数理解析研究所講究録 第2122巻 2019年 45-55

Global bifurcation structure of a limiting system to the SKT competition model with cross-diffusion *

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1 Introduction

This is a joint work with Yuan Lou (The Ohio State University), Wei-Ming Ni (The Chinese University of Hong Kong and University of Minnesota), Tatsuki Mori (Osaka University), and Shota Yamakawa (Ryukoku University).

We have been interested in the cross-diffusion system

$$
\text{(P)}\begin{cases}
u_t = \Delta[(d_1+\alpha_{11}u+\alpha_{12}v)u] + u(a_1-b_1u-c_1v) & \text{in } \Omega\times(0,\infty), \quad (1.1)\\
v_t = \Delta[(d_2+\alpha_{21}u+\alpha_{22}v)v] + v(a_2-b_2u-c_2v) & \text{in } \Omega\times(0,\infty), \quad (1.2)\\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,\infty), \quad (1.3)\\
u(x,0)=u_0(x)\ge 0,\ \ v(x,0)=v_0(x)\ge 0 & \text{in } \Omega, \quad (1.4)
\end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$.

This mathematical model was proposed by Shigesada, Kawasaki and Teramoto [8] in 1979 to investigate segregation phenomena of two competing species with each other in the same habitat area. Here, $u = u(x,t)$ and $v = v(x,t)$ represent the densities of two competing species, d_1 and d_2 are their diffusion coefficients, a_1 and a_2 denote the intrinsic growth rates of these two species, b_1 and c_2 account

*S. Yotsutani was supported by Grant-in-Aid. for Scientific Research (C) 15K04972. This work was supported by the Joint Research Center for Science and Technology of Ryukoku University in 2018.

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for intra-specific competitions while b_2 and c_1 account for inter-specific competitions. The constants α_{11} and α_{22} represent intra-specific population pressures, also known as self-diffusion rates, and α_{12} and α_{21} are the coefficients of inter-specific population pressures, also known as cross-diffusion rates.

The effect of cross-diffusion affects the population pressure between two different kinds. It is an interesting problem to see whether this effect may give rise to a spatial segregation or not, and clarify its mechanism.

We should remark that it is well known that the important quantities involving the constants a_i, b_i, c_i $(i = 1, 2)$ are only

$$A := \frac{a_1}{a_2}, \quad B := \frac{b_1}{b_2}, \quad C := \frac{c_1}{c_2}. \tag{1.5}$$

It seems natural to consider the following two cases separately: the "strong competition" case $B < C$ and the "weak competition" case $C < B$. The behavior of solution in case $B < C$ is very different from $C > B$.

We refer to [7] and [8] for further details of this model.

A lot of research works are done by the singular perturbation method, which started from a theoretical research by Mimura [5]. Kan-on [1] obtained some criteria on the stability of those non-constant solutions of (P). However, it is not easy to clarify the global structure of stationary solutions and stability of stationary solutions.

Lou and Ni [2], [3] started to investigate N-dimensional case and general diffusion coefficients. To investigate the cross-diffusion effects, let us put $\alpha_{11} = \alpha_{21} = \alpha_{22} = 0$ and $r := \alpha_{12}/d_1$. We have

$$(\mathrm{TP}_\mathrm{r}^\mathrm{N}) \begin{cases} u_t = d_1\Delta[(1+rv)u] + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0,\infty), & (1.6) \\ v_t = d_2\Delta v + v(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0,\infty), & (1.7) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), & (1.8) \\ u(x,0) = u_0(x) \geq 0, \ v(x,0) = v_0(x) \geq 0 & \text{in } \Omega, & (1.9) \end{cases}$$

where $u = u(x,t)$ and $v = v(x,t)$. Then, the stationary problem of $(\mathrm{TP}_\mathrm{r}^\mathrm{N})$ is

$$(\mathrm{S}_\mathrm{r}^\mathrm{N}) \begin{cases} d_1\Delta[(1+rv)u] + u(a_1 - b_1 u - c_1 v) = 0 & \text{in } \Omega, & (1.10) \\ d_2\Delta v + v(a_2 - b_2 u - c_2 v) = 0 & \text{in } \Omega, & (1.11) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, & (1.12) \\ u \geq 0, \ v \geq 0 & \text{in } \Omega, & (1.13) \end{cases}$$

where $u = u(x)$ and $v = v(x)$.

