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On the soliton decomposition of solutions for the energy critical parabolic equation

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1 Problem and main results

1.1 Problem and basic facts

In this note, we are concerned with the existence of the global bounds for the Sobolev norm of time-global solutions for a semilinear parabolic equations involving the critical Sobolev exponent.

Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a smooth domain and let $\dot{H}^1(\Omega)$ be a homogeneous Sobolev space defined as a closure of $C_0^\infty(\Omega)$ by the homogeneous Sobolev norm $\|\nabla \cdot\|_2$, where $\|\cdot\|_r$ denotes the standard L^r-norm. Let $2^* := \frac{2N}{N-2}$ be the critical Sobolev exponent of the Sobolev embedding $\dot{H}^1 \hookrightarrow L^p$. It is known that $\dot{H}^1 \hookrightarrow L^{2^*}$ is continuous but fails to be compact. We consider

$$\text{(P)} \qquad \begin{cases} \partial_t u = \Delta u + u|u|^{p-2} & \text{in } \Omega \times (0, T_m), \\ u|_{t=0} = u_0 & \text{in } \Omega \end{cases}$$

with the homogeneous boundary condition

$$u = 0 \text{ on } \partial\Omega \times (0, T_m)$$

if $\partial\Omega \neq \emptyset$, where $u_0 \in L^\infty \cap H^1$ for the sake of simplicity, T_m denotes the maximal existence time of the classical solution u of (P). A solution with $T_m = \infty$ is called as a *time-global* solution. In the main body of this note, we assume $p = 2^*$, $\Omega = \mathbb{R}^N$ and $u_0 \geq 0$.

In this note, we are concerned with the validity of the following global bounds for time global solutions u:

$$\sup_{t>0} \|\nabla u(t)\|_2 < \infty. \tag{1.1}$$

As is shown in the proof of Theorem 1.2, the analysis of the validity of a bound of the form (1.1) is a first step for the analysis of the asymptotic behavior of a time-global solution u.

Note that by the decreasing property of the energy J along the orbit of u (see (1.8) below), (1.1) is equivalent to

$$\sup_{t>0} \|u(t)\|_p < \infty. \tag{1.2}$$

The aim of this note is to introduce an argument to establish the validity of (1.2) for the case where $p = 2^*$, $\Omega = \mathbb{R}^N$ and u is a nonnegative time-global solution of (P).

Time-local existence of a solution We review basic facts concering the time local existence of solutions of (P) which is needed in proving main results. For the proof of facts stated in this paragraph, see e.g. Brezis-Cazenave [1], Ruf-Terraneo [21], and Weissler [25].

We consider the solution of (P) in the following sense:

$$u \in C^{2,1}(\mathbb{R}^N \times (0, T_m)) \cap C^1((0, T_m); L^2) \cap C([0, T_m); H^1). \tag{1.3}$$

The solution in this class is easily constructed. Indeed, since $u_0 \in L^\infty$, the existence of a classical solution of (P) is a standard fact and for $u_0 \in H^1$, a solution $u \in C^1((0, T_m); L^2) \cap C([0, T_m); H^1)$ is constructed, see e.g. in Brezis-Cazenave [1], Weissler [25] and Ruf-Terraneo [21].

Since u in the class (1.3) is a classical solution, it satisfies the blow-up alternative in L^∞-sense:

$$\text{if } T_m < \infty, \text{ then } \lim_{t \to T_m} \|u(t)\|_\infty = \infty. \tag{1.4}$$

It is also well known that this class of solution satisfies the integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t ds e^{(t-s)\Delta} u(s)|u(s)|^{p-2} \tag{1.5}$$

associated with (P).

The energy structure By multiplying $\partial_t u$ to both sides of (P) and integrating over $\mathbb{R}^N$, we (formally) obtain the energy equality

$$\|\partial_t u(t)\|_2^2 = -\frac{d}{dt}J(u(t)), \tag{1.6}$$

where J denotes the energy functional assocated with (P) defined by

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u(t)\|_p^p.$$

It is known that solutions u of (P) satisfying (1.3) actually satisfy (1.6) for any $t \in (0, T_m)$.

