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The constrained total variation flow.

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Abstract

In this paper we review some recent results about the total variation flow for functions into manifolds. After recalling some well-know results about existence, uniqueness and qualitative properties for the unconstrained case, the focus will be in obtaining a good notion of solution to the flow. We will show how existence and sometimes uniqueness of solutions can be obtained in some different scenarios.

Keywords. total varation flow, quasilinear parabolic system, Riemannian manifold, existence, uniqueness, image processing

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1 Introduction

Throughout the paper we use standard notations for the space of functions of bounded variation as in [3], to which we refer. We also make use of the following notation:

$$X(U,\mathscr{N}) = \{\mathbf{w} \in X(U,\mathbb{R}^N) \colon \mathbf{w}(\mathbf{y}) \in \mathscr{N} \text{ for a. e. } \mathbf{y} \in U\},$$

where U is any domain in $\mathbb{R}^l$, $l = 1,2,\dots$ and $X(U,\mathbb{R}^N)$ is a subspace of $L^1_{loc}(U,\mathbb{R}^N)$. We will also use Einstein summation convention.

1.1 The Total Variation Flow

Let $\Omega \subset \mathbb{R}^m$ be an open, bounded Lipschitz domain. The total variation functional (TV) is defined in the space $L^1(\Omega)$ as follows:

$$u \mapsto TV(u) = \int_\Omega |Du| := \sup\left\{ \int_\Omega u(x)\mathrm{div}\varphi(x)\,dx \ : \varphi \in (C^1_0(\Omega))^m, \|\varphi\|_\infty \leq 1 \right\}. \tag{1}$$

This energy functional has received a lot of attention since the seminal paper in total variation denoising by W. Rudin, S. Osher and E. Fatemi ([32]). In this paper, the authors proposed to minimize the TV functional into the class of bounded variation functions satisfying the following two constraints, as an algorithm to denoise an observed noisy image $f \in L^2(\Omega)$:

(a) $\displaystyle\int_\Omega u(x)\,dx = \int_\Omega f(x)\,dx$

(b) $\displaystyle\int_\Omega (u(x) - f(x))^2\,dx = \sigma^2$

Here, the constraints (a) and (b) ensure that the real image is corrupted by a white noise (with zero mean and known variance σ^2).

This constrained minimization problem is equivalent to the following unconstrained one, as shown by A. Chambolle and P. L. Lions [18] (with $\lambda \in \mathbb{R}$ being a Lagrange multiplier corresponding to the variance constraint):

$$\min_{u\in BV(\Omega)\cap L^2(\Omega)} \mathscr{E}(u) := \left(\int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u-f)^2\,dx \right); \tag{2}$$

i.e. minimization of the total variation functional together with an L^2 fidelity term.

We observe that functional $\mathscr{E}$ is lower semicontinuous and convex. Therefore, existence and uniqueness of a minimizer $u^\lambda \in BV(\Omega) \cap L^2(\Omega)$ is guaranteed. Moreover, since this energy functional is the sum of a convex functional (the TV functional) and a Freéchet differentiable one, therefore its subdifferential is given by $\partial E(u) = \partial TV(u) + \lambda(u-f)$. Therefore, the minimizer satisifies the corresponding Euler-Lagrange equation:

$$\lambda(u-f) \in -\partial TV(u). \tag{3}$$

The gradient descent flow of TV with respect to the L^2 convergence is given by

$$u_t \in -\partial TV(u). \tag{4}$$

We observe that, by properness, lower semicontinuity and convexity, the subdifferential is a maximal monotone operator in $L^2(\Omega)$. Therefore, we can apply Brezis' abstract result [13] o obtain a unique strong solution:

Theorem 1.1 *For any $u_0 \in L^2(\Omega)$, there exists a unique function $u \in C(0,\infty;L^2(\Omega))$, locally Lipschitz continuous, such that* (4) *holds a.e. $t > 0$, and, for all $t > 0$, $u(t) \in BV(\Omega)$ and*

$$\frac{d^+u}{dt} + (\partial TV)^0(u) = 0,$$

where $(\partial TV)^0(u)$ denotes the canonical seletion of $(\partial TV)(u)$; i.e. the element with smallest norm.

We now turn our attention on how to obtain the solution u to (4) by an Euler implicit scheme from the functional $\mathscr{E}$: Suppose that $u^0 \in BV(\Omega) \cap L^2(\Omega)$ and let $T > 0$, $K \geq 1$, $h := T/K$, $t_n := nh$. Then, letting $\lambda = \frac{1}{h}$ in (2), we define inductively $u^n := \operatorname{argmin}_{u \in BV(\Omega) \cap L^2(\Omega)} \mathscr{E}(u_{n-1})$; i.e:

$$u^n = J_h^n(u_0),$$

with $J_h := (Id + h\partial TV)^{-1}$ being the resolvent of ∂TV. Then, one defines a piecewise constant in time interpolation

$$u^K(t) := u^0 \chi_{[0,t_1]}(t) + \sum_{n=1}^{K-1} u^n \chi_{(t_n,t_n+1]}(t). \tag{5}$$

Therefore, one expects that, when $h \to 0$, u^K would converge to u. In fact, this is implied by Crandall-Ligget's Theorem [19], and we have:

Theorem 1.2 *For any $u^0 \in L^2(\Omega)$, and $t > 0$, assuming the notation in the previous paragraph,*

$$\lim_{h \to 0^+, nh \to t} J_h^n u^0 = S(t)u^0,$$

where $S(t)$ is the semigroup generated by $-\partial TV$ given by Theorem 1.1. Moreover, the convergence is uniform on compact intervals of $[0,\infty[$.

Note that the semigroup solution obtained in Theorems 1.1 and 1.2 is an abstract notion of solution. If one wants to study some qualitative properties of the solution such as regularity, asymptotic behavior or even explicit solutions, one needs to characterize the subdifferential of the total variation. An easy computation shows that, formally,

$$\partial TV(u) = -\mathrm{div}\left(\frac{Du}{|Du|}\right).$$

Last expression is just a formal one, because $|Du|$ may vanish and moreover, because $|Du|$ is a Radon measure. The proper characterization of the subdifferential was the main result obtained by F. Andreu, C. Ballester, V. Caselles and J. Mazón in [7]:

Theorem 1.3 *Let $u \in L^2(\Omega) \cap BV(\Omega)$ and $v \in L^2(\Omega)$. Then, $v \in \partial TV(u)$ if, and only if, there exists a vector field*

$$z \in X_2(\Omega) := \{w \in L^\infty(\Omega;\mathbb{R}^N) : \mathrm{div} z \in L^2(\Omega)\}$$

such that $\|z\|_\infty \le 1$, $v = -\mathrm{div} z$ in $\mathscr{D}'(\Omega)$,

$$\int_\Omega vu = \int_\Omega |Du|$$

and

$$[z,\nu^\Omega] = 0 \quad \text{on } \partial\Omega.$$

Here $[z,\nu^\Omega]$ denotes the weak trace of the normal component of the vector field z (see [9] for details.)

