Is the Spin-Boson model renormalisable?

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1 Introduction

The purpose of these notes is to give an explanation of the results obtained in [2]. In that paper, the authors consider the Spin-Boson model, which is a very popular model for a qubit coupled to a radiation field. It is proven, that ultraviolet renormalisation in the Spin-Boson model cannot be done in the same way as Edward Nelson renormalized the Nelson model in the paper [8]. More precisely, it is proven, that if Edward Nelsons method worked then the limiting operator is independent of the qubit so it would be physically uninteresting. It should be noted, that the proof does not exclude other (more exotic) renormalisation methods. It might be possible to take the coupling constant to 0 as the ultraviolet cut-off is removed and then end up at a physically useful model.

2 Notation and definitions

The following introduction is taken almost directly from [2]. Throughout this paper, $\mathcal{H}$ will always denote a separable Hilbert space. Write $\mathcal{H}^{\otimes n}$ for the n-fold tensor product of $\mathcal{H}$ and let $\mathcal{H}^{\otimes_s n} \subset \mathcal{H}^{\otimes n}$ be the subspace of symmetric tensors. The bosonic (or symmetric) Fock space is defined as

$$\mathcal{F}_b(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n}.$$

If $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ where $(\mathcal{M}, \mathcal{F}, \mu)$ is a σ-finite measure space then $\mathcal{H}^{\otimes_s n} = L^2_{sym}(\mathcal{M}^n, \mathcal{F}^{\otimes n}, \mu^{\otimes n})$. We will write an element $\psi \in \mathcal{F}_b(\mathcal{H})$ in terms of its coordinates as $\psi = (\psi^{(n)})$ and define the vacuum $\Omega = (1, 0, 0, \dots)$. For $g \in \mathcal{H}$ one defines the annihilation operator $a(g)$ and the creation operator $a^\dagger(g)$ on symmetric tensors in $\mathcal{F}_b(\mathcal{H})$ by $a(g)\Omega = 0$, $a^\dagger(g)\Omega = g$ and

$$\begin{aligned} a(g)(f_1 \otimes_s \cdots \otimes_s f_n) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle g, f_i \rangle f_1 \otimes_s \cdots \otimes_s \widehat{f_i} \otimes_s \cdots \otimes_s f_n \\ a^\dagger(g)(f_1 \otimes_s \cdots \otimes_s f_n) &= \sqrt{n+1}\, g \otimes_s f_1 \otimes_s \cdots \otimes_s f_n \end{aligned}$$

where $\widehat{f_i}$ means that f_i is omitted from the tensor product. These operators extend to closed operators in $\mathcal{F}_b(\mathcal{H})$ and $(a(g))^* = a^\dagger(g)$. Furthermore, we have the canonical commutation relations:

$$\overline{[a(f), a(g)]} = 0 = \overline{[a^\dagger(f), a^\dagger(g)]} \text{ and } \overline{[a(f), a^\dagger(g)]} = \langle f, g \rangle.$$

We also define the field operators

$$\varphi(g) = \overline{a(g) + a^\dagger(g)}.$$

Let A be a selfadjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(A)$. Then we define the second quantisation of A to be the selfadjoint operator

$$d\Gamma(A) = 0 \oplus \bigoplus_{n=1}^{\infty} \overline{\sum_{k=1}^{n} (1\otimes)^{k-1} A (\otimes 1)^{n-k} \Big|_{\mathcal{H}^{\otimes_s n}}}. \tag{2.1}$$

The number operator is defined as $N = d\Gamma(1)$. If $\mathcal{K}$ is another Hilbert space and $U : \mathcal{H} \to \mathcal{K}$ is a bounded operator with $\|U\| \leq 1$ then we define

$$\Gamma(U) = 1 \oplus \bigoplus_{n=1}^{\infty} U^{\otimes n} \mid_{\mathcal{H}^{\otimes_s n}} .$$

We will write $d\Gamma^{(n)}(A) = d\Gamma(A) \mid_{\mathcal{H}^{\otimes_s n}}$ and $\Gamma^{(n)}(U) = \Gamma(U) \mid_{\mathcal{H}^{\otimes_s n}}$ throughout the text. If ω is a multiplication operator then $d\Gamma^{(n)}(\omega)$ is the multiplication operator defined by the map $\omega_n(k_1, \dots, k_n) = \omega(k_1) + \cdots + \omega(k_n)$. The following three results are essential standard results (see [1], [2], [6]).