In the main body of this note, we assume that $p = 2^*$, $\Omega = \mathbb{R}^N$ and u is a nonnegative time-global solution of (P). In this case, the concavity argument (this name comes from the concavity of a part $-\frac{1}{p}\|u\|_p^p$ in the energy functional) of Payne-Sattinger [22] and Levine [17] for bounded domain together with the comparison argument implies that

$$\lim_{t\to\infty} J(u(t)) \geq 0 \tag{1.7}$$

and (1.6) and (1.7) imply the existence of $d \geq 0$ satisfying

$$J(u_0) \geq J(u(t)) \downarrow d \text{ as } t \to \infty, \tag{1.8}$$

see Mizoguchi [18, Lemma 2.4].

Remark 1.1

In this note, we assume that the nonnegativity of solution of (P) which is only used to assure (1.8), in other words, to exclude the existence of a solution satisfying

$$T_m = \infty \text{ and } \lim_{t\to\infty} J(u(t)) = -\infty. \tag{1.9}$$

For bounded Ω, we can exclude the existence of such solutions by using the concavity argument. In an unbounded domain case, we also rely on the comparison argument to exclude a solution as in (1.9) and we need the nonnegativity assumption of solutions for the comparision. ∎

1.2 Known results and motivation for main results

The investigation of the global bounds of the form (1.1) is initiated in Ôtani [20] in the setting of an abstract evolution equation theory governed by subdifferential operators. The systematic analysis of the asymptotics of time-global solutions is first introduced by Henry [10].

For a subcritical problem on a bounded domain, i.e., problem (P) with $p < 2^*$ and bounded Ω, Ôtani [20] obtained (1.1) for p in the subcritical range. Later, more detailed analysis was done, see e.g. Cazenave-Lions [3], Giga [9], Fila [7], Ikehata-Suzuki [11] and references therein. In all these works, it is proved that every (time-global) solution has a time-global bounds (1.1) in the subcritical case. Also, based on this global bounds, it is proved that

$$\text{every time-global solution is attracted to a set of stationary solutions,} \tag{1.10}$$

see e.g. Cazenave-Haraux [2, §9]. We also discuss in this note how to obtain this fact, see Proposition 2.1 below. As for a subcritical problem on the entire domain, see e.g. Kawanago [16], Cortázar-del Pino-Elgueta [5], Feireisl-Petzeltová [6], Chill-Jendoubi [4] and references therein.

There is not so much result on the case $p = 2^*$, a critical problem. As for the asymptotics of time-global solution, it is pointed out in Ni-Sacks-Tavantzis [19] that (P) with bounded domain admits a time-global *weak* solution which is unbounded in L^∞-sense. Since the solution treated in [19] is a weak global solution, it is not clear whether the solution blows-up in finite time or not in a classical sense. Later, it is proved in [13] that there exists an unbounded, time-global, radially symmetric and nonnegative *classical* solution u of (P) on a ball or on the entire domain which behaves like

$$u(\cdot,t) - \|u(t)\|_\infty U(\|u(t)\|_\infty^{\frac{2}{N-2}} \cdot) = o(1) \text{ in } \dot{H}^1 \tag{1.11}$$

as $t \to \infty$, where U is a unique nonnegative nontrivial stationary solution of (P) (in $\mathbb{R}^N$) with $\|U\|_\infty = 1$ (U is called a Talenti function, see [24] and e.g. [23, §I]). This results shows that the solution u behaves like a scaling of a nontrivial stationary solution of (P). Since $\dot{H}^1$-norm is invariant under the scaling appeared in (1.11) (see Propoition 2.1 below), we have

$$\|\nabla u(t)\|_2^2 = \|\nabla(\|u(t)\|_\infty U(\|u(t)\|_\infty^{\frac{2}{N-2}} \cdot))\|_2 + o(1) = \|\nabla U\|_2^2 + o(1) \tag{1.12}$$

as $t \to \infty$, thus (1.1) holds for this solution. Based on this fact, it is proved in [13] that the time-global bounds (1.1) is true for any time-global, radially symmetric and nonnegative solution u of (P) in ball or $\mathbb{R}^N$. For the validity of (1.1) for another case, see e.g. [12] and references therein.

The asymptotics (1.11) suggests that the general asymptotic behavior in the critical case is not so simple as in the subcritical case (1.10). Indeed, for (P) on a ball, it is proved in [13] that there holds $\|u(t)\|_\infty \to \infty$ as $t \to \infty$, hence a solution in (1.11) concentrates at the origin as $t \to \infty$ while the Sobolev norm is bounded (1.12). Observe that this u does not converges to any function in the strong $\dot{H}^1$-topology, since $u(t) \rightharpoonup 0$ as $t \to \infty$ in $\dot{H}^1$ (this comes from $u(x,t) \to 0$ a.e. x as $t \to \infty$ by (1.11)) while $\|\nabla u(t)\|_2^2 \not\to 0$ which is obvious from (1.12). Hence, in the critical case, some time-global solution exhibit different behavior from the absorbtion to a set of stationary solution and the validity of (1.1) for general time-global solution is an open problem so far.