With this characterization at hand, the authors showed existence and uniqueness of entropy solutions to the total variation flow even for data in $L^1(\Omega)$. This result was the seed for many different studies on the total variation flow and related equations (see, for a reference, the excellent monograph [8]). Existence and uniquenesss of solutions to different problems were obtained: the nonhomogenous Dirichlet problem [6], the Cauchy problem [11] or the anisotropic case [30]. Moreover, different qualititave properties were deeply studied; explicit solutions, asymptotic profiles, extinction in finite time ([5],[12] or characterization of convex calibrable sets ([2],[14]) among others.

Concerning regularity of the solutions, the most important results were obtained in [16],[17] and [15], which we summarize here:

Theorem 1.1 *Let $u^0 \in L^2(\Omega) \cap BV(\Omega)$ and let u be the solution of the Total Variation Flow with initial condition u^0. Then, $u(t) \in L^\infty(\Omega) \cap BV(\Omega)$ for any $t > 0$ and*

$$J_{u(t)} \subseteq J_{u(s)} \subseteq J_{u^0}, \quad \mathscr{H}^{N-1} - \text{a.e. for any } 0 < s < t.$$

Moreover,

$$|u(t)^+ - u(t)^-| \le |u(s)^+ - u(s)^-| \le |(u^0)^+ - (u^0)^-|, \mathscr{H}^{N-1} - a.e. x \in J_{u_0}.$$

Finally, if u^0 is uniformly continuous, then, for any $t > 0$, $u(t)$ is uniformly continuous with modulus $\omega_{u(t,\cdot)} \le \omega_{u^0}$.

This means that uniform continuity is preserved, that no new jumps are created through the evolution and that size of jumps cannot increase in time.

1.2 The Constrained Total Variation Flow

Once the (unconstrained and scalar) total variation flow is well understood, we will try to generalize the above results in the case that the solutions are constrained to take values into a Riemannian manifold.

Let $(\mathscr{N}, g)$ be a complete, connected smooth n-dimensional Riemannian manifold (without boundary). Throughout the paper, without loss of generality [31, 27], we will treat it as an isometrically embedded submanifold in the Euclidean space $\mathbb{R}^N$. Given an open, bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$ we consider the formal steepest descent flow with respect to the L^2 distance of the functional $\mathrm{TV}_\Omega^{\mathscr{N}}$: the total variation functional constrained to functions taking values in $\mathscr{N}$, given for smooth $\mathbf{u}$ by

$$\mathrm{TV}_\Omega^{\mathscr{N}}[\mathbf{u}] = \int_\Omega |\nabla \mathbf{u}|. \tag{6}$$

Important examples of manifolds in image processing are the following ones: the case $\mathscr{N} \subseteq \mathbb{S}^{N-1}$, which appears in denoising of optical flows [33] or color images [34], the space of isometries $SO(3) \times \mathbb{R}^3$ [28] appears in the denoising of camera trajectories, the cylinder $\mathbb{R}^2 \times \mathbb{S}^1$ plays a role in denoising in the LCh color space [35] and the space of positive definite symmetric matrices (diffusion tensors) $Sym_+(3)$ naturally appears in brain image processing [35].

Now, if one tries to apply the analogous to the Euler implicit scheme for the flow, one is forced to solve the following minimization problem at each time step:

$$\min_{\mathbf{u} \in BV(\Omega;\mathscr{N})} \mathscr{E}_f^{\mathscr{N}}(\mathbf{u}) := \int_\Omega |D\mathbf{u}| + \frac{1}{2h} \int_\Omega \mathrm{dist}^2_{\mathscr{N}}(u, f)\, dx. \tag{7}$$

Several questions appear immediately when having a look at (7): The first one is the proper meaning of the expression $\int_\Omega |Du|$ for functions $\mathbf{u} \in BV(\Omega;\mathscr{N})$. Second one is, is there a unique minimizer to the functional? Next, would the (analogous to) Crandall-Ligget's Theorem still work? And finally, can we charaterize the (equivalent to) the subdifferential of the functional (7)?

The first question was answered by M. Giaquinta and Mucci. We recall, to the extent we need, the result in [26]

Theorem 1.2 *Assume that $\mathscr{N}$ is compact and topologically trivial. Define $TV^{\mathscr{N}}$ as the relaxed functional of the total variation for functions in $L^1(\Omega;\mathscr{N})$; i.e.:*

$$TV^{\mathscr{N}}(\mathbf{u}) := \inf\left\{ \liminf_{k\to\infty} \int_\Omega |D\mathbf{u}_k| : \mathbf{u}_k \in C^1(\Omega;\mathscr{N}), \mathbf{u}_k \rightharpoonup \mathbf{u} \text{ weakly in the BV sense} \right\}.$$

It turns out that $BV(\Omega;\mathcal{N})$ *coincides with the set of* $L^1(\Omega;\mathcal{N})$ *functions with bounded* $TV^{\mathcal{N}}$. *Supposing that* $\mathbf{u}\in BV(\Omega;\mathcal{N})$ *satisfies that*

$$\mathrm{dist}_{\mathcal{N}}(\mathbf{u}(x)^-,\mathbf{u}(x)^+) < \mathrm{inj}_{\mathcal{N}} \quad \text{for any } x\in J_{\mathbf{u}}, \tag{8}$$

where $\mathrm{inj}_{\mathcal{N}}$ *denotes the inectivity radius of* $\mathcal{N}$*, it follows that*

$$TV^{\mathcal{N}}(\mathbf{u}) := \int_\Omega |\nabla\mathbf{u}(x)|\,dx + \int_\Omega d|D^c\mathbf{u}| + \int_{J_{\mathbf{u}}} \mathrm{dist}(\mathbf{u}^+,\mathbf{u}^-)\,d\mathcal{H}^{N-1}. \tag{9}$$

Therefore, for functions $\mathbf{u}\in BV(\Omega;\mathcal{N})$ satisfying (8), we identify $\int_\Omega |D\mathbf{u}|$ with the expression at (9).