They obtained limiting systems as $r \to \infty$ for $(\mathrm{TP}_{\mathrm{r}}^{\mathrm{N}})$ and $(\mathrm{S}_{\mathrm{r}}^{\mathrm{N}})$. One of limiting systems as $r \to \infty$ are as follows. The time-dependent limiting system is

$$
(\mathrm{TP}_{\infty}^{\mathrm{N}}) \begin{cases} \dfrac{\partial}{\partial t}\displaystyle\int_\Omega \frac{\tau}{v} dx = \int_\Omega \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) dx & \text{in } (0,\infty), \quad (1.14) \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau & \text{in } \Omega \times (0,\infty), \quad (1.15) \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \quad (1.16) \\ v(0,t) = v_0(x) > 0 & \text{in } \Omega, \quad (1.17) \end{cases}
$$

where $v = v(x,t)$ and $\tau = \tau(t)$ are unknown positive functions, and $\tau(t)/v(x,t)$ corresponds to $u(x,t)$. The stationary limiting system is

$$
(\mathrm{S}_{\infty}^{\mathrm{N}}) \begin{cases} \displaystyle\int_\Omega \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) dx = 0, & (1.18) \\ d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau = 0 \quad \text{in } \Omega, & (1.19) \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, & (1.20) \\ v(x) > 0, \quad \text{in } \Omega, & (1.21) \end{cases}
$$

where $v = v(x)$ is an unknown positive function, τ is an unknown positive constant.

For one-dimension $\Omega := (0,1)$, the limiting system corresponding $(\mathrm{TP}_{\infty}^{\mathrm{N}})$ and $(\mathrm{SP}_{\infty}^{\mathrm{N}})$ are

$$
(\mathrm{TP}_{\infty}^{1}) \begin{cases} \dfrac{\partial}{\partial t}\left(\displaystyle\int_0^1 \frac{\tau}{v} dx\right) = \int_0^1 \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) dx & \text{in} (0,1) \times (0,\infty), \quad (1.22) \\ \dfrac{\partial v}{\partial t} = d_2 v_{xx} + v(a_2 - c_2 v) - b_2 \tau & \text{in } (0,1), \quad (1.23) \\ v_x(0,t) = 0, \quad v_x(1,t) = 0, & \text{in } (0,\infty), \quad (1.24) \\ v(x,0) = v_0(x) > 0, & \text{in } (0,1), \quad (1.25) \end{cases}
$$

and

$$
(\mathrm{S}_{\infty,\mathrm{general}}^{1}) \begin{cases} \displaystyle\int_0^1 \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) dx = 0, & (1.26) \\ d_2 v_{xx} + v(a_2 - c_2 v) - b_2 \tau = 0 \quad \text{in } (0,1), & (1.27) \\ v_x(0) = 0, \quad v_x(1) = 0, & (1.28) \\ v(x) > 0 \quad \text{in } (0,1). & (1.29) \end{cases}
$$

Lou, Ni and Yotsutani [4] obtained existence and non-existence of non-constant steady state solutions, the asymptotic shape of solutions, and almost clarified the structure of solutions of $(\mathrm{S}_{\infty,\mathrm{general}}^{1})$.

In what follows, we concentrate on the monotone increasing case $v_x(x) > 0$ to understand the essence of structure of $(\mathrm{S}^1_{\infty,\mathrm{general}})$.

Now, we introduce a (S^1_∞) as follows:

$$(\mathrm{S}^1_\infty)\begin{cases}\displaystyle\int_0^1 \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right)dx = 0, & (1.30)\\ d_2 v_{xx} + v(a_2 - c_2 v) - b_2\tau = 0 \quad \text{in } (0,1), & (1.31)\\ v_x(0) = 0, \quad v_x(1) = 0, & (1.32)\\ v(x) > 0, \quad v_x(x) > 0 \quad \text{in } (0,1). & (1.33)\end{cases}$$

2 Results

We first explain results in [4] for (S^1_∞). As for the existence and non-existence, the following theorems are obtained:

Theorem A (Existence, weak competition)**.** *Suppose that $C \le B$.*

(i) If $B \le A$ then there exists a solution (v, τ) of (S^1_∞).

(ii) If $(B + 3C)/4 < A < B$, then there exists a solution of (S^1_∞). for $d_2 \in (0, \frac{2A-(B+C)}{B-C} \cdot \frac{a_2}{\pi^2})$.