We claim in this note that, in spite of these evidences which indicate the difference between the subcritical and the critical case, general nonnegative time-global solution of (P) with $p = 2^*$ and $\Omega = \mathbb{R}^N$ satisfy (1.1) (Theorem 1.1 below). Moreover, we will cralify the fact that, different from the subcritical case, time-global solutions behave like a finite number of superposition of rescaled and translated starionary solutions (Theorem 1.2 below) as is implied by the asymptotics (1.11) in a ball.

1.3 Main results

In this note, we show the validity of (1.1) for nonnegative global-in-time solution of (P) without the assumption of radial symmetry, and give an asymptotic behavior of time-global solutions.

Theorem 1.1 (Global bounds for the critical case)
Let u be a nonnegative time-global solution of (P) *with $p = 2^*$ and $\Omega = \mathbb{R}^N$. Then there holds $\sup_{t>0} \|\nabla u(t)\|_2 < \infty$.* ∎

Remark 1.2 (For the general case)
For (P) on general smooth domain Ω with $p = 2^*$, we have $\limsup_{t\to\infty} \|\nabla u(t)\|_2 < \infty$ if

$$\liminf_{t\to\infty} \|\nabla u(t)\|_2 < \infty, \tag{1.13}$$

see [14]. Therefore, for an arbitrary time-global solution of u, we have either

$$\limsup_{t\to\infty} \|\nabla u(t)\|_2 < \infty$$

or

$$\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty.$$

For a bounded Ω, we always have (1.13). For $\Omega = \mathbb{R}^N$ with $p = 2^*$, we have the alternative

$$\limsup_{t\to\infty} \|\nabla u(t)\|_2 < \infty$$

or

$$\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty \text{ and } \lim_{t\to\infty} J(u(t)) = -\infty. \tag{1.14}$$

since $\lim_{t\to\infty} J(u(t)) > -\infty$ implies (1.13) as is shown by the same argument for bounded Ω above. The existence of a sign-changing solution u satsifying (1.14) is an open problem.

∎

Remark 1.3 (An extension of a class of initial data)
We can considerably enlarge the admissible class of initial datum, see e.g. Brezis-Cazenave [1] and Ruf-Terraneo [21]. ∎

Based on the Theorem 1.1, we can clarify the following asymptotics of nonnegative time-global solutions of (P). For a Banach space X and for $A \subset X$, let $\operatorname{dist}_X(u, A) := \inf_{v\in A} \|u - v\|_X$.

Theorem 1.2 (Asymptotics for the critical case)
Let a time-global solution u of (P) *with $p = 2^*$ and $\Omega = \mathbb{R}^N$ satisfies*

$$\sup_{t>0} \|\nabla u(t)\|_2 < \infty. \tag{1.15}$$

Let $E_\infty(u_0)$ be a set defined by

$$\begin{aligned} &E_\infty(u_0) \\ :=\ &\left\{ \sum_{j=1}^{n} (\lambda^j)^{\frac{N-2}{2}} \varphi^j(\lambda^j(\cdot - y^j));\ \varphi^j \text{ is a stationary solution of (P)}, \right. \\ &\left. (\lambda^j)_{j=1}^n \subset \mathbb{R}_+,\ (y^j)_{j=1}^n \subset \mathbb{R}^N,\ n \in \mathbb{N}\cup\{0\} \text{ with } \sum_{j=1}^{n} J(\psi^j) \le J(u_0) \right\}. \end{aligned}$$

Then there holds

$$\mathrm{dist}_{L^{2^*}}(u(t), E_\infty(u_0)) \to 0 \tag{1.16}$$

as $t \to \infty$ *(Note that all* ψ^j *may be a trivial solution).*

Remark 1.4 (For nonnegative solutions)

If u is a nonnegative solution of (P), the conclusion of Theorem 1.2 holds since (1.15) follows from Theorem 1.1. In this case, ψ^j in the definition of $E_\infty(u_0)$ can be taken as nonnegative functions and the convergence in (1.16) can be improved to that in $\dot{H}^1$, see [14]. It is not clear whether we can improve the convergence in (1.16) to $\dot{H}^1$ for sign-changing case. ∎