Recall that for any two points $\mathbf{p}_1,\mathbf{p}_2\in\mathcal{N}$, the condition $\mathrm{dist}_{\mathcal{N}}(\mathbf{p}_1,\mathbf{p}_2) < \mathrm{inj}_{\mathcal{N}}$ implies that there is exactly one minimizing geodesic in $\mathcal{N}$ joining $\mathbf{p}_1$ and $\mathbf{p}_2$. This imposes a restriction when working with BV solutions; i.e, we will have to assume that our initial data (and therefore our solutions) take values into a subset of the manifold where unique minimizing geodesics exist or, at least, that jumps are not too far apart. However, we observe that in the case of nonpositive sectional curvature, the injectivity radius is infinite and no restriction is imposed.

Second question, related to convexity, is much more delicate and we still don't know the answer. Suppose that TV is λ-convex along geodesics and that $\mathcal{N}$ has nonpositive sectional curvature. Then, by the results of U. Mayer proved in the context of NPC-spaces [29] , for sufficiently small $h>0$, there exists a unique minimizer to problem (7). Then, one can define the resolvent $J_h : BV(\Omega;\mathcal{N})\cap L^2(\Omega;\mathcal{N}) \to BV(\Omega;\mathcal{N})\cap L^2(\Omega;\mathcal{N})$ by

$$J_h(f) := \mathrm{argmin}_{u\in BV(\Omega;\mathcal{N})\cap L^2(\Omega;\mathcal{N})} \mathcal{E}_f^{\mathcal{N}}(\mathbf{u}).$$

One crucial step in the proof of Crandall-Ligget's Theorem is the nonexpansitivity of the resolvent; i.e.

$$\mathrm{dist}_{\mathcal{N}}(J_h(\mathbf{u}),J_h(\mathbf{v})) \le \mathrm{dist}_{\mathcal{N}}(\mathbf{u},\mathbf{v}),\ \text{for any } \mathbf{u},\mathbf{v}\in BV(\Omega;\mathcal{N})\cap L^2(\Omega;\mathcal{N}).$$

Mayer proved, that under the conditions above concerning convexity and curvature, the resolvent is nonexpansive and then, the corresponding approximating solutions (piecewise constant in time as in (5)) converge to a function uniformly in compact time intervals and the exponential formula defines a contraction semigroup on $\overline{D(TV^{\mathcal{N}})}$. Moreover, he also showed that the $TV^{\mathcal{N}}$ functional is convex along geodesics under the assumption that $\mathcal{N}$ has nonpositive sectional curvature. Therefore, the solution to the flow can be abstractly obtained in this particular case.

It turns out that we are not able to ensure even uniqueness of the minimizer in the general case and therefore to define the resolvent. In fact, in [24] we show the following

Example 1.3 *Let* $\mathcal{N}=S^{N-1}$. *Then, the functional* $TV^{\mathcal{N}}$ *is not* $\lambda-$*convex for any* $\lambda\in\mathbb{R}$

Therefore, we will try to show existence of solutions to the gradient flow for $TV^{\mathscr{N}}$ without making use of the time discretization. Instead, we will focus on the parabolic Euler-Lagrange system associated to the gradient descent flow in $L^2(\Omega;\mathscr{N})$ and we will work directly on it.

Given a point $\mathbf{p}\in\mathscr{N}$, we denote by

$$\pi_{\mathbf{p}}:\, T_{\mathbf{p}}\mathbb{R}^N \equiv \mathbb{R}^N \to T_{\mathbf{p}}\mathscr{N}$$

the orthogonal projection onto the tangent space of $\mathscr{N}$ at $\mathbf{p}$, $T_{\mathbf{p}}\mathscr{N}$. After a formal computation of the first variation of (6) at $\mathbf{u}$, supposing that $\mathbf{u}$ is smooth, one obtains that the flow in a time interval $[0,T[$ starting with initial datum $\mathbf{u}_0$ is formally given by the system

$$\mathbf{u}_t = \pi_{\mathbf{u}}\left(\operatorname{div}\tfrac{D\mathbf{u}}{|D\mathbf{u}|}\right) = \operatorname{div}\tfrac{D\mathbf{u}}{|D\mathbf{u}|} + \mathscr{A}(\mathbf{u})\left(\tfrac{D\mathbf{u}}{|D\mathbf{u}|}, D\mathbf{u}\right) \quad \text{in }]0,T[\times\Omega, \tag{10}$$

$$\nu^{\Omega}\cdot\tfrac{D\mathbf{u}}{|D\mathbf{u}|} = 0 \quad \text{in }]0,T[\times\partial\Omega, \tag{11}$$

$$\mathbf{u}(0,\cdot)\equiv\mathbf{u}^0, \tag{12}$$

where $\mathscr{A}(\mathbf{p})(\mathbf{X},\mathbf{Y})$ denotes the second fundamental form at a point $\mathbf{p}\in\mathscr{N}$ acting on $\mathbf{X},\mathbf{Y}\in T_{\mathbf{p}}\mathscr{N}$. Observe that, in general one expects to have some energy estimates ensuring that $\mathbf{u}_t\in L^\infty([0,T);L^2(\Omega))$, therefore, even for the case of BV-solutions a system like the following one could be the right substitute to (10):

$$\mathbf{u}_t = \operatorname{div}\mathbf{Z} + \mu = (\operatorname{div}\mathbf{Z})^a + \mu^a = \pi_{\mathbf{u}}(\operatorname{div}\mathbf{Z})^a, \tag{13}$$

where superindex a represents the Radon-Nikodym derivative of the Radon measure with respect to the Lebesgue measure. The main task would be to characterize the tensor field $\mathbf{Z}$ representing the quotient $\frac{D\mathbf{u}}{|D\mathbf{u}|}$ and/or the measure μ representing $\mathscr{A}(\mathbf{u})(\mathbf{Z},D\mathbf{u})$ for a generic manifold $\mathscr{N}$.

The rest of the paper will consist in reviewing the main results in [25],[23] and [24], showing existence of solutions and characterization of $\mathbf{Z}$ and/or μ in different scenarios. In Section 2, heavily using the symmetries of the sphere, we define a notion of solution to the flow when the target manifold is a hyperoctant of the sphere $\mathscr{N}=\mathbb{S}^{N-1}_+$ and we construct solutions after a suitable regularization of the energy functional and a delicate passage to the limit. In Section 3, we define and obtain local in time regular solutions to the flow when the initial data are Lipschitz continuous functions. These results will be the basis of future studies for BV initial data. In fact, in Section 4, we will use them to obtain global existence of suitable defined solutions for a generic BV initial data when the domain is an interval. Moreover, in case the target manifold has non-positive sectional curvature, solutions are shown to be unique.

2 BV-solutions in case of $\mathbb{S}_+^{N-1}$

In this Section, we review the results in [25]. For the sake of clarity of explanation and in order to avoid some technicalities about multivectors, we will just treat the case that $N = 3$ and we refer to [25] for the general case.