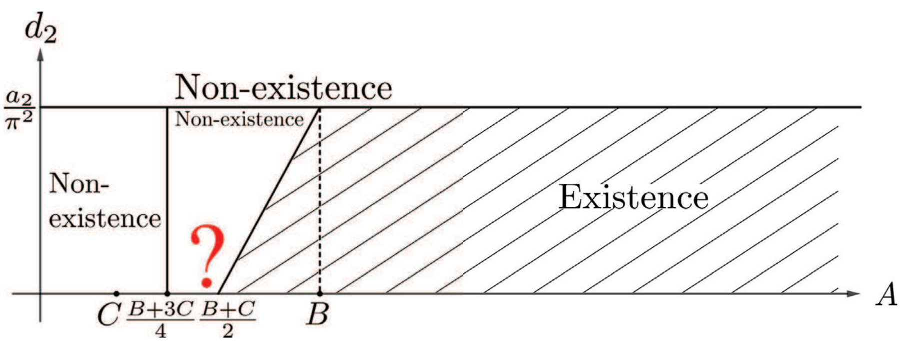

Figure 1: Existence and non-existence of solutions of (S^1_∞) for $C \le B$.

Theorem B (Non-Existence, weak competition)**.** *Suppose that $C \le B$.*

(i) If $d_2 > a_2/\pi^2$, then there exists no solution of (S^1_∞).

(ii) If $(B + 3C)/4 < A < B$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of (S^1_∞) for $d_2 \in (d_2^*, a_2/\pi^2)$.

(iii) If $A \le (B + 3C)/4$, there exists no solution of (S^1_∞).

Figure 1 shows the existence and non-existence region of solutions of (S^1_∞) in the case $C \leq B$ assured by theorems A and B. Here, horizontal axis is A, vertical axis is d_2. For the case d_2 sufficiently close to 0 and $(B+3C)/4 < A < (B+C)/2$, existence and non-existence of solutions of (S^1_∞) are not clear.

Figure 2 shows the existence and non-existence region of solutions of (S^1_∞) in the case $B < C$ assured by theorems C and D. For the case $0 < d_2 < ((B+C-2A)/(C-B)) \cdot (a_2/\pi^2)$ and $B < A < (B+C)/2$, existence and non-existence of solutions of (S^1_∞) also are not clear.

Theorem C (Existence, strong competition)**.** *Suppose that $B < C$. If*

$$\max\left\{0, \frac{B+C-2A}{C-B} \cdot \frac{a_2}{\pi^2}\right\} < d_2 < \frac{a_2}{\pi^2}, \tag{2.1}$$

then there exists a solution (v, τ) of (S^1_∞) .

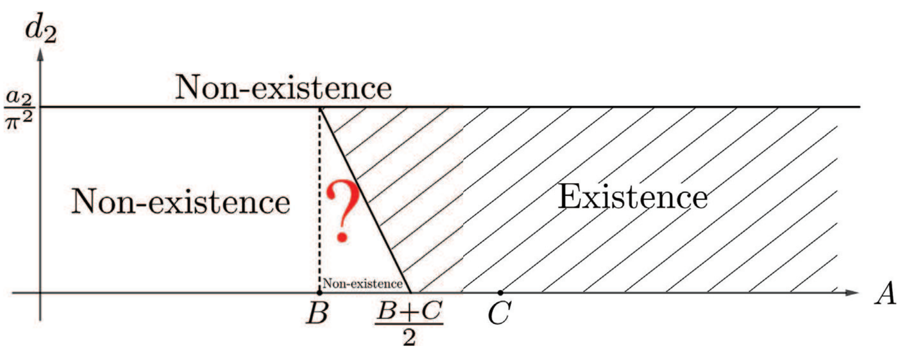

Figure 2: Existence and non-existence of solutions of (S^1_∞) for $B < C$.

Theorem D (Non-Existence, strong competition)**.** *Suppose that $B < C$.*

(i) If $d_2 > a_2/\pi^2$, then there exists no solution of (S^1_∞) .

(ii) If $B \leq A < (B+C)/2$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of (S^1_∞) for $d_2 \in (0, d_2^*]$.

(iii) If $A < B$, there exists no solution of (S^1_∞) .

In [9], Lou, Ni and Yotsutani conjectured that the situation of existence, non-existence and the uniqueness drastically changes at $C = (7/3)B$. For the case $B < C \leq (7/3)B$, the uniqueness seems to hold as shown in Figures 3 and 4. Recently, we have found a mathematical proof of this case.