Lemma 2.1. *Let $\omega \geq 0$ be a selfadjoint operator defined on the Hilbert space $\mathcal{H}$ and let $m = \inf(\sigma(\omega))$. For $n \geq 1$ we have*

$$\sigma(d\Gamma^{(n)}(\omega)) = \overline{\{\lambda_1 + \cdots + \lambda_n \mid \lambda_i \in \sigma(\omega)\}},$$
$$\inf(\sigma(d\Gamma^{(n)}(\omega))) = nm.$$

Furthermore, $d\Gamma(\omega)$ will have compact resolvents if and only if ω has compact resolvents.

Lemma 2.2. *Let $U : \mathcal{H} \to \mathcal{K}$ be unitary, A be a selfadjoint operator on $\mathcal{H}$, $V \in \mathcal{U}(\mathcal{H})$ and $f \in \mathcal{H}$. Then $\Gamma(U)$ is unitary and*

$$\Gamma(U) d\Gamma(A) \Gamma(U)^* = d\Gamma(UAU^*).$$
$$\Gamma(U) W(f, V) \Gamma(U)^* = W(Uf, UVU^*).$$
$$\Gamma(U) \varphi(f) \Gamma(U)^* = \varphi(Uf).$$

Furthermore, $\Gamma(U)(f_1 \otimes_s \cdots \otimes_s f_n) = Uf_1 \otimes_s \cdots \otimes_s Uf_n$ and $\Gamma(U)\Omega = \Omega$.

Lemma 2.3. *Let $\omega \geq 0$ be selfadjoint and injective. If $g \in \mathcal{D}(\omega^{-1/2})$ then $\varphi(g)$ is $d\Gamma(\omega)^{1/2}$ bounded. In particular, $\varphi(g)$ is $N^{1/2}$ bounded. We have the following bound*

$$\|\varphi(g)\psi\| \leq 2 \|(\omega^{-1/2} + 1) g\| \|(d\Gamma(\omega) + 1)^{1/2} \psi\|$$

which holds on $\mathcal{D}(d\Gamma(\omega)^{1/2})$. In particular, $\varphi(g)$ is infinitesimally $d\Gamma(\omega)$ bounded.

We now introduce the Weyl representation. For any $g \in \mathcal{H}$ we define the corresponding exponential vector

$$\epsilon(g) = \sum_{n=0}^{\infty} \frac{g^{\otimes n}}{\sqrt{n!}}. \tag{2.2}$$

One may prove that if $\mathcal{D} \subset \mathcal{H}$ is a dense subspace then $\{\epsilon(f) \mid f \in \mathcal{D}\}$ is a linearly independent and total subset of $\mathcal{F}_b(\mathcal{H})$. Write $\mathcal{U}(\mathcal{H})$ for the set of unitary maps from $\mathcal{H}$ into $\mathcal{H}$. Let $U \in \mathcal{U}(\mathcal{H})$ and $h \in \mathcal{H}$. Then there is a unique unitary map $W(h, U)$ such that

$$W(h, U)\epsilon(g) = e^{-\|h\|^2/2 - \langle h, Ug \rangle} \epsilon(h + Ug) \quad \forall g \in \mathcal{H}.$$