For a smooth map $\mathbf{u} : \Omega \to \mathbb{S}_+^2$, the gradient flow with respect to the L^2 distance of the $TV^{\mathbb{S}^2+}$ functional formally reads as

$$\begin{cases} \mathbf{u}_t = \operatorname{div}\left(\dfrac{D\mathbf{u}}{|D\mathbf{u}|}\right) + \mathbf{u}|D\mathbf{u}|, \quad \mathbf{u} \in \mathbb{S}_+^2 & \text{in } Q_T = (0,T) \times \Omega \\ \dfrac{D\mathbf{u}}{|D\mathbf{u}|} \cdot \nu = 0 & \text{on } S_T = (0,T) \times \partial\Omega \\ \mathbf{u}(0,\cdot) = \mathbf{u}^0(\cdot), \quad \mathbf{u}^0 \in \mathbb{S}_+^2 & \text{in } \Omega, \end{cases} \tag{14}$$

Before introducing our notion of solution, we need to give some auxiliary definitions:

Definition 2.1 *Let $\mathbf{u} \in BV(\Omega;\mathbb{S}_+^{N-1})$. The geodesic representative $\mathbf{u}_g : \Omega \backslash (S_{\mathbf{u}} \backslash J_{\mathbf{u}}) \to \mathbb{S}_+^{N-1}$ of $\mathbf{u}$ is defined by*

$$\mathbf{u}_g = \begin{cases} \mathbf{u}^* & \text{on } \Omega \backslash S_{\mathbf{u}} \\ \mathbf{u}^*/|\mathbf{u}^*| & \text{on } J_{\mathbf{u}}. \end{cases}$$

Note that $\mathbf{u}_g \in BV(\Omega;\mathbb{S}_+^{N-1})$ since $\mathbf{u}^+$ and $\mathbf{u}^-$ are $\mathscr{H}^{m-1}$-measurable on $J_{\mathbf{u}}$ (see [3, Prop. 3.69]). Hence, the following Radon measures is well defined:

$$\mathbf{u}_g|D\mathbf{u}| := \mathbf{u}\left(|\nabla \mathbf{u}|\mathscr{L}^m + |D^c\mathbf{u}|\right) + \mathbf{u}_g|\mathbf{u}^+ - \mathbf{u}^-|\,\mathscr{H}^{m-1} \llcorner_{J_{\mathbf{u}}}. \tag{15}$$

We are now ready to introduce the concept of solution for (14).

Definition 2.2 *Let $T > 0$, and $\mathbf{u}^0 \in BV(\Omega;\mathbb{S}_+^2)$. A function*

$$\mathbf{u} \in L^\infty(0,T;BV(\Omega;\mathbb{R}^N)) \cap C(0,T;L^1(\Omega;\mathbb{R}^N)), \quad \mathbf{u}_t \in L^2(0,T;L^2(\Omega;\mathbb{R}^N))$$

is a solution to (14) in Q_T if $\mathbf{u}(0) = \mathbf{u}^0$, $\mathbf{u} \in \mathbb{S}_+^2$ a.e. in Q_T, and there exists a matrix-valued function $\mathbf{Z} \in L^\infty(Q_T, \mathbb{R}^{N\times m})$, with $\|\mathbf{Z}\|_\infty \le 1$ such that

$$\mathbf{u}_t(t) - \operatorname{div}\mathbf{Z}(t) = \mathbf{u}(t)_g\,|D\mathbf{u}(t)| \quad \text{as measures for a.e. } t \in [0,T], \tag{16}$$

$$\mathbf{u}_t(t) \wedge \mathbf{u}(t) = \operatorname{div}(\mathbf{Z}(t) \wedge \mathbf{u}(t)) \quad \text{in } L^2(\Omega;(\mathbb{R}^{N\times m})) \quad \text{for a.e. } t \in [0,T], \tag{17}$$

$$\mathbf{Z}^T \cdot \mathbf{u} = 0 \quad \textit{a.e. in } Q_T, \tag{18}$$

and

$$[\mathbf{Z}(t), \nu] = 0 \quad \mathscr{H}^{m-1}\textit{-a.e. on } \partial\Omega \textit{ for a.e. } t \in [0,T]. \tag{19}$$

Our main result is the following existence theorem.

Theorem 2.3 *For any $T > 0$ and any $\mathbf{u}^0 \in BV(\Omega;\mathbb{S}_+^{N-1})$ there exists a solution $\mathbf{u}$ to (14) in the sense of Definition 2.2.*

The strategy to prove Theorem 2.3 can be summarized as follows. First of all, we regularize the energy functional and we consider, for a given $\varepsilon > 0$, and $\alpha > m$,

$$J_\alpha^\varepsilon(\mathbf{v}) := \varepsilon^\alpha \int_\Omega |\nabla \mathbf{v}(x)|^2 dx + \int_\Omega \sqrt{|\nabla \mathbf{v}(x)|^2 + \varepsilon^2}\,\mathrm{d}x, \quad \mathbf{v} \in W^{1,2}(\Omega;\mathbb{R}^N),$$

The gradient flow corresponding to functional J_α^ε restricted to functions valued in $\mathbb{S}_+^2$ is the following one:

$$\begin{cases} \mathbf{u}_t^\varepsilon = \mathrm{div}\mathbf{Z}^\varepsilon + \mu^\varepsilon & \text{in } Q_T \\ [\mathbf{Z}^\varepsilon, \nu] = 0 & \text{in } S_T, \end{cases} \tag{20}$$

where

$$\mathbf{Z}^\varepsilon = \varepsilon^\alpha \nabla \mathbf{u}^\varepsilon + \frac{\nabla \mathbf{u}^\varepsilon}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}} \quad \text{and} \quad \mu^\varepsilon = \varepsilon^\alpha \mathbf{u}^\varepsilon |\nabla \mathbf{u}^\varepsilon|^2 + \mathbf{u}^\varepsilon \frac{|\nabla \mathbf{u}^\varepsilon|^2}{\sqrt{|\nabla \mathbf{u}^\varepsilon|^2 + \varepsilon^2}}, \tag{21}$$

The following result concerning existence and uniqueness of solutions and energy estimates to the system (20) was obtained in [10]