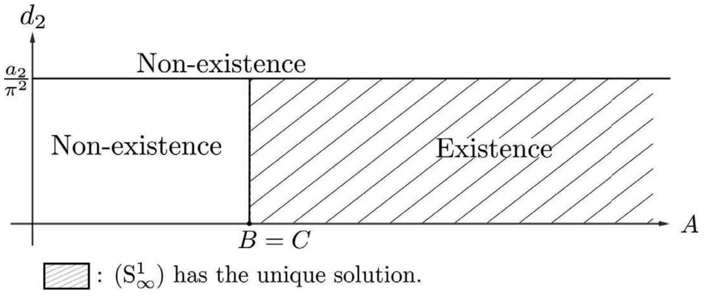

Figure 3: $C = B$.

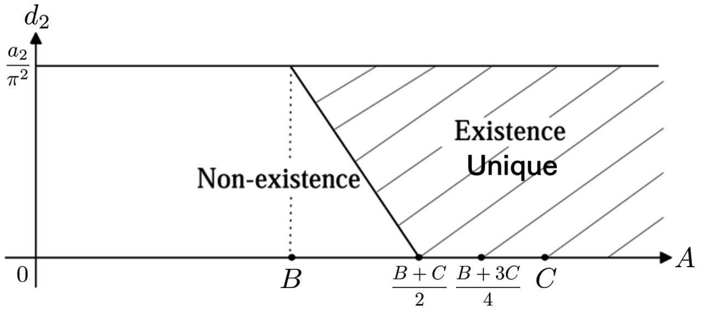

Figure 4: Existence and non-existence of solutions of (S^1_∞) for $B < C \leq (7/3)B$.

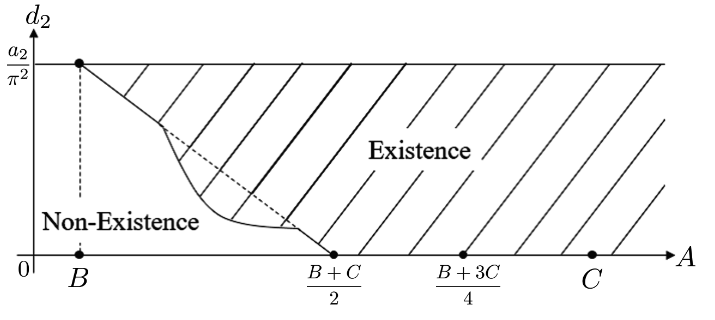

Figure 5: Existence and non-existence of solutions of (S^1_∞) for $C > (7/3)B$.

On the other hand, for the case $C > (7/3)B$, the existence region becomes wider as shown in Figure 5. In [6], Mori, Suzuki and Yotsutani have obtained precise numerical results with the stability and instability for this case

As explained above, existence, non-existence and multiplicity of solutions for the case $B \leq C$ are precisely understood.

However, it is not clarified the case $C < B$. Therefore, we investigate this case. Figure 6 show existence, non-existence and multiplicity of non-constant solutions for (S^1_∞) obtained by numerical computation.

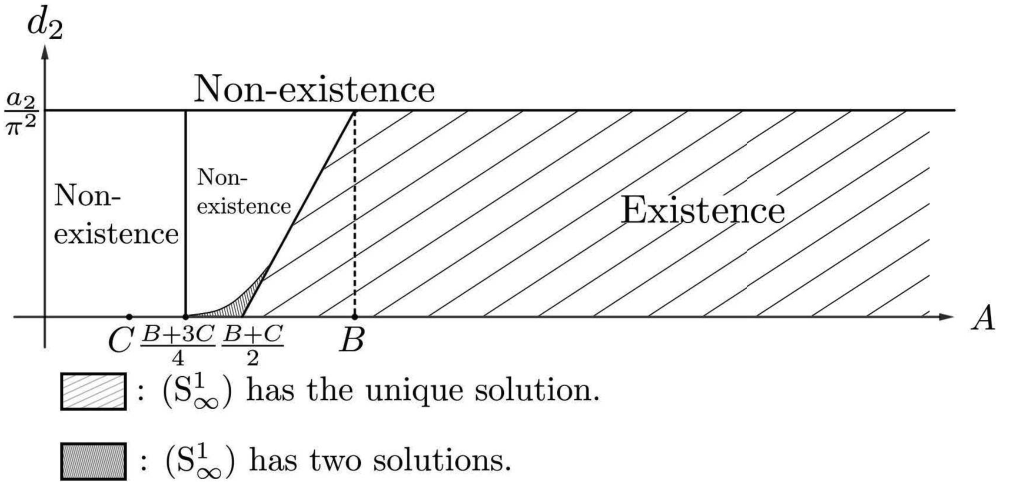

Figure 6: $0 < C < B$.