One may easily check that $(h, U) \mapsto W(h, U)$ is strongly continuous and that

$$W(h_1, U_1) W(h_2, U_2) = e^{-i\mathrm{Im}(\langle h_1, U_1 h_2 \rangle)} W((h_1, U_1)(h_2, U_2)),$$

where $(h_1, U_1)(h_2, U_2) = (h_1 + U_1 h_2, U_1 U_2)$. If A is a selfadjoint operator on $\mathcal{H}$ and $f \in \mathcal{H}$ we have

$$e^{itd\Gamma(A)} = \Gamma(e^{itA}) = W(0, e^{itA})$$
$$e^{it\varphi(if)} = W(tf, 1).$$

We have the following lemma (see [6]):

Lemma 2.4. *Let $f, h \in \mathcal{H}$ and $U \in \mathcal{U}(\mathcal{H})$. Then*

$$\begin{aligned} W(h,U)\varphi(g)W(h,U)^* &= \varphi(Ug) - 2\mathrm{Re}(\langle Ug, h\rangle) \\ W(h,U)a(g)W(h,U)^* &= a(Ug) - \langle Ug, h\rangle \\ W(h,U)a^\dagger(g)W(h,U)^* &= a^\dagger(Ug) - \langle h, Ug\rangle. \end{aligned}$$

Furthermore, if ω is selfadjoint, non negative and injective on $\mathcal{H}$ and $h \in \mathcal{D}(\omega U^*)$ then

$$W(h,U)d\Gamma(\omega)W(h,U)^* = d\Gamma(U\omega U^*) - \varphi(U\omega U^* h) + \langle h, U\omega U^* h\rangle$$

on the domain $\mathcal{D}(d\Gamma(U\omega U^*))$.

The last essential ingredient is the lemmas In what follows we consider two fixed Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We will need the following two lemmas (see [9]).

Lemma 2.5. *There is a unique isomorphism $U : \mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2)$ such that $U(\epsilon(f \oplus g)) = \epsilon(f) \otimes \epsilon(g)$. If ω_i is selfadjoint on $\mathcal{H}_i$, V_i is unitary on $\mathcal{H}_i$ and $f_i \in \mathcal{H}_i$ then*

$$\begin{aligned} UW(f_1 \oplus f_2, V_1 \oplus V_2)U^* &= W(f_1, V_1) \otimes W(f_2, V_2) \\ Ud\Gamma(\omega_1 \oplus \omega_2)U^* &= d\Gamma(\omega_1) \otimes 1 + 1 \otimes d\Gamma(\omega_2) \\ U\varphi(f_1, f_2)U^* &= \varphi(f_1) \otimes 1 + 1 \otimes \varphi(f_2) \\ Ua(f_1, f_2)U^* &= a(f_1) \otimes 1 + 1 \otimes a(f_2) \\ Ua^\dagger(f_1, f_2)U^* &= a^\dagger(f_1) \otimes 1 + 1 \otimes a^\dagger(f_2). \end{aligned}$$

Lemma 2.6. *There is a unique isomorphism*

$$U : \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2) \to \mathcal{F}(\mathcal{H}_1) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_1) \otimes S_n(\mathcal{H}_2^{\otimes n})$$

such that

$$U(w \otimes \{\psi_2^{(n)}\}_{n=0}^{\infty}) = \psi^{(0)} w \oplus \bigoplus_{n=1}^{\infty} w \otimes \psi_2^{(n)}.$$

Let A be a selfadjoint operator on $\mathcal{F}(\mathcal{H}_1)$ and B be selfadjoint on $\mathcal{F}(\mathcal{H}_2)$ such that B is reduced by all of the subspaces $S_n(\mathcal{H}_2^{\otimes n})$. Write $B^{(n)} = B \mid_{S_n(\mathcal{H}_2^{\otimes n})}$. Then

$$U(A \otimes 1 + 1 \otimes B)U^* = A + B^{(0)} \oplus \bigoplus_{n=1}^{\infty} (A \otimes 1 + 1 \otimes B^{(n)})$$

$$UA \otimes BU^* = A \otimes B = B^{(0)} A \oplus \bigoplus_{n=1}^{\infty} A \otimes B^{(n)}.$$