Proposition 2.4 *Let $\varepsilon > 0$, $T > 0$ and $\alpha > m$. If $\mathbf{u}_0^\varepsilon \in W^{1,2}(\Omega;\mathbb{S}^{N-1})$, then there exists*

$$\mathbf{u}^\varepsilon \in L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^N)) \cap W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^N))$$

such that $\mathbf{u}^\varepsilon(0,\cdot) = \mathbf{u}_0^\varepsilon$,

$$|\mathbf{u}^\varepsilon| = 1 \quad \text{a.e. in } Q_T, \tag{22}$$

and $\mathbf{u}^\varepsilon$ is a weak solution to (20)*. Furthermore, the following holds:*

$$(\mathbf{Z}^\varepsilon)^T \cdot \mathbf{u}^\varepsilon = 0 \quad \text{a.e. in } Q_T, \tag{23}$$

$$\mathbf{u}_t^\varepsilon \cdot \mathbf{u}^\varepsilon = 0 \quad \text{a.e. in } Q_T, \tag{24}$$

$$\mathbf{u}_t^\varepsilon \wedge \mathbf{u}^\varepsilon = \mathrm{div}(\mathbf{Z}^\varepsilon \wedge \mathbf{u}^\varepsilon), \tag{25}$$

$$J_\alpha^\varepsilon(\mathbf{u}^\varepsilon(t)) + \int_0^t \int_\Omega |\mathbf{u}_t^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s \le J_\alpha^\varepsilon(\mathbf{u}_0) \quad \text{for a.e. } t \in [0,T], \tag{26}$$

and a positive ε-independent constant C exists such that

$$\|\mathrm{div}\mathbf{Z}^\varepsilon\|_{L^2(0,T;L^1(\Omega;\mathbb{R}^N))} \le C, \tag{27}$$

$$\|\mathrm{div}(\mathbf{Z}^\varepsilon \wedge \mathbf{u}^\varepsilon)\|_{L^2(0,T;L^2(\Omega;\Lambda_2(\mathbb{R}^N)))} \le C, \tag{28}$$

$$\varepsilon^{\frac{\alpha}{2}} \|\nabla \mathbf{u}^\varepsilon(t)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{N\times m}))} \le C. \tag{29}$$

We next showed that any function $\mathbf{u}^0 \in BV(\Omega;\mathbb{S}^2_+)$ can be approximated by functions $(\mathbf{u}^0)^\varepsilon$ in $W^{1,2}(\Omega;\mathbb{S}^2_+)$ in such a way that the initial energy is controlled. Therefore, by using Proposition 2.4 and standard compactness arguments, we obtain convergence of $\mathbf{u}^\varepsilon$, $\mathbf{Z}^\varepsilon$, and μ^ε to $\mathbf{u}$, $\mathbf{Z}$, and μ. The functions $\mathbf{u}$ and $\mathbf{Z}$ can be seen to satisfy the regularity assumptions in Definition 2.2 and, for a.e. $t \in [0,T]$, (17),(18),(19) and

$$\mathbf{u}_t(t) - \operatorname{div}\mathbf{Z}(t) = \mu(t) \quad \text{in } \mathscr{M}(\Omega;\mathbb{R}^N).$$

The only remaining task is to characterize the vector measure μ. First of all, we showed that

$$|\mu(t)| \leq |D\mathbf{u}(t)| \quad \text{as measures for a.e. } t \in [0,T] \tag{30}$$

Next, we deduce, using some vectorial identities that, in order to identify

$$\mu(t) = \mathbf{u}(t)_g |D\mathbf{u}(t)| \quad \text{for a.e. } t \in [0,T]$$

it suffices to show next inequality:

$$\mathbf{u}(t)_g \cdot \frac{\mu(t)}{|D\mathbf{u}(t)|} \geq 1 \quad \text{for a.e. } t \in [0,T], \tag{31}$$

where $\frac{\mu(t)}{|D\mathbf{u}(t)|}$ denotes the Radon-Nikodým derivative of $\mu(t)$ with respect to $|D\mathbf{u}(t)|$. For the diffuse part of μ, this follows from a relaxation result in [1], applied to each of the components of the energy functional

$$\mathscr{F}(\mathbf{v}) := \int_\Omega \mathbf{v}(x)|\nabla \mathbf{v}(x)|\,\mathrm{d}x$$

For the jump part, after a blow-up procedure as in [20] and a dimensional reduction argument as in [21] we showed that

$$\mathbf{u}(t)_g \cdot \frac{\mu(t)}{|D\mathbf{u}(t)|} \geq \frac{1}{|\mathbf{u}(t)^+(x) - \mathbf{u}(t)^-(x)|} \inf_{\gamma \in \tilde{\Gamma}_N} \int_0^1 \mathbf{u}(t)_g(x) \cdot \gamma(s)|\gamma'(s)|\,\mathrm{d}s \tag{32}$$

for a.e. t and $\mathscr{H}^{m-1}$-a.e. $x \in J_{\mathbf{u}(t)}$, where

$$\tilde{\Gamma}_N := \left\{\gamma \in W^{1,1}((0,1);\mathbb{S}^2_+) : \ \gamma(0) = \mathbf{u}(t)^-(x), \ \gamma(1) = \mathbf{u}(t)^+(x)\right\} \tag{33}$$

It turns out that the minimum in (32) is achieved by the standard geodesic on $\mathbb{S}^2_+$ connecting $\mathbf{u}_-$ and $\mathbf{u}_+$. However, the analysis of (32) is delicate since one wants to minimize a non-convex functional (it always possesses a second smooth critical point, which is shown not to be a shortest path) in a manifold with boundary. The identification of the shortest path yields the lower bound (31) on the jump part, and we conclude the proof of existence.

3 The regular 1-harmonic flow

In this Section we review some results in [23] about existence, uniqueness and some qualitative properties of the solutions to the flow in the case that the initial data (and therefore the solutions) are Lipschitz continuous.

Our notion of solution to the system is the following one:

Definition 3.1 *Let $T\in]0,\infty]$. We say that*

$$\mathbf{u}\in W^{1,2}_{loc}([0,T[\times\overline{\Omega},\mathscr{N}) \text{ with } \nabla\mathbf{u}\in L^{\infty}_{loc}([0,T[\times\overline{\Omega},\mathbb{R}^{m\cdot N})$$

is a (regular) solution to (10) (in $[0,T[$) if there exists $\mathbf{Z}\in L^{\infty}(]0,T[\times\Omega,\mathbb{R}^{m\cdot N})$ with $\operatorname{div}\mathbf{Z}\in L^2_{loc}([0,T[\times\overline{\Omega},\mathbb{R}^N)$ satisfying

$$\mathbf{Z}\in\tfrac{\nabla\mathbf{u}}{|\nabla\mathbf{u}|}, \tag{34}$$

$$\mathbf{u}_t=\pi_{\mathbf{u}}(\operatorname{div}\mathbf{Z}) \tag{35}$$

$\mathscr{L}^{1+m}-$ a. e. in $]0,T[\times\Omega$. We say that a regular solution $\mathbf{u}$ to (10) satisfies the (homogeneous) Neumann boundary condition (11) if

$$\nu^{\Omega}\cdot\mathbf{Z}=0 \quad \mathscr{L}^1\otimes\mathscr{H}^{m-1}-\text{a. e. in }]0,T[\times\partial\Omega \tag{36}$$