3 Representation of solutions

We explain the representation of solutions of (S^1_∞) , since it is very efficient for investigating the solution structure of (S^1_∞) .

Let us introduce a notations. Jacobi's elliptic function $\mathrm{sn}(x, k)$ defined by

$$\mathrm{sn}^{-1}(z, k) = \int_0^z \frac{d\xi}{\sqrt{1-k^2\xi^2}\sqrt{1-\xi^2}} \tag{3.1}$$

for $-1 \le z \le 1$. The complete elliptic integrals of the first, second and third kind are defined by

$$K(k) := \int_0^1 \frac{d\xi}{\sqrt{1-k^2\xi^2}\sqrt{1-\xi^2}}, \quad E(k) := \int_0^1 \frac{\sqrt{1-k^2\xi^2}}{\sqrt{1-\xi^2}}\,d\xi, \tag{3.2}$$

and

$$\Pi(\nu, k) := \int_0^1 \frac{d\xi}{(1+\nu\xi^2)\sqrt{1-k^2\xi^2}\sqrt{1-\xi^2}} \tag{3.3}$$

for $0 \le k < 1$ and $-1 < \nu$, respectively.

In what follows in (S^1_∞), we will concentrate on the case

$$b_1 = 1 \quad \text{and} \quad a_2 = b_2 = c_2 = 1. \tag{3.4}$$

In fact, we get from (S^1_∞).

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{\bar v}\left(\frac{A}{B} - \frac{\bar\tau}{\bar v} - \frac{C}{B}\bar v\right) dx = 0, & (3.5) \\ \bar d_2 \bar v_{xx} + \bar v(1-\bar v) - \bar\tau = 0 \qquad \text{in } (0,1), & (3.6) \\ \bar v_x(0) = 0, \quad \bar v_x(1) = 0, & (3.7) \\ \bar v(x) > 0, \quad \bar v_x(x) > 0 \qquad \text{in } (0,1) & (3.8) \end{cases}$$

by employing the following change of variables

$$\bar v := \frac{c_2}{a_2}\cdot v, \quad \bar\tau := \frac{b_2 c_2}{a_2^2}\cdot \tau, \quad \bar d_2 := \frac{d_2}{a_2}. \tag{3.9}$$

Thus, without lose of generality, we may consider the case $b_1 = 1$ and $a_2 = b_2 = c_2 = 1$.

Now, we introduce an auxiliary problem to investigate (S^1_∞) with $b_1 = a_2 = b_2 = c_2 = 1$. Let $d_2 > 0$ be given. Unknowns are a function $v = v(x)$ and a constant $\tau > 0$.

$$\text{(E)} \begin{cases} d_2 v_{xx} + v(1-v) - \tau = 0 \qquad \text{in } (0,1), & (3.10) \\ v(x) > 0 \ \text{ in } [0,1] \ \text{ and } \ v_x(x) > 0 \qquad \text{in } (0,1), & (3.11) \\ v_x(0) = 0, \quad v_x(1) = 0 \ \text{ and } \ \tau > 0. & (3.12) \end{cases}$$

Exact solutions of (E) are given in the following proposition.

Proposition 3.1. *(E) has a solution if and only if $d_2 \in (0, 1/\pi^2)$. All solutions $(v(x), \tau)$ of (E) are represented by*

$$v(x; d_2, h) = \alpha + (\beta - \alpha)\,\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}), \tag{3.13}$$

$$\tau(d_2, h) = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{3} = \frac{1}{4} - 4\,d_2^2\,(h^2 - h + 1)K(\sqrt{h})^4, \tag{3.14}$$

where

$$\alpha = \frac{1}{2} - 2d_2 K(\sqrt{h})^2\,(h+1)\,, \tag{3.15}$$

$$\beta = \frac{1}{2} + 2d_2 K(\sqrt{h})^2\,(2h-1)\,, \tag{3.16}$$

$$\gamma = \frac{1}{2} + 2d_2 K(\sqrt{h})^2\,(2-h)\,. \tag{3.17}$$

Here $\tilde{h}$ is the unique solution of an equation

$$(h+1)K(\sqrt{h})^2 = \frac{1}{4d_2} \tag{3.18}$$

in h, $K(\sqrt{h})$ is the complete elliptic integral of the 1st kind, and $\mathrm{sn}(\cdot,\cdot)$ *is Jacobi's elliptic function.*

