,1]\} & \text{if } q\in R_{i,i+1} \\ & \text{for some } i\in\{0,1,\dots,N_\gamma-1\}.\end{cases}$$

By Corollary 3.2, $c(y-z)\notin\mathcal{W}_\gamma$ implies that $x^*-z\neq 0$, and thus $x^*-z\in\bigcup_{i=0}^{N_\gamma-1}Q_i\cup\bigcup_{i=0}^{N_\gamma-1}R_{i,i+1}$. By combining the above, we obtain the followings.

(i) If $x^*-z\in R_{i,i+1}$, then $x^*=y-(1/c)(\lambda n_i+(1-\lambda)n_{i+1})$ for some $\lambda\in[0,1]$.

(ii) If $x^*-z\in Q_i$, then $x^*=y-(1/c)n_i$.

Let us consider the case (i). In this case, we have

$$-c(x^*-y)=\lambda n_i+(1-\lambda)n_{i+1}$$

for some $\lambda\in[0,1]$. By taking an inner product between n_i or n_{i+1} and the above, we obtain

$$-c(x^*-y)\cdot n_i=\lambda|n_i|^2+(1-\lambda)n_{i+1}\cdot n_i, \tag{3.10}$$
$$-c(x^*-y)\cdot n_{i+1}=\lambda n_i\cdot n_{i+1}+(1-\lambda)|n_{i+1}|^2. \tag{3.11}$$

Note that $x^*-z\in R_{i,i+1}$ implies $n_i\cdot(x^*-z)=n_{i+1}\cdot(x^*-z)$. Then, (3.10) and (3.11) yield the following;

$$c(y-z)\cdot n_i-\lambda|n_i|^2-(1-\lambda)n_{i+1}\cdot n_i=c(y-z)\cdot n_{i+1}-\lambda n_i\cdot n_{i+1}-(1-\lambda)|n_{i+1}|^2.$$

We calculate the above and obtain

$$\lambda=\lambda_i=\frac{c(y-z)\cdot(n_i-n_{i+1})-n_i\cdot n_{i+1}+|n_{i+1}|^2}{|n_i-n_{i+1}|^2}\in[0,1].$$

Moreover, we have $x^*=y-(1/c)\xi_i$, and then $(x^*-z)\cdot n_i=(x^*-z)\cdot n_{i+1}\geq 0$. This implies

$$\begin{aligned}(c(y-z)-\xi_i)\cdot\xi_i&=\left(c\left(x^*-\frac{1}{c}\xi_i-z\right)-\xi_i\right)\cdot\xi_i\\ &=c\left[\lambda_i\left(x^*-z\right)\cdot n_i+\left(1-\lambda_i\right)\left(x^*-z\right)\cdot n_{i+1}\right]\geq 0.\end{aligned}$$

We next consider the case (ii). In this case we have

$$(x^*-z)\cdot n_i>(x^*-z)\cdot n_j\quad\text{for } j\neq i.$$

Since $x^*=y-(1/c)n_i$, the formula (3.8) yields that

$$\begin{aligned}\lambda_i&=\frac{c(y-z)\cdot(n_i-n_{i+1})-n_i\cdot n_{i+1}+|n_{i+1}|^2}{|n_i-n_{i+1}|^2}\\ &=\frac{c(x^*-z+(1/c)n_i)\cdot(n_i-n_{i+1})-n_i\cdot n_{i+1}+|n_{i+1}|^2}{|n_i-n_{i+1}|^2}\\ &=\frac{c(x^*-z)\cdot(n_i-n_{i+1})+|n_i-n_{i+1}|^2}{|n_i-n_{i+1}|^2}>1.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\lambda_{i-1} &= \frac{c(y-z)\cdot(n_{i-1}-n_i) - n_{i-1}\cdot n_i + |n_i|^2}{|n_{i-1}-n_i|^2} \\ &= \frac{c(x^* - z + (1/c)n_i)\cdot(n_{i-1}-n_i) - n_{i-1}\cdot n_i + |n_i|^2}{|n_{i-1}-n_i|^2} \\ &= \frac{c(x^*-z)\cdot(n_{i-1}-n_i)}{|n_{i-1}-n_i|^2} < 0. \ \square\end{aligned}$$

By Lemma 3.3, either the following (A) or (B) occurs when $c(y-z) \notin \mathcal{W}_\gamma$:

(A) There exists $i \in \{0, 1, \dots, N_\gamma - 1\}$ such that $\lambda_i \in [0, 1]$ and $(y - (1/c)\xi_i)\cdot \xi_i \geq 0$.