Here, $\frac{\nabla\mathbf{u}}{|\nabla\mathbf{u}|}$ is understood as a multifunction

$$\frac{\nabla\mathbf{u}}{|\nabla\mathbf{u}|}:(t,\mathbf{x})\mapsto\begin{cases}\frac{\nabla\mathbf{u}(t,\mathbf{x})}{|\nabla\mathbf{u}(t,\mathbf{x})|} & \text{if } \nabla\mathbf{u}(t,\mathbf{x})\neq\mathbf{0}\\ B(0,1)\subset\mathbb{R}^m\times T_{\mathbf{u}(t,\mathbf{x})}\mathscr{N} & \text{if } \nabla\mathbf{u}(t,\mathbf{x})=\mathbf{0}\end{cases}$$

The validity of Definition 3.1 is supported by the well-posedness results that we obtain. First of all, regular solutions are unique. The proof of this result is standard once one assumes that there is a bound on the Lipschitz norm of the solutions and we refer to [23] for details.

Theorem 3.2 *Suppose that $\mathbf{u},\mathbf{v}$ are two regular solutions to (10, 11) in $[0,T[$, $T\in]0,\infty[$ such that $\mathbf{u}(0,\cdot)=\mathbf{v}(0,\cdot)=\mathbf{u}^0$. Then $\mathbf{u}\equiv\mathbf{v}$.*

The existential theory depends on the sectional curvature $K_{\mathscr{N}}$ of $\mathscr{N}$ or, equivalently, on the Riemannian curvature tensor $\mathscr{R}^{\mathscr{N}}$ of $\mathscr{N}$. We denote by $K_{\mathscr{N}}$ the supremum of sectional curvature over $\mathscr{N}$, i. e.

$$K_{\mathscr{N}}=\sup\left\{\frac{\mathbf{v}\cdot\mathscr{R}^{\mathscr{N}}_{\mathbf{p}}(\mathbf{v},\mathbf{w})\mathbf{w}}{|\mathbf{v}|^2|\mathbf{w}|^2-(\mathbf{v}\cdot\mathbf{w})^2}\;\middle|\;\mathbf{p}\in\mathscr{N},\mathbf{v},\mathbf{w}\in T_{\mathbf{p}}\mathscr{N}\text{ linearly independent}\right\}. \tag{37}$$

Theorem 3.3 *Suppose that Ω is convex, the embedding of $\mathscr{N}$ in $\mathbb{R}^N$ is closed and $K_{\mathscr{N}}<\infty$. Given $\mathbf{u}^0\in W^{1,\infty}(\Omega,\mathscr{N})$, we denote $T_\dagger=(K_{\mathscr{N}}\|\nabla\mathbf{u}_0\|_{L^\infty})^{-1}$ if $K_{\mathscr{N}}>0$ and $T_\dagger=+\infty$ otherwise. There exists a regular solution $\mathbf{u}$ to (10, 11, 12) in $[0,T_\dagger[$ satisfying the energy inequality*

$$\operatorname*{ess\,sup}_{t\in[0,T_\dagger[}\int_\Omega|\nabla\mathbf{u}(t,\cdot)|+\int_0^{T_\dagger}\int_\Omega\mathbf{u}_t^2\leq\int_\Omega|\nabla\mathbf{u}_0|. \tag{38}$$

Let us comment on the strategy to prove Theorem 3.3:

First of all, we approximate the system in a similar way to the one for the case $\mathbb{S}^2_+$, but without the elliptic regularization given by the laplacian:

$$\mathbf{u}^\varepsilon_t = \pi_{\mathbf{u}^\varepsilon}\left(\operatorname{div}\frac{\nabla\mathbf{u}^\varepsilon}{\sqrt{\varepsilon^2+|\nabla\mathbf{u}^\varepsilon|^2}}\right) \quad \text{in }]0,T[\times\Omega, \tag{39}$$

$$\nu^\Omega\cdot\nabla\mathbf{u}^\varepsilon = \vec{0} \quad \text{in }]0,T[\times\partial\Omega, \tag{40}$$

$$\mathbf{u}^\varepsilon(0,\cdot) = \mathbf{u}^0. \tag{41}$$

As a first step, we obtain the following Bochner formula:

Lemma 3.4 *Let* $\mathbf{u}\in C^{\frac{3+\alpha}{2},3+\alpha}_{loc}(\overline{\Omega}_{[0,T[},\mathscr{N})$ *satisfy* (39)*. Then, on* $]0,T[\times\Omega$*,*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}|^2 = (\nabla\mathbf{u}:\nabla\mathbf{Z}_i)_{x^i} - (\pi_{\mathbf{u}}\nabla^2\mathbf{u}):\nabla\mathbf{Z} + \mathbf{Z}_i\cdot\mathscr{R}^{\mathscr{N}}_{\mathbf{u}}(\mathbf{u}_{x^i},\mathbf{u}_{x^j})\mathbf{u}_{x^j}, \tag{42}$$

where $\mathbf{Z}^\varepsilon = \frac{\nabla\mathbf{u}^\varepsilon}{\sqrt{\varepsilon^2+|\nabla\mathbf{u}^\varepsilon|^2}}$.

This formula, together with the following energy estimate (here $v := \sqrt{\varepsilon^2+|\nabla\mathbf{u}^\varepsilon|^2}$)

$$\sup_{t\in[0,T[}\int_\Omega v(t,\cdot) + \int_0^T\int_\Omega \mathbf{u}_t^2 \le \int_\Omega v_0, \tag{43}$$

permits us to show the following uniform a priori Lipschitz bound:

Lemma 3.5 *Let* $\mathbf{u}\in C^{\frac{3+\alpha}{2},3+\alpha}_{loc}(\overline{\Omega}_{[0,T[},\mathscr{N})$ *satisfy (39-41).*

(i) If $K_{\mathscr{N}}\in]0,\infty[$*, then*

$$\|v(t,\cdot)\|_{L^\infty} \le \frac{\|v_0\|_{L^\infty}}{1-t\,K_{\mathscr{N}}\,\|v_0\|_{L^\infty}} \tag{44}$$

for $t\in]0,\min(T_\dagger,T)[$*, where* $T_\dagger := (K_{\mathscr{N}}\,\|v_0\|_{L^\infty})^{-1}$.