3 The Spin-Boson model

Let $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define $e_1 = (1, 0)$ and $e_{-1} = (0, 1)$. The Spin-Boson Hamiltonian is given by

$$H_\eta(v, \omega) := \eta \sigma_z \otimes 1 + 1 \otimes d\Gamma(\omega) + \sigma_x \otimes \varphi(v),$$

which is here parametrised by $v \in \mathcal{H}, \eta \in \mathbb{C}$ and ω selfadjoint on $\mathcal{H}$. We will also need the fiber operators:

$$F_\eta(v, \omega) = \eta \Gamma(-1) + d\Gamma(\omega) + \varphi(v).$$

acting in $\mathcal{F}_b(\mathcal{H})$. If the spectra are real we define

$$\begin{aligned} E_\eta(v,\omega) &:= \inf(\sigma(H_\eta(v,\omega))) \\ \mathcal{E}_\eta(v,\omega) &:= \inf(\sigma(F_\eta(v,\omega))). \end{aligned}$$

For ω selfadjoint on $\mathcal{H}$ we define

$$m(\omega) = \inf\{\sigma(\omega)\} \quad \text{and} \quad m_{\text{ess}}(\omega) = \inf\{\sigma_{\text{ess}}(\omega)\}.$$

Standard perturbation theory and Lemma 2.3 yields:

Proposition 3.1. *Let $\omega \geq 0$ be selfadjoint and injective, $v \in \mathcal{D}(\omega^{-1/2})$ and $\eta \in \mathbb{C}$. Then the operators $F_\eta(v,\omega)$ and $H_\eta(v,\omega)$ are closed on the respective domains*

$$\begin{aligned} \mathcal{D}(F_\eta(v,\omega)) &= \mathcal{D}(d\Gamma(\omega)) \\ \mathcal{D}(H_\eta(v,\omega)) &= \mathcal{D}(1 \otimes d\Gamma(\omega)). \end{aligned}$$

Given any core $\mathcal{D}$ of ω the linear span of the following sets

$$\begin{aligned} \mathcal{J}(\mathcal{D}) &:= \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{f_1 \otimes_s \cdots \otimes_s f_n \mid f_j \in \mathcal{D}\} \\ \widetilde{\mathcal{J}}(\mathcal{D}) &:= \{f_1 \otimes f_2 \mid f_1 \in \{e_1, e_{-1}\}, f_2 \in \mathcal{J}(\mathcal{D})\} \end{aligned}$$

is a core for $F_\eta(v,\omega)$ and $H_\eta(v,\omega)$ respectively. Furthermore, both operators are selfadjoint and semibounded if $\eta \in \mathbb{R}$ and they have compact resolvents if ω has compact resolvents.

From the paper [2] we find the following theorem:

Theorem 3.2. *Let $\phi = (\phi_1, \phi_{-1}) = e_1 \otimes \phi_1 + e_{-1} \otimes \phi_{-1}$ be an element in $\mathcal{F}_b(\mathcal{H})^2 = \mathcal{F}_b(\mathcal{H}) \oplus \mathcal{F}_b(\mathcal{H}) \approx \mathbb{C}^2 \otimes \mathcal{F}_b(\mathcal{H})$. Write $\phi_i = (\phi_i^{(k)})$ for $i \in \{-1, 1\}$. Let $i \in \{-1, 1\}$. Define $\widetilde{\phi}_i = (\widetilde{\phi}_i^{(k)})$ where*

$$\widetilde{\phi}_i^{(k)} = \begin{cases} \phi_i^{(k)} & k \text{ is even} \\ \phi_{-i}^{(k)} & k \text{ is odd} \end{cases}$$

and $V(\phi_1, \phi_{-1}) = (\widetilde{\phi}_1, \widetilde{\phi}_{-1})$. Then

(1) *V is unitary with $V^* = V$.*

(2) *If $\omega \geq 0$ is selfadjoint and injective then $V 1 \otimes d\Gamma(\omega) V^* = 1 \otimes d\Gamma(\omega)$. Furthermore, if $\eta \in \mathbb{R}$ and $v \in \mathcal{D}(\omega^{-1/2})$ then*