Remark 1.5 (Meaning of the asymptotics in the critical case)

We here discuss the intuitive meaning of the result in Theorem 1.2. For the simplicity, let us consider a nonnegative solution of (P). From Theorem 1.2, we see that for any time sequence (t_n) with $t_n \to \infty$, there exists a subsequence (denoted by the same symbol), $n \in \mathbb{N}$, $(\lambda_n^j)_{j=1}^n \subset \mathbb{R}_+$, $(y_n^j)_{j=1}^n$ such that

$$u(\cdot, t_n) - \sum_{j=1}^{n} (\lambda_n^j)^{\frac{N-2}{2}} \varphi^j(\lambda_n^j(\cdot - y_n^j)) = o(1) \text{ in } \dot{H}^1 \tag{1.17}$$

as $n \to \infty$, where U is a unique nonnegative stationary solution of (P) (in $\mathbb{R}^N$), see Propostition 3.1.

Note that (P) is invariant under the spatial translations, i.e., if $u(x,t)$ satisfies (P), then $u(x-y,t)$ also satisfies (P) with initial $u_0(x-y)$ for any $y \in \mathbb{R}^N$. Also, (P) has a scale invariance under $u(x,t) \mapsto \mu^{\frac{2}{p-2}} u(\mu x, \mu^2 t)$, where $\mu \in \mathbb{R}_+$, see Proposition 2.1 below. The peculiarity of the critical case $p = 2^*$ is that, only in this case the energy function J is also invariant under the scaling. In other words, only in the critical case, the evolution equation structure and the variatioinal strucures are both invariant under the scaling. The relation (1.17) says that time-global solutions behave like as a superposition of rescaled stationary solutions by reflecting this invariance. This behavior is out of the scope of "the absorbtion to a set of equilibrium", a postulate (1.10) in the subcritical case. ∎

2 Preliminaries

We introduce preliminary facts which will be needed in the proof of Theorem 1.1 and Theorem 1.2.

2.1 Scaling invariance and the existence of a balanced time sequence

In this subsection, we check the invariance property of (P) and J under the scaling with x, t and u and introduce an existence of time sequence (t_n) satsifying

$$\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_p^p + o(1) \tag{2.1}$$

as $n \to \infty$.

Let u be a solution of (P) and let $\mu > 0$. For any $x_0 \in \mathbb{R}^N$ and $t_0 > 0$, let

$$y := \mu(x - x_0), \quad s := \mu^2(t - t_0), \quad \mu^{\frac{2}{p-2}} u_{\mu,x_0}(y,s) = u(x,t). \tag{2.2}$$

Then it is easy to see that

Proposition 2.1 (Scale invariance)
Let $\delta > 0$. Then u_{μ,x_0} satisfies

$$\partial_s u_{\mu,x_0} = \Delta_y u_{\mu,x_0} + u_{\mu,x_0}|u_{\mu,x_0}|^{p-2} \text{ in } \mathbb{R}^N \times [0,\delta]$$

if and only if u satisfies

$$\partial_t u = \Delta_x u + u|u|^{p-2} \text{ in } \mathbb{R}^N \times \left[t_0, t_0 + \frac{\delta}{\mu^2}\right].$$

Moreover, we have

$$\begin{aligned}
&\mu^{\frac{N-2}{p-2}(2^*-p)} \int_0^\delta \|\partial_s u_{\mu,x_0}\|_2^2 ds = \int_{t_0}^{t_0+\frac{\delta}{\mu^2}} \|\partial_t u\|_2^2 dt,\\
&\mu^{\frac{N-2}{p-2}(2^*-p)} \|\nabla u_{\mu,x_0}(s)\|_2 = \|\nabla u(t)\|_2,\\
&\mu^{\frac{N-2}{p-2}\left(\frac{2}{N-2}(r-p)+2^*-p\right)} \|u_{\mu,x_0}(s)\|_r = \|u(t)\|_r.
\end{aligned}$$

Remark 2.1 (The peculiarity of the critical problem)
The proposition above says that the problem (P) is always invariant under the scaling and the translation (2.2). The important feature of the critical

case is that only in this case, the energy structure, i.e., $L^2(I;L^2)$, $\dot{H}^1$ and L^p-norms, is also invariant, i.e., there hold

$$\begin{aligned}
&\int_0^\delta \|\partial_s u_{\mu,x_0}\|_2^2 ds = \int_{t_0}^{t_0+\frac{\delta}{\mu^2}} \|\partial_t u\|_2^2 dt,\\
&\|\nabla u_{\mu,x_0}(s)\|_2 = \|\nabla u(t)\|_2,\\
&\|u_{\mu,x_0}(s)\|_{2^*} = \|u(t)\|_{2^*},\\
&(\|u_{\mu,x_0}(s)\|_2 = \mu\|u(t)\|_2).
\end{aligned}$$