(ii) If $K_{\mathscr{N}}\le 0$*, then for* $0<t<T<T_\dagger := +\infty$ *there holds*

$$\|v(t,\cdot)\|_{L^\infty} \le \|v_0\|_{L^\infty}. \tag{45}$$

As a second step, we unconstrain the problem. For this, we construct a totally geodesic embedding of $(\mathscr{N},g)$ into $(\mathbb{R}^N,h)$. The gradient flow of the unconstrained functional $\int_\Omega|\nabla\mathbf{u}|_h$ is

$$u^i_t = \operatorname{div}\frac{\nabla u^i}{\sqrt{\varepsilon^2+|\nabla\mathbf{u}|_h^2}} + \frac{1}{\sqrt{\varepsilon^2+|\nabla\mathbf{u}|_h^2}}\Gamma^i_{jk}(\mathbf{u})u^j_{x^l}u^k_{x^l}, \tag{46}$$

$$v^\Omega \cdot \nabla u^i = 0, \tag{47}$$

where $i = 1,\ldots,N$ and Γ^i_{jk} are the Christoffel symbols of $(\mathbb{R}^N, h)$. As h restricted to $T\mathscr{N}$ coincides with g, the system (46, 47) is identical to (39, 47) as long as the range of $\mathbf{u}$ is contained in $\mathscr{N}$. Under some additional compatibility conditions on the initial data, we obtain that for any $\varepsilon > 0$ the system (39-41) has a unique solution

$$\mathbf{u}^\varepsilon \in C_{loc}^{\frac{3+\alpha}{2},3+\alpha}(\overline{\Omega}_{[0,T_\dagger[}, \mathscr{N})$$

where $T_\dagger = T_\dagger(\|\nabla \mathbf{u}_0\|_{L^\infty}, K_{\mathscr{N}}) \in]0,\infty]$ is defined in Lemma 3.5.

Thanks to the uniform a priori bounds obtained, for good initial data, by standard compactness arguments and letting $\varepsilon \to 0^+$ one easily obtains that there exists a solution to the flow. The general case of Lipschitz initial data is proven by approximation with $C^\infty(\overline{\Omega}; \mathscr{N})$ satisfying the compatibility conditions and this finishes the proof of existence of solutions.

Finally, in the next result we show some qualitative properties of the solutions: the existence of invariant regions to the flow and that, in the case of nonpositive sectional curvature, solutions exist globally and they converge to a point in finite time. Here, we denote by $B_g(\mathbf{p}, R)$ the ball centered at $\mathbf{p} \in \mathscr{N}$ of radius $R > 0$ with respect to the metric induced by g on $\mathscr{N}$.

Theorem 3.6 *Let $\mathbf{p}_0 \in \mathscr{N}$, $\mathbf{u}^0 \in W^{1,\infty}(\Omega, \mathscr{N})$ and $\mathbf{u}$ be a regular solution to (10, 11, 12) in $[0,T[$. Suppose that $\mathbf{u}^0(\Omega) \in \overline{B_g(\mathbf{p}_0, R)}$, $R > 0$. There exist a constant $R_* = R_*(\mathscr{N}, \mathbf{p}_0) > 0$ such that if $R < R_*$, then $\mathbf{u}(t,\Omega) \in \overline{B_g(\mathbf{p}_0, R)}$ for $t \in]0,T[$. In the case that $K_{\mathscr{N}} \le 0$, the solution exists global in time. Moreover, there exists $T_* = T_*(\mathbf{u}_0) \in [0,\infty[$ and $\mathbf{u}_* = \mathbf{u}_*(\mathbf{u}^0) \in \mathscr{N}$ such that $\mathbf{u}(t,\cdot) \equiv \mathbf{u}_*$ for $t \ge T_*$.*

4 BV-solutions in the case of curves

In this Section we collect some new results in the case that Ω is an open bounded interval $I \subset \mathbb{R}$. Despite that image processing application are limited (let us mention the case that $\mathscr{N} = SO(3) \times \mathbb{R}^3$ for denoising camera trajectories), we find this case as mathematically interesting and enlightening concerning the identification of the tensor field $\mathbf{Z}$.

Given any two points $\mathbf{p}_1, \mathbf{p}_2 \in \mathscr{N}$ such that there is exactly one minimizing geodesic in $\mathscr{N}$ joining $\mathbf{p}_1$ and $\mathbf{p}_2$, we will denote its arclength parametrization by $\gamma_{\mathbf{p}_1}^{\mathbf{p}_2}$. Furthermore, we denote $\boldsymbol{T}_{\mathbf{p}_1}^{\mathbf{p}_2} = \left(\gamma_{\mathbf{p}_1}^{\mathbf{p}_2}\right)'(\mathbf{p}_1)$. In the next definition of solution we completely characterize the vector field $\mathscr{H}^{m-1}$ a.e; i.e. pointwise for curves.

Definition 4.1 *Let $\mathbf{u}^0 \in BV(I, \mathscr{N})$ and $T > 0$. Let $\mathbf{u} \in H^1(0,T; L^2(I)) \cap L^\infty(0,T; BV(I))$ be such that $\mathbf{u}(t,\cdot)$ satisfies* (8) *for a. e. $t \in]0,T[$. We say that $\mathbf{u}$ is a strong solution to (10,11) in $[0,T[$ with initial datum $\mathbf{u}^0$ if there exists $\mathbf{z} \in L^\infty(]0,T[\times I) \cap L^2(0,T; BV(I))$ such that for a. e. $t \in]0,T[$ there holds*

$$\mathbf{u}_t(t,\cdot) = \pi_{\mathbf{u}(t,\cdot)} \mathbf{z}_x^a(t,\cdot) \quad \mathscr{L}^1\text{-a. e. in } I, \tag{48}$$

$$\mathbf{z}^{\pm}(t,\cdot) \in T_{\mathbf{u}^{\pm}(t,\cdot)}\mathscr{N} \quad \textit{in } I, \tag{49}$$

$$|\mathbf{z}(t,\cdot)| \leq 1 \quad \textit{in } I, \tag{50}$$

$$\mathbf{z}(t,\cdot) = \frac{\mathbf{u}_x(t,\cdot)}{|\mathbf{u}_x(t,\cdot)|} \quad |\mathbf{u}_x(t,\cdot)|\text{-}\textit{a. e. in } I \setminus J_{\mathbf{u}(t,\cdot)}, \tag{51}$$

$$\mathbf{z}^{\pm}(t,\cdot) = \pm \boldsymbol{T}_{\mathbf{u}^{\mp}(t,\cdot)}^{\mathbf{u}^{\pm}(t,\cdot)} \quad \textit{on } J_{\mathbf{u}(t,\cdot)}, \tag{52}$$

$$\mathbf{z}(t,\cdot) = 0 \quad \textit{on } \partial I \tag{53}$$

and

$$\mathbf{u}(0,\cdot) = \mathbf{u}^0. \tag{54}$$

Our first result in this case is uniqueness of solutions when $\mathscr{N}$ has nonpositive sectional curvature. This is in fact implied by the following