$$V H_\eta(v,\omega) V^* = F_{-\eta}(v,\omega) \oplus F_\eta(v,\omega).$$

(3) *Let $\omega \geq 0$ be selfadjoint and injective, $\eta \in \mathbb{R}$ and $v \in \mathcal{D}(\omega^{-1/2})$. Then $E_\eta(v,\omega) = \mathcal{E}_{-|\eta|}(v,\omega)$ and $H_\eta(v,\omega)$ has a ground state if and only if the operator $F_{-|\eta|}(v,\omega)$ has a ground state. This is the case if $m(\omega) > 0$, and it is non degenerate if $\eta \neq 0$. Also*

$$\begin{aligned} \inf(\sigma_{\text{ess}}(F_{|\eta|}(v,\omega))) &= \mathcal{E}_{-|\eta|}(v,\omega) + m_{\text{ess}}(\omega) \\ \inf(\sigma_{\text{ess}}(H_\eta(v,\omega))) &= E_\eta(v,\omega) + m_{\text{ess}}(\omega) \end{aligned}$$

and $\mathcal{E}_{|\eta|}(v,\omega) > \mathcal{E}_{-|\eta|}(v,\omega)$ if and only if both $\eta \neq 0$ and $m(\omega) \neq 0$.

(4) *Let $\omega \geq 0$ be selfadjoint and injective, $\eta \in \mathbb{R}$ and $v \in \mathcal{D}(\omega^{-1/2})$. If ϕ is a ground state for $H_\eta(v,\omega)$ then*

$$V\phi = \begin{cases} e_{-\text{sign}(\eta)} \otimes \psi & \eta \neq 0 \\ e_{-1} \otimes \psi_{-1} + e_1 \otimes \psi_1 & \eta = 0 \end{cases}$$

where ψ is a ground state for $F_{-|\eta|}(v,\omega)$ and ψ_1, ψ_{-1} are either 0 or a ground state for $F_0(v,\omega)$.

At this point we should look at an example:

Example 3.3. The physically correct model we have $\mathcal{H} = L^2(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), \lambda_3)$ where λ_3 is the Lebesgue measure and $\mathcal{B}(\mathbb{R}^3)$ is the Borel σ-algebra. Furthermore, $\omega(k) = \sqrt{|k|^2+m^2}$ and $v_{g,\Lambda} = g\omega^{-1/2}1_{\{|k|\leq\Lambda\}}$ for some $m \geq 0$, $g > 0$ and $\Lambda > 0$. In this case, $m(\omega) = m = m_{\mathrm{ess}}(\omega)$ and $\sigma(\omega) = [m, \infty) = \sigma_{\mathrm{ess}}(\omega)$. Note that all assumptions in Theorem 3.2 are fulfilled.

4 Result and Interpretation

Throughout this section ω will always denote an injective, non negative and selfadjoint operator on $\mathcal{H}$. Furthermore, we will write $m = m(\omega)$ and $m_{\mathrm{ess}} = m_{\mathrm{ess}}(\omega)$. The main technical result is the following theorem:

Theorem 4.1. *Let $\{v_g\}_{g\in(0,\infty)} \subset \mathcal{D}(\omega^{-1/2})$ and P_ω denote the spectral measure corresponding to ω. For each $\widetilde{m} > 0$ we define $P_{\widetilde{m}} = P_\omega((\widetilde{m}, \infty))$ and $\overline{P}_{\widetilde{m}} = 1 - P_{\widetilde{m}} = P_\omega([0, \widetilde{m}])$. Assume that there is $\widetilde{m} > 0$ such that:*

(1) *$\{\overline{P}_{\widetilde{m}} v_g\}_{g\in(0,\infty)}$ converges to $v \in \mathcal{D}(\omega^{-1/2})$ in the graph norm of $\omega^{-1/2}$.*

(2) *$\|\omega^{-1}P_{\widetilde{m}}v_g\|$ diverges to ∞ as g tends to infinity.*