This is one of the origin of the noncompactness for the evolution and the variational structure. ∎

Proposition 2.2 (Existence of a balanced time sequence [12])
Let u be a nonnegative time-global solution of (P) with $p = 2^*$ and $\Omega = \mathbb{R}^N$. Then there exists $t_n \to \infty$ such that $\|\nabla u(t_n)\|_2^2 - \|u(t_n)\|_p^p = o(1)$ as $n \to \infty$.

Proof of Proposition 2.2.
Let $\tau_n \to \infty$ be a sequence such that

$$\lim_{n\to\infty} \|u(\tau_n)\|_2 = \limsup_{t\to\infty} \|u(t)\|_2 (\leq \infty).$$

We define $\lambda_n > 0$ by

$$\lambda_n^2 := \frac{1}{\|u(\tau_n)\|_2^2} \tag{2.3}$$

and define y, s, u_n by $y := \lambda_n x$, $s := \lambda_n^2(t - \tau_n)$ and $u_n(y,s) := \lambda_n^{\frac{N-2}{2}} u(x,t)$. Observe that

$$\|u_n(0)\|_2^2 = \lambda_n^2 \|u(\tau_n)\|_2^2 = 1 \tag{2.4}$$

by Proposition 2.1 and (2.3). Then by Proposition 2.1, (1.6) and (1.8), there holds

$$\begin{aligned}
\int_0^\delta ds \|\partial_s u_n\|_2^2 &= -J(u_n(\delta)) + J(u_n(0))\\
&= -J\left(u\left(\tau_n + \frac{\delta}{\lambda_n^2}\right)\right) + J(u(\tau_n))\\
&= -d + d + o(1) = o(1)
\end{aligned} \tag{2.5}$$

as $n \to \infty$ for any $\delta > 0$, thus

$$\|u_n(\sigma) - u_n(0)\|_2 \leq \int_0^\sigma \|\partial_s u_n(s)\|_2 ds \leq \sqrt{\delta} \left(\int_0^\sigma \|\partial_s u_n(s)\|_2^2 ds \right)^{\frac{1}{2}} = o(1)$$

as $n \to \infty$, uniformly in $\sigma \in [0, \delta]$. This relation together with (2.4) yields

$$\|u_n(\sigma)\|_2^2 \leq 2\|u_n(0)\|_2^2 = 2, \quad \sigma \in [0, \delta]$$

for large n. Again by (2.5), we can find $\eta \in [0, \delta]$ such that

$$\|\partial_s u_n(\eta)\|_2 = o(1), \tag{2.6}$$

as $n \to \infty$, passing subsequences if necessary. Since u_n satisfies (P) due to Proposition 2.1, by multplying u_n to (P) and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned} \left| -\|\nabla u_n(\eta)\|_2^2 + \|u_n(\eta)\|_p^p \right| &\leq \left| \int \partial_s u_n(\eta) u_n(\eta) \right| \\ &\leq \|\partial_s u_n(\eta)\|_2 \|u_n(\eta)\|_2 = o(1) \end{aligned} \tag{2.7}$$

as $\to \infty$, where we used (2.6) in the last line. Let $t_n := \tau_n + \frac{\eta}{\lambda_n^2}$. Then from (2.7) and Proposition 2.1, we obtain

$$\|\nabla u(t_n)\|_2^2 = \|\nabla u_n(\eta)\|_2^2 = \|u_n(\eta)\|_p^p + o(1) = \|u(t_n)\|_p^p + o(1),$$

which implies the conclusion. ∎

2.2 A profile decomposition of Gérard-Jaffard

In order to analyze the asymptotic behavior of time-global solutions in the critical case, we rely on the following compactness device, see Gérard [8, THÉORÈME 1.1, REMARQUES 1.2.(b)], see also Jaffard [15, Theorem 1].

Proposition 2.3 (Profile decomposition)
Let $(u_n) \subset \dot{H}^1(\mathbb{R}^N)$ be a bounded sequence. Then there exist $(\lambda_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}^N$ $(j = 1, \cdots)$, $(\psi^j)_{j\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^N)$ such that, for

$$\psi_n^j(x) := \left(\lambda_n^j\right)^{\frac{N-2}{2}} \psi^j(\lambda_n^j(x - x_n^j)),$$

there hold the following.

