Unimodality for classical and free Brownian motions with initial distributions

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Abstract. This is a summary of the paper [5]. The main result is that classical and free Brownian motions with initial distributions are unimodal for sufficiently large time, under some assumption on the initial distributions. The assumption is almost optimal in some sense. Some related results, problems and conjectures are discussed.

1. Introduction

A Borel measure μ on $\mathbb{R}$ is *unimodal* if there exist $a \in \mathbb{R}$ and a function $f\colon \mathbb{R} \to [0,\infty)$ which is non-decreasing on $(-\infty, a)$ and non-increasing on (a,∞), such that

$$\mu(dx) = \mu(\{a\})\delta_a + f(x)\,dx. \tag{1.1}$$

The most outstanding result on unimodality is Yamazato's theorem [8] saying that all classical selfdecomposable distributions are unimodal. After this result, Hasebe and Thorbjørnsen proved the free analog of Yamazato's result in [4]: all freely selfdecomposable distributions are unimodal. The unimodality has several other similarities between the classical and free probability theories [3]. However, dissimilarity also appears. For example, classical compound Poisson processes are likely to be non-unimodal in large times [7], while free Lévy processes with compact support become unimodal in large times [3].

In this paper, we mainly focus on the unimodality of classical and free Brownian motions with initial distributions and discuss the similarity/dissimilarity between them. Namely, for classical probability we analyze the unimodality of the distribution $\mu * N(0,t)$. In free probability theory, the semicircle distribution

$$S(0,t) = \frac{1}{2\pi t}\sqrt{4t-x^2}\,1_{(-2\sqrt{t},2\sqrt{t})}(x)\,dx \tag{1.2}$$

is the free analog of the normal distribution $N(0,t)$. Free Brownian motion with initial distribution μ is defined as a non-commutative process with free independent increments, distributed as μ at time 0 and as $\mu \boxplus S(0,t)$ at time $t > 0$, where $\boxplus$ is called free convolution. Free Brownian motion can be understood as a large Hermitian matrix-valued Brownian motion, and then its eigenvalue distribution at time $t \geq 0$ converges

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to $\mu \boxplus S(0,t)$ as the size of the matrices tends to infinity. For further details on free convolution and $\mu \boxplus S(0,t)$ see [1, 2].

2. Free Brownian motion

In our studies, we first consider the symmetric Bernoulli distribution $\mu := \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$ as an initial distribution and discuss the unimodality of $\mu \boxplus S(0,t)$. This measure is known to be Lebesgue absolutely continuous and its density $p_t(x)$ is described by Biane [2]. Then we can see that the probability distribution $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4$ and it is not unimodal for $0 < t < 4$; see Figures 1–6.

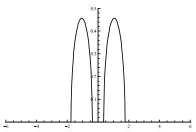

FIGURE 1. $p_{0.25}(x)$

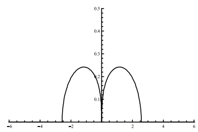

FIGURE 2. $p_1(x)$

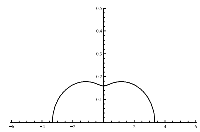

FIGURE 3. $p_2(x)$

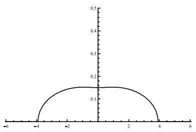

FIGURE 4. $p_3(x)$

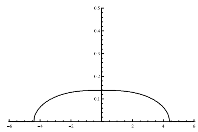

FIGURE 5. $p_4(x)$

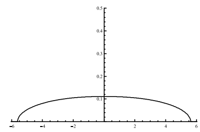

FIGURE 6. $p_7(x)$

This computation leads to a natural problem: for which class of probability measures μ on $\mathbb{R}$ does the distribution $\mu \boxplus S(0,t)$ become unimodal for sufficiently large time? We answer to this problem as follows.

Theorem 2.1. (1) Let μ be a compactly supported probability measure on $\mathbb{R}$ and $D_\mu := \sup\{|x-y| : x, y \in \operatorname{supp}(\mu)\}$. Then $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4D_\mu^2$.

(2) Let $f\colon \mathbb{R} \to [0,\infty)$ be a Borel measurable function. Then there exists a probability measure μ on $\mathbb{R}$ such that $\mu \boxplus S(0,t)$ is not unimodal for any $t > 0$ and

$$\int_{\mathbb{R}} f(x)\, d\mu(x) < \infty. \tag{2.1}$$

Note that such a measure μ is not compactly supported by (1).

The function f can grow very fast such as e^{x^2}, and so, a tail decay of the initial distribution does not imply large time unimodality.

3. Classical Brownian motion

The corresponding classical problem is natural, that is, for which class of initial distributions on $\mathbb{R}$ does Brownian motion become unimodal for sufficiently large time $t > 0$? Again, starting from an elementary example, we can show by calculus that

$$\frac{1}{2}(\delta_{-1} + \delta_1) * N(0,t) \tag{3.1}$$

is unimodal if and only if $t \geq 1$; see Figures 7-12. For more general initial distributions, we obtained the following results.