(B) There exists $i \in \{0, 1, \dots, N_\gamma - 1\}$ such that $\lambda_i > 1$ and $\lambda_{i-1} < 0$.

Hence, by combining Corollary 3.2 and Lemma 3.3, we obtain the following scheme to calculate the minimizer $d^{k,j+1}$ as (3.4). See [10] for the practical way to construct γ° from a polyhedral γ.

Scheme (Sh) to calculate $d^{k,j+1}$ of (3.4).

Set $y = \nabla u^{k,j+1}(x) + b^k(x)$ and $z = \nabla\theta(x)$.

- If $c(y-z) \in \mathcal{W}_\gamma$, i.e., $\gamma^\circ(y-z) \leq 1/c$, then set

$$d^{k,j+1} = z = \nabla\theta(x).$$

- If $\gamma^\circ(y-z) > 1/c$, then set $d^{k,j+1}$ as follows: Set λ_i and ξ_i by (3.8) and (3.9), respectively.
  - If there exists $i \in \{0, 1, \dots, N_\gamma - 1\}$ such that $\lambda_i \in [0, 1]$ and $(y-(1/c)\xi_i)\cdot\xi_i \geq 0$, then set
  $$d^{k,j+1}(x) = y - \frac{1}{c}\xi_i = \nabla u^{k,j+1}(x) + b^k(x) - \frac{1}{c}\xi_i.$$
  - Otherwise, find $i \in \{0, 1, \dots, N_\gamma - 1\}$ satisfying $\lambda_i > 1$ and $\lambda_{i-1} < 0$, and then set
  $$d^{k,j+1}(x) = y - \frac{1}{c}n_i = \nabla u^{k,j+1}(x) + b^k(x) - \frac{1}{c}n_i.$$

4. Numerical results

In this section, we propose some numerical results on the evolution of spirals by (1.3) by our scheme.

For a given initial curve $\Gamma_0 \subset \mathbb{R}^n$, we give a continuous initial data $u_0 \in C(\overline{W})$ satisfying $\Gamma_0 = \{u_0 - \theta \equiv 0\}$. Then, by repeating the algorithm proposed in §2.2, we obtain a sequence of minimizers $u_n := \arg\min_u E(u; u_{n-1})$ and a family of the spirals $\Gamma_n = \{u_n - \theta \equiv 0\}$. One can find two loops to obtain u_{n+1} from u_n;

(Outer) Finding u_{n+1} from u_n by (3.1)–(3.2),

(Inner) Finding (u_n^{k+1}, d^{k+1}) from (u_n^k, d^k) by (3.3)–(3.4).

See Algorithm 1 for the summary of our algorithm. In the both Outer and Inner loops, we calculate $F_\mu(u_n^{k,j}, d^{k,j}; u_n)$ or $F_\mu(u_n^k, d^k; u_n)$ to check if they are the minimizer of $F_\mu(w, d; u_n)$ with $\psi = \gamma(\nabla(u_n - \theta))$. However, it is failed when the set $\{x \in \overline{W};\ \gamma(\nabla(u_n - \theta)) = 0\}$ has nonempty interior. To avoid this issue, we introduce an approximation of $F_\mu(w, d; g)$ by cut-off only for checking the condition breaking the loop; define

$$F_{\mu,\alpha}(w, d; g) := \int_W \rho_c \gamma(d - \nabla\theta) dx - \int_W f w dx + \frac{1}{2h} \left\| \frac{w - g}{\max\{\gamma(\nabla(g - \theta)), \alpha\}} \right\|_{L^2}^2 + \frac{\mu}{2} \|d - \nabla w\|_{L^2}^2,$$

with a positive constant $\alpha \ll 1$. Then, to break the inner and outer loop, we check if

$$\text{(for Inner)} \quad |F_{\mu,\alpha}(u_n^{k,j+1}, d^{k,j+1}; u_n) - F_{\mu,\alpha}(u_n^{k,j}, d^{k,j}; u_n)| < \varepsilon_{in},$$
$$\text{(for Outer)} \quad |F_{\mu,\alpha}(u_n^{k+1}, d^{k+1}; u_n) - F_{\mu,\alpha}(u_n^k, d^k; u_n)| < \varepsilon_{out}$$

for some $\varepsilon_{in}, \varepsilon_{out} \ll 1$. The equation (3.5)–(3.6) can be solved by SOR method. Remark