(a) *There holds*

$$\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} + \frac{|x_n^i - x_n^j|}{\lambda_n^i} \to \infty \text{ as } n \to \infty \text{ for } i \neq j.$$

(b) *For any* $l \in \mathbb{N}$, *there holds*

$$\lim_{l\to\infty} \lim_{n\to\infty} \|r_n^l\|_{2^*} = 0,$$

where $r_n^l := u_n - \sum_{j=1}^l \psi_n^j$.
(c) *There hold*

$$\begin{aligned} \|\nabla u_n\|_2^2 &= \sum_{j=1}^l \|\nabla\psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1), \\ \|u_n\|_{2^*}^{2^*} &= \sum_{j=1}^l \|\psi^j\|_{2^*}^{2^*} + \|r_n^l\|_{2^*}^{2^*} + o(1) \end{aligned}$$

as $n \to \infty$.

Remark 2.2 (The meaning of the profile decompisition)
As is mentioned in Proposition 2.1, norms of $\dot{H}^1$ and L^{2^*} have a scale and translation invariance in the sence that $\|\nabla u_{\lambda,y}\|_2 = \|\nabla u\|_2$ and $\|u_{\lambda,y}\|_{2^*} = \|u\|_{2^*}$, where

$$u_{\lambda,y}(x) = \lambda^{\frac{N-2}{2}} u(\lambda(x-y)), \quad \lambda \in \mathbb{R}_+,\ y \in \mathbb{R}^N. \tag{2.8}$$

By using this invariance, it is easy to construct a bounded sequence $(u_n) \subset \dot{H}^1$ which is not strongly convergent in L^{2^*}. Indeed, let

$$u_n(x) := \lambda_n^{\frac{N-2}{2}} \varphi(\lambda_n(x - x_n)), \quad \lambda_n \to \infty, \quad (x_n) \subset \mathbb{R}^N,$$

where $\varphi \in C_0^\infty$. Then it is easy to see that is bounded in $\dot{H}^1$ since $\|\nabla u_n\|_2 = \|\nabla\varphi\|_2$ by the scale invariance mentioned above and, $u_n(x) \to 0$ a.e. $x \in \mathbb{R}^N$ as $n \to \infty$. These together with the Sobolev embedding imply $u_n \rightharpoonup 0$ in L^{2^*} but (u_n) cannot be strongly convergent to 0 in L^{2^*} since $\|u_n\|_{2^*} = \|\varphi\|_{2^*}$ again by the scale invariance.

The proposition above says that the converse is also true, i.e., the lack of the compactness of $\dot{H}^1 \hookrightarrow L^{2^*}$ only comes from the invariance above. Namely, for bounded sequence (u_n) in $\dot{H}^1$, if one substract finitely many "profiles" which are the rescaling and a translation of $\varphi^j(\cdot)$, then the remainder term r_n^l tends to 0 strongly in L^{2^*} as $n \to \infty$. Moreover, by (a), the rescalings and translations are "mutually orthogonal" in $\dot{H}^1$. Namely, if one consider, for fixed $l \in \mathbb{N}$,

$$\begin{aligned} v_n^{j_0}(y) &:= \left(\frac{1}{\lambda_n^{j_0}}\right)^{\frac{N-2}{2}} u_n \left(x_n^{j_0} + \frac{y}{\lambda_n^{j_0}}\right) \\ &= \psi^{j_0}(y) + \sum_{i \neq j_0, 1 \leq i \leq l} \left(\frac{\lambda_n^i}{\lambda_n^{j_0}}\right)^{\frac{N-2}{2}} \psi^i \left(\frac{\lambda_n^i}{\lambda_n^{j_0}} \left[y + \frac{x_n^{j_0} - x_n^i}{\lambda_n^i}\right]\right) \\ &\quad + \left(\frac{1}{\lambda_n^{j_0}}\right)^{\frac{N-2}{2}} r_n^l \left(x_n^{j_0} + \frac{y}{\lambda_n^{j_0}}\right) \end{aligned}$$

which is a scale back of u_n focusing on the j_0-th "bubble", then $v_n^{j_0} \rightharpoonup \psi^{j_0}$ in $\dot{H}^1$ by virtue of (a), i.e., bubbles other than the j_0-th one "disappears" from the asymptotics of $v_n^{j_0}$. ∎

3 Proof of main results

In this section, we always assume that u is a time-global solution of (P) with $p = 2^*$ and $\Omega = \mathbb{R}^N$ satisfying (1.8) with finite d (if u is a nonnegative solution, then this assumption is satisfied, see (1.7)).