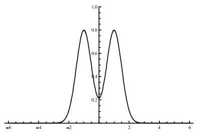

FIGURE 7. $t = 0.25$

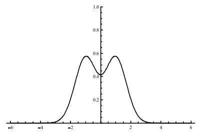

FIGURE 8. $t = 0.5$

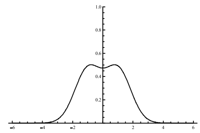

FIGURE 9. $t = 0.75$

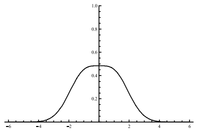

FIGURE 10. $t = 1$

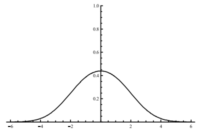

FIGURE 11. $t = 2$

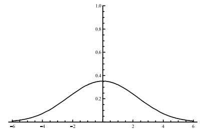

FIGURE 12. $t = 4$

Theorem 3.1. (1) Let μ be a probability measure on $\mathbb{R}$ such that

$$\alpha := \int_{\mathbb{R}} e^{\varepsilon x^2}\, d\mu(x) < \infty \tag{3.2}$$

for some $\varepsilon > 0$. Then the distribution $\mu * N(0,t)$ is unimodal for all $t \geq 36\varepsilon^{-1}\log(2\alpha)$.

(2) There exists a probability measure μ on $\mathbb{R}$ satisfying that

$$\int_{\mathbb{R}} e^{A|x|^p}\, d\mu(x) < \infty \quad \text{for all } A > 0 \text{ and } 0 < p < 2 \tag{3.3}$$

such that $\mu * N(0,t)$ is not unimodal for any $t > 0$.

Thus, in the classical case, the tail decay (3.2) is sufficient and almost necessary to guarantee the large time unimodality.

Remark 3.2. If μ is unimodal then $\mu * N(0,t)$ is unimodal for all $t > 0$. This is a consequence of the strong unimodality of the normal distribution $N(0,t)$, in contrast with the failure of freely strong unimodality of the semicircle distribution (see Theorem 4.2).

One open problem is the following.

Problem 3.3. Estimate the position of the mode of classical Brownian motion with initial distributions satisfying the assumption (3.2). The proof of Theorem 3.1 (1) in [5] shows that for $t \geq 36\varepsilon^{-1}\log(2\alpha)$, the mode is located in the interval $[-\sqrt{t}/2, \sqrt{t}/2]$. How about free Brownian motion?

4. Related results, problems and conjectures

It is known that there two unimodal distributions whose classical convolution is not unimodal. Then a probability measure μ is said to be strongly unimodal if the convolution $\mu * \nu$ is unimodal for every unimodal distribution ν. Ibragimov showed that μ is strongly unimodal if and only if μ is Lebesgue absolutely continuous on a (finite or infinite) interval and the density is log concave. On the other hand, the classical convolution of two symmetric unimodal distributions is again (symmetric and) unimodal. For further details see Sato's book [6].

We discuss those facts in the context of free probability. The first basic observation is the following.

Proposition 4.1. There two unimodal distributions whose free convolution is not unimodal.

Actually, in the proof we take a Cauchy distribution μ and another distribution ν. Then the interesting relation $\mu * \nu = \mu \boxplus \nu$ holds. Since the density of the Cauchy distribution is not log concave, we can find ν such that $\mu * \nu$ is not unimodal.

We say that a probability measure μ is freely strongly unimodal if the free convolution $\mu \boxplus \nu$ is unimodal for every unimodal distribution ν. We obtained the following.

Theorem 4.2. Let λ be a probability measure with finite variance, not being a delta measure. Then λ is not freely strongly unimodal.

By contrast, there are many strongly unimodal distributions with finite variance in classical probability including the normal distributions and exponential distributions. Thus the notion of strong unimodality breaks the similarity between the classical and free probability theories.

One open question is:

Problem 4.3. Is there a probability measure, not being a Dirac delta, which is freely strongly unimodal?

In the classical case convolution preserves symmetric unimodal distributions as mentioned. In this direction we obtained the following partial result.

Proposition 4.4. If μ is symmetric unimodal then so is $\mu \boxplus S(0,t)$.

The proof heavily depends on Biane's formula for the density of $\mu \boxplus S(0,t)$. We pose the full analogy as a conjecture.

Conjecture 4.5. If μ and ν are symmetric unimodal then so is $\mu \boxplus \nu$.

Finally, we would like to mention that we proved an analogue of Theorem 3.1 for Cauchy distributions and a partial analogue for Lévy distributions. Interested readers can consult [5].

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