Algorithm 1: Minimizing movement without distance function

Input: $\Gamma_0 \subset \overline{W}$ and $u_0 \in C(\overline{W})$ such that $\Gamma_0 = \{u_0 - \theta \equiv 0\}$.
Output: $\Gamma(T) = \{u(T) - \theta \equiv 0\}$ with some function $u(T)\colon \overline{W} \to \mathbb{R}$.
(Time step) **for** $n = 0, 1, \ldots, [T/h] - 1$ **do**
  Set $g = u_n$;
  Initialize $u_n^0 = u_n$, $b^0 = d^0 = 0$;
  (Outer loop) **for** $k = 0, 1, \ldots$ **do**
    Initialize $u_n^{k,0} = u_n^k$, $d^{k,0} = d^k$;
    (Inner loop) **for** $j = 0, 1, 2, \ldots$ **do**
      Solve (3.5)–(3.6) with $\psi = \gamma(\nabla(g - \theta))$ to obtain $u_n^{k,j+1}$;
      Calculate $d^{k,j+1}$ with Scheme (Sh);
      if $|F_{\mu,\alpha}(u_n^{k,j+1}, d^{k,j+1}; g) - F_{\mu,\alpha}(u_n^{k,j}, d^{k,j}; g)| < \varepsilon_{in}$ **then** break;
    Set $u_n^{k+1} = u_n^{k,j+1}$, $d_n^{k+1} = d^{k,j+1}$;
    Set $b^{k+1} = b^k + \nabla u_n^{k+1} - d^{k+1}$;
    if $|F_{\mu,\alpha}(u_n^{k+1}, d^{k+1}; g) - F_{\mu,\alpha}(u_n^k, d^k; g)| < \varepsilon_{out}$ **then** break;
  Set $u_{n+1} = u_n^k$;

that we do not apply the cut-off of $\psi = \gamma(\nabla(u_n - \theta))$ for (3.5).

Finally, we present some numerical results on the evolution of spirals by (1.3) with $f \equiv 2$ and $\rho_c = 0.04$, i.e.,

$$V_\gamma = 2 - 0.04 H_\gamma \tag{4.1}$$

with Algorithm 1. Here we set $\Omega = [-2.5, 2.5]^2$ and the numerical lattice on Ω as $D = \{x_{i,j} = (i\Delta x, j\Delta x) \in \Omega;\ -100 \le i, j \le 100\}$, so that the mesh size is $\Delta x = 0.025$. We impose that $a_j \in D$, and set $r = (2 - 10^{-8})\Delta x$. The time span and the number of time steps are chosen as $h = 0.02 \times \Delta x$ and 2000 steps, and then $T = 1.0$. For the split Bregman method, we set $\mu = \rho_c$, $\varepsilon_{in} = 10^{-2}$, $\varepsilon_{out} = 10^{-5}$, and $\alpha = 10^{-8}$.

Figure 1 and 2 presents profiles of spirals with two centers; $N = 2$, $a_1 = (-0.75, 0)$, $a_2 = (0.75, 0)$. The settings for Figure 1 and Figure 2 are as follows.

$$\begin{aligned}
&(\text{Figure 1})\colon \gamma(p) = \gamma_1(p) = |p_1| + |p_2|, \text{ i.e., } N_\gamma = 4,\ \theta_j = \frac{\pi}{4} + \frac{\pi j}{2},\ r_j = \sqrt{2}, \\
&\qquad \theta(x) = \arg(x - a_1) + \arg(x - a_2), \quad u_0 \equiv 0. && (4.2) \\
&(\text{Figure 2})\colon \gamma(p) = \sqrt{2}\gamma_\infty(p) = \sqrt{2}\max\{|p_1|, |p_2|\}, \text{ i.e., } N_\gamma = 4,\ \theta_j = \frac{\pi j}{2},\ r_j = \sqrt{2}, \\
&\qquad \theta(x) = \arg(x - a_1) - \arg(x - a_2), \quad u_0 \equiv -\pi. && (4.3)
\end{aligned}$$