Let (t_n) be any time sequence with

$$\text{(A)} \quad t_n \to \infty \text{ as } n \to \infty \text{ and } \sup_{n \in \mathbb{N}} \|u(t_n)\|_{2^*} < \infty.$$

By (A) and (1.8), we also have

$$\sup_{n \in \mathbb{N}} \|\nabla u(t_n)\|_2 < \infty,$$

hence $u_n := u(t_n)$ satisfies the assumption of Proposition 2.3. The key claim to have main results is the following:

Proposition 3.1 (Profiles are stationary solutions)
ψ^j appeared in Proposition 2.3 for $(u(t_n))$ is a stationary solution of (P).

The proof of Proposition 3.1 is rather technical, see [14].

We assume Proposition 3.1 is correct and prove Theorem 1.1 and Theorem 1.2.

3.1 Proof of Theorem 1.1

We start with the following:

Proposition 3.2 (Liminf is finite in the critical case)
There holds

$$\liminf_{t\to\infty} \|u(t)\|_{2^*}^{2^*} \le \frac{d}{\frac{1}{2}-\frac{1}{p}},$$

where $d = \lim_{t\to\infty} J(u(t))(> -\infty)$.

Proof of Proposition 3.2.
By Proposition 2.2, we have the existence of (t_n) satisfying $t_n \to \infty$ and

$$\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_{2^*}^{2^*} + o(1)$$

as $n \to \infty$. Combining this with (1.8), the decreasing property of the energy with finite limit d, we see that

$$d = J(u(t_n)) + o(1) = \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{2^*}\|u(t_n)\|_{2^*}^{2^*} = \left(\frac{1}{2}-\frac{1}{2^*}\right)\|u(t_n)\|_{2^*}^{2^*} + o(1)$$

as $n \to \infty$, hence the conclusion follows. ∎

Next we prove:

Proposition 3.3 (Non-oscilation theorem for $\|u(t)\|_p$ in the critical case)
Let (t_n) *be a time sequence satisfying the assumption* (A). *Then there holds*

$$\|u(t_n)\|_p^p \le \frac{d}{\frac{1}{2}-\frac{1}{p}} + o(1)$$

as $n \to \infty$*, where* $d = \lim_{t\to\infty} J(u(t))(> -\infty)$.

Proof of Proposition 3.3.
By the assumption and (1.8), we see $\sup_{n\in\mathbb{N}} \|\nabla u(t_n)\|_2 < \infty$. This together with Proposition 2.3 yields the existence of $(\lambda_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_n^j)_{j\in\mathbb{N}} \subset$

$\mathbb{R}^N$ $(j = 1, \cdots)$ and $(\psi^j)_{j\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^N)$ such that the conclusion of Proposition 2.3 holds. Moreover, for any $l \in \mathbb{N}$, we have

$$\|u(t_n)\|_{2^*}^{2^*} = \sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*} + \|r_n^l\|_{2^*}^{2^*} + o(1),$$

$$\|\nabla u(t_n)\|_2^2 = \sum_{j=1}^{l} \|\nabla\psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1)$$

as $n \to \infty$ by (b) and for any $\varepsilon > 0$, taking n large, we see that

$$\|r_n^l\|_{2^*}^{2^*} < \varepsilon.$$

by (c). Proposition 3.1 says that ψ^j is a stationary solution of (P) for each $j \in \mathbb{N}$. Hence we see that

$$-\Delta\psi^j = \psi^j|\psi^j|^{2^*-2} \text{ in } \mathbb{R}^N.$$

Multiplying ψ^j to both sides and integrating over $\mathbb{R}^N$, we obtain

$$\|\nabla\psi^j\|_2^2 = \|\psi^j\|_{2^*}^{2^*}. \tag{3.1}$$

Then we have

$$\begin{aligned} d + o(1) &= J(u(t_n)) = \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{2^*}\|u(t_n)\|_{2^*}^{2^*} \\ &= \frac{1}{2}\left(\sum_{j=1}^{l} \|\nabla\psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1)\right) - \frac{1}{p}\left(\sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*} + \|r_n^l\|_{2^*}^{2^*} + o(1)\right) \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*} + \frac{1}{2}\|\nabla r_n^l\|_2^2 - \frac{1}{p}\|r_n^l\|_{2^*}^{2^*} + o(1) \end{aligned}$$