Now, we note that (1.30) with $b_1 = 1$ is rewritten as

$$\frac{\tau \displaystyle\int_0^1 \frac{1}{v^2}dx + c_1}{\displaystyle\int_0^1 \frac{1}{v}dx} = a_1. \tag{3.19}$$

Thus, let us define a function $\tilde{a}_1(h; d_2, c_1)$ by

$$\tilde{a}_1(h; d_2, c_1) := \frac{\tau \displaystyle\int_0^1 \frac{1}{v(x; d_2, h)^2}dx + c_1}{\displaystyle\int_0^1 \frac{1}{v(x; d_2, h)}dx}. \tag{3.20}$$

$\tilde{a}_1(h; d_2, c_1)$ is explicitly given in the following proposition.

Proposition 3.2. *Let $d_2 \in (0, 1/\pi^2)$, $h \in (0, \tilde{h}(d_2))$. It holds that*

$$\begin{aligned}
&\tilde{a}_1(h; d_2, c_1) \\
&= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{6\alpha\beta\gamma\Pi\left(\frac{\beta-\alpha}{\alpha}, \sqrt{h}\right)} \\
&\quad \cdot \left((\gamma-\alpha)\alpha E(\sqrt{h}) - \alpha\gamma K(\sqrt{h}) + (\alpha\beta+\beta\gamma+\gamma\alpha)\Pi\left(\frac{\beta-\alpha}{\alpha}, \sqrt{h}\right)\right) \\
&\quad + \frac{\alpha K(\sqrt{h})c_1}{\Pi\left(\frac{\beta-\alpha}{\alpha}, \sqrt{h}\right)},
\end{aligned} \tag{3.21}$$

where α, β and γ are defined by (3.15), (3.16) and (3.17) respectively. Here, $K(\cdot)$, $E(\cdot)$ and $\Pi(\cdot,\cdot)$ are the complete elliptic integral of the 1st, 2nd and 3rd kind, respectively.

We explain the reason that the existence and non-existence regions change at $c_1 = 7/3$ $(C/B = 7/3)$. We obtain

$$\tilde{a}_1(h; d_2, c_1) = \frac{1}{2}\left(d_2\pi^2(1-c_1) + (1+c_1)\right) + \tilde{a}_{1,2}\cdot h^2 + \cdots, \tag{3.22}$$

by Taylor's expansion of (3.21) in h, where

$$\tilde{a}_{1,2} := \frac{3d_2\pi^2}{64(1-\pi^2 d_2)^2}\Big((35+13c_1)\pi^4 d_2^2 - 14\pi^2(c_1-1)d_2 + (c_1-1)\Big). \tag{3.23}$$

We check the sign of the coefficient $\tilde{a}_{1,2}$. We get $d_2 = d_+$ and d_- by solving

$$(35+13c_1)\pi^4 d_2^2 - 14\pi^2(c_1-1)d_2 + (c_1-1) = 0, \tag{3.24}$$

where

$$d_+ := \frac{7(c_1-1) + 2\sqrt{3(c_1-1)(3c_1-7)}}{\pi^2(35+13c_1)} \tag{3.25}$$

and

$$d_- := \frac{7(c_1-1) - 2\sqrt{3(c_1-1)(3c_1-7)}}{\pi^2(35+13c_1)}. \tag{3.26}$$

Thus,

$$\tilde{a}_{1,2} < 0 \quad \text{for} \quad 0 < c_1 < 1, \qquad 0 < d_2 < d_+, \tag{3.27}$$
$$\tilde{a}_{1,2} \geq 0 \quad \text{for} \quad 1 \leq c_1 \leq 7/3, \quad 0 < d_2 < 1/\pi^2, \tag{3.28}$$
$$\tilde{a}_{1,2} \geq 0 \quad \text{for} \quad c_1 > 7/3, \qquad d_+ \leq d_2 < 1/\pi^2, \tag{3.29}$$
$$\tilde{a}_{1,2} < 0 \quad \text{for} \quad c_1 > 7/3, \qquad d_- < d_2 < d_+, \tag{3.30}$$
$$\tilde{a}_{1,2} \geq 0 \quad \text{for} \quad c_1 > 7/3, \qquad 0 < d_2 \leq d_-. \tag{3.31}$$

Therefore, the behavior of $\tilde{a}_1(h, d_2, c_1)$ near $h = 0$ is drastically change at $c_1 = 1$ and $c_1 = 7/3$.

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