for $p = (p_1, p_2)$. Note that $\gamma_1^\circ = \gamma_\infty$ and $\gamma_\infty^\circ = \gamma_1$, and then $\mathcal{W}_{\gamma_1} = [-1, 1]^2$ is a square and $\mathcal{W}_{\gamma_\infty}$ is a diagonal square whose vertices are $(\pm 1, 0)$ and $(0, \pm 1)$. As in Figure 1 and 2, one can find the evolution of square spiral with topological change reflecting the shape of $\mathcal{W}_{\gamma_1}$ and $\mathcal{W}_{\gamma_\infty}$, respectively. Moreover, we can choose an initial curve which is not parallel to the facets of $\mathcal{W}_\gamma$ as in Figure 2. Note that such an initial curve is not admissible which is defined in [9], although the admissibility is required in the discrete model due to [9]. Our scheme can be applied to the evolution which violates the admissibility of the curve in finite time.

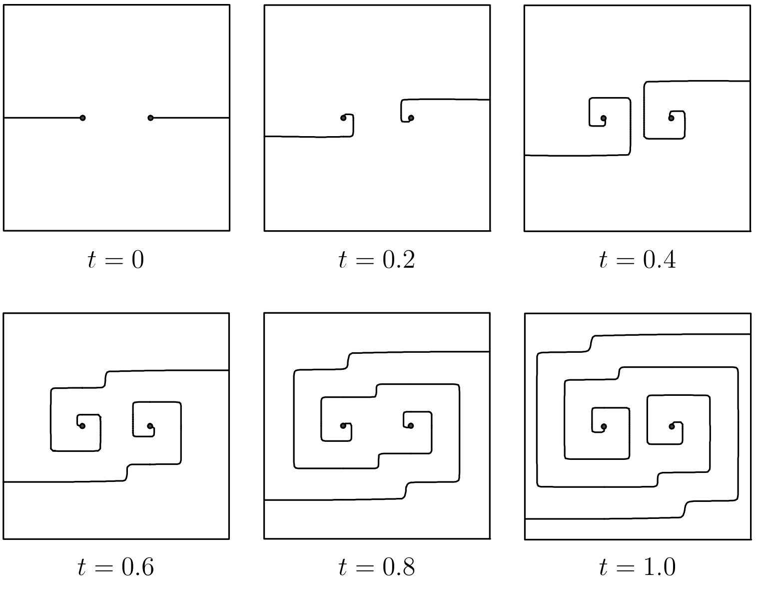

Figure 1: Profiles of spirals evolving by (4.1) with (4.2).

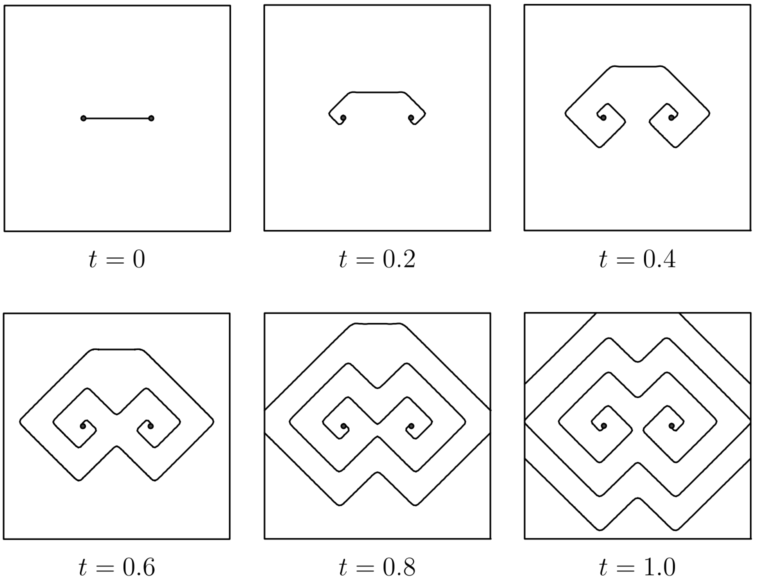

Figure 2: Profiles of spirals evolving by (4.1) with (4.3).

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