as $n \to \infty$, hence

$$\begin{aligned} d + o(1) + \frac{1}{2}\varepsilon &\geq d + o(1) + \frac{1}{2}\|r_n^l\|_{2^*}^{2^*} \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*}\right) + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|u(t_n)\|_{2^*}^{2^*} + o(1), \end{aligned}$$

thus

$$\frac{d + o(1) + \frac{1}{2}\varepsilon}{\frac{1}{2} - \frac{1}{2^*}} \geq \|u(t_n)\|_{2^*}^{2^*}$$

as $n \to \infty$, thus the conclusion. ∎

End of the proof of Theorem 1.1 Now assume that $\limsup_{t\to\infty} \|u(t)\|_{2^*}^{2^*} = \infty$. Then this assumption and Proposition 3.2 yield the existence of (t_n) satisfying $t_n \to \infty$ and

$$\|u(t_n)\|_{2^*}^{2^*} = 2\frac{d}{\frac{1}{2} - \frac{1}{p}} \text{ for any } n \tag{3.2}$$

as $n \to \infty$. Then since (t_n) satisfies the assumption (A), Proposition 3.3 implies

$$\|u(t_n)\|_{2^*}^{2^*} \leq \frac{d}{\frac{1}{2} - \frac{1}{p}},$$

which contradicts (3.2). This completes the proof.

3.2 Proof of Theorem 1.2

Let us assume, on the contrary, the conclusion does not hold. Then there exists a time sequecne (t_n) and $\varepsilon > 0$ satsifying $t_n \to \infty$ and

$$\mathrm{dist}_{L^{2^*}}(u(t_n), E_\infty(u_0)) \geq \varepsilon. \tag{3.3}$$

By Theorem 1.1, we know $\sup_n \|u(t_n)\|_{2^*} < \infty$. Hence Proposition 2.3 and Proposition 3.1 yields the existence of $(\lambda_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}^N$ $(j = 1, \cdots)$, a family of stationary solution $(\psi^j)_{j\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^N)$ of (P) whixh satisfy (a) – (c) of Proposition 2.3. Take any $l \in \mathbb{N}$. Then for large n, we see that

$$u(t_n) = \sum_{j=1}^{l} (\lambda_n^j)^{\frac{N-2}{2}} \psi^j(\lambda_n^j(\cdot - x_n^j)) + r_n^l \tag{3.4}$$

and

$$\|r_n^l\|_{2^*} < \frac{\varepsilon}{2}. \tag{3.5}$$

Note that

$$w := \sum_{j=1}^{l} (\lambda_n^j)^{\frac{N-2}{2}} \psi^j(\lambda_n^j(\cdot - x_n^j)) \in E_\infty(u_0).$$

Let (t_n) be a time sequence satisfying $t_n \to \infty$ as $n \to \infty$. Then by (1.8) and Proposition 2.3 (b) and (c), passing to subsequence if necessary, we have

$$\begin{aligned} J(u_0) &\geq \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{p}\|u(t_n)\|_{2^*}^{2^*} \\ &= \frac{1}{2}\left(\sum_{j=1}^{l} \|\nabla \psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1)\right) - \frac{1}{p}\left(\sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*} + \|r_n^l\|_{2^*}^{2^*} + o(1)\right) \\ &\geq \frac{1}{2}\sum_{j=1}^{l} \|\nabla \psi^j\|_2^2 - \frac{1}{p}\sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*} + o(1) = \sum_{j=1}^{l} J(\psi^j) + o(1) \end{aligned} \tag{3.6}$$

as $n \to \infty$ for any $l \in \mathbb{N}$. This together with (3.4) and (3.5) imply

$$\mathrm{dist}_{L^{2^*}}(u, E_\infty(u_0)) \leq \|u(t_n) - w\|_{2^*} = \|r_n^l\|_{2^*} < \frac{\varepsilon}{2},$$

which contradicts to (3.3). This completes the proof of Theorem 1.2.

Remark 3.1

By (3.6) and the fact that $J(\psi^j) \geq S^{\frac{N}{2}}$ for a stationary solution ψ^j of (P), where $S := \inf_{u \in \dot{H}^1 \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$ is the best Sobolev constant, we see that the number of j for which $\psi^j \neq 0$ is at most $\frac{d}{S^{\frac{N}{2}}}$. ∎

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