Lemma 4.2 *Suppose that* $\mathscr{K}_{\mathscr{N}} \leq 0$ *and let* $\mathbf{u}$ *be a solution to* (10)–(11) *in the sense of Definition 4.1. Then,*

$$\frac{1}{2}\frac{d}{dt}\int_I dist^2_{\mathscr{N}}(\mathbf{u},\mathbf{v}) + \int_I |\mathbf{u}_x| \leq \int_I |\mathbf{v}_x|, \quad \forall \mathbf{v} \in BV(I;\mathscr{N}).$$

We observe that the abstract solution to the flow constructed by Mayer's Crandall-Ligget's extension is shown to satisfy exactly this inequality (see [4, Theorem 4.04]) and that solutions satisfying it are unique. Therefore, our solution coincides with Mayer's solution in the case of nonpositive sectional curvature and we have given a characterization of the (analogous to) subdifferential of $TV^{\mathscr{N}}$.

We next show global existence of solutions for initial data such that jumps are not too far apart. We will consider $\mathbf{u}^0 \in BV(I;\mathscr{N})$ satisfying

$$\mathrm{dist}((\mathbf{u}^0)^+,(\mathbf{u}^0)^-) \leq 2\min\{\frac{1}{2}\mathrm{inj}_{\mathscr{N}}, r_{\mathscr{N}}\} \quad \text{at } J_{\mathbf{u}^0} \tag{55}$$

where $r_{\mathscr{N}}$ denotes the convexity radius of $\mathscr{N}$.

Theorem 4.1 *Given any* $\mathbf{u}^0 \in BV(I,\mathscr{N})$ *satisfying* (55) *and* $T > 0$, *there exists a strong solution to (10,11) in* $[0,T[$ *with initial datum* $\mathbf{u}^0$.

Our method to prove Theorem 4.1 consist in a two-step approximation procedure. First of all, by Theorem 1.2 we consider a properly smoothed initial datum $(\mathbf{u}^0)^{\delta} \in W^{1,\infty}(I;\mathscr{N})$ strictly converging to $\mathbf{u}^0$ as $\delta \to 0^+$. Then, by Theorems 3.3 and 3.2, we obtain existence and uniqueness of a regular solution in the sense of Definition 3.1 and satisfying the energy inequality (38). Moreover, note that in the case of 1-D domain, the term involving the Riemannian tensor in Bochner's formula (42) vanishes. Therefore, as in the case of nonpositive sectional curvature, the obtained bounds are uniform in time and therefore, we can show that the solution exists globally in time; i.e. there exists

$$\mathbf{u}^\delta \in W^{1,2}_{loc}([0,+\infty[\times\bar{I},\mathscr{N}) \text{ with } \mathbf{u}^\delta_x \in L^\infty_{loc}([0,+\infty[\times\bar{I},\mathbb{R}^N)$$

and $\mathbf{z}^\delta \in L^\infty(]0,+\infty[\times I,\mathbb{R}^N)$ with $\mathbf{z}^\delta_x \in L^2_{loc}([0,T[\times\overline{\Omega},\mathbb{R}^N)$ satisfying all the conditions in Definition 3.1 with $T=\infty$ and the following energy inequality:

$$\operatorname*{ess\,sup}_{t>0}\int_\Omega |\nabla \mathbf{u}^\delta(t,\cdot)| + \int_0^\infty\int_\Omega \mathbf{u}_t^2 \le \int_\Omega |\nabla \mathbf{u}^\delta_0|. \tag{56}$$

At this point we prove a completely local estimate which was proven to be true for the case of $\mathscr{N}=\mathbb{R}$; i.e. the 1-dimensional scalar total variation flow ([22]):

$$|\mathbf{u}^\delta_x(t,x)| \le |\mathbf{u}^\delta_x(0,x)| \qquad \text{for all } x\in I. \tag{57}$$

As in the case of $\mathbb{S}^2$, the energy inequality (56), suffices to obtain a triple of limits $\mathbf{u} \in H^1(0,\infty;L^2(I))\cap L^\infty(0,\infty;BV(I)),\mathbf{z}\in L^\infty(]0,\infty[\times I)\cap L^2(0,\infty;BV(I))$ and $\mu \in L^\infty([0,\infty[;\mathscr{M}(I))$ such that

$$\mathbf{u}_t = \mathbf{z}_x + \mu. \tag{58}$$

The identification of $\mathbf{z}$ and μ, which is the hardest task, would be possible, thanks to the powerful tool given by the completely local estimate 57.

Indeed, we use it to uniformly control the distance of the approximated solutions to their average in order to show that (51) holds. Moreover, we also use it to show continuity in time of the functions $\mathbf{u}^\pm(\cdot,x_0)$ at jump points. Finally, it also allows us to show that, close to jump points of the solution (and therefore of the initial data by (57)), approximating solutions are close to the geodesic joining $\mathbf{u}^+$ and $\mathbf{u}^-$. This is the key to obtain the identification of $\mathbf{z}$ at jump points ((52)). We refer to [24] for the details.

5 Conclusion

We have studied the manifold constrained total variation flow in three different cases.

(a) For the case of Lipschitz initial data, we have shown local existence and uniqueness of regular solutions with no further assumption on $\mathscr{N}$.

(b) For the case of BV-initial data we have shown:

  (b1) In the case of $\mathscr{N}=\mathbb{S}^{N-1}_+$, we have obtained global existence of solutions after a complete characterization of the limiting system of equations. Here we adopted an extrinsic point of view and strongly used the symmetries of $\mathscr{N}$.

(b2) In the case of curves, we have obtained global existence of solutions and we have completely characterized the vector field $\mathbf{z}$. Moreover, uniqueness also holds in case that $\mathscr{N}$ has nonpositive sectional curvature. Here, we have used the extremely useful estimate (57).

In order to extend our study in $[(b)]$ to the $m-$dimensional domain case, we cannot expect to have a tool such as (57). Indeed, such an estimate is false in general (one can consider examples of bending for the scalar total variation as in [2]). One possibility will be to try to define an intrinsic notion of Anzellotti pairing (a kind of scalar product between bounded vector fields with measure divergence and BV functions) and to use the corresponding intrinsic integration by parts formula (see [9] for the usual Anzellotti pairing). This pairing was shown to be an essential tool in the unconstrained case for the characterization of the subdifferential of the total variation as in Theorem 1.3.

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