Then the g-dependent family of operators given by

$$\begin{aligned}\widetilde{F}_{\eta,\widetilde{m}}(v_g, \omega) :&= W(\omega^{-1}P_{\widetilde{m}}v_g, 1)F_\eta(v_g, \omega)W(\omega^{-1}P_{\widetilde{m}}v_g, 1)^* + \|\omega^{-1/2}P_{\widetilde{m}}v_g\|^2 \\ &= \eta W(2\omega^{-1}P_{\widetilde{m}}v_g, -1) + d\Gamma(\omega) + \varphi(\overline{P}_{\widetilde{m}}v_g)\end{aligned} \tag{4.1}$$

is uniformly bounded below by $-|\eta| - \sup_{g\in(0,\infty)}\|\omega^{-1/2}\overline{P}_{\widetilde{m}}v_g\|^2$. Furthermore, $\{\widetilde{F}_{\eta,\widetilde{m}}(v_g, \omega)\}_{g\in(0,\infty)}$ converges to $d\Gamma(\omega) + \varphi(v)$ in norm resolvent sense as g tends to ∞.

The assumption in part (1) is critical. Divergence where ω is small can lead to problems (see [3] for a counter example). The following Corollary easily proved:

Corollary 4.2. *Assume $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ and ω is a multiplication operator on this space. Let $v : \mathcal{M} \to \mathbb{C}$ is measurable and $\{\chi_g\}_{g\in(0,\infty)}$ be a collection of functions from $\mathbb{R}$ into $[0, 1]$. Assume $g \mapsto \chi_g(x)$ is increasing and converges to 1 for all $x \in \mathbb{R}$. Assume furthermore that $k \mapsto \chi_g(\omega(k))v(k) \in \mathcal{D}(\omega^{-1/2})$ and that there is $\widetilde{m} > 0$ such that $\widetilde{v} := 1_{\{\omega\leq\widetilde{m}\}}v \in \mathcal{D}(\omega^{-1/2})$. If $k \mapsto \omega(k)^{-1}v(k)1_{\{\omega>1\}}(k) \notin \mathcal{H}$ there are unitary maps $\{U_g\}_{g\in(0,\infty)}$ and $\{V_g\}_{g\in(0,\infty)}$ independent of η such that:*

(1) *$\{U_gF_\eta(v_g, \omega)U_g^* + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2\}_{g\in(0,\infty)}$ converges in norm resolvent sense to the operator $d\Gamma(\omega) + \varphi(\widetilde{v})$ as g tends to infinity.*

(2) *$\{V_gH_\eta(v_g, \omega)V_g^* + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2\}_{g\in(0,\infty)}$ is uniformly bounded below and converges in norm resolvent sense to the operator*

$$\widetilde{H} := (d\Gamma(\omega) + \varphi(\widetilde{v})) \oplus (d\Gamma(\omega) + \varphi(\widetilde{v}))$$

as g tends to ∞. In particular,

$$(H_\eta(v_g, \omega) + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2+i)^{-1} - (H_0(v_g, \omega) + \|\omega^{-1/2}1_{\{\omega>\widetilde{m}\}}v_g\|^2+i)^{-1}$$

will converge to 0 in norm as g tends to ∞.

Example 4.3. We continue Example 3.3. Let us consider self-energy renormalisation schemes as invented in [8]. In such schemes proves that

$$H_\eta(v_{g,\Lambda}, \omega) - E_\eta(v_{g,\Lambda}, \omega)$$

converges in strong or uniform resolvent sense to an operator $H_\eta^{\mathrm{Ren}}(v_g, \omega)$. Using Corollary 4.2 we see:

(1) $\Lambda \mapsto E_\eta(v_{g,\Lambda},\omega) + \|\omega^{-1/2}1_{\{\omega>1\}}v_{g,\Lambda}\|^2$ *has a limit independent of* η.

(2) $(H_\eta(v_{g,\Lambda},\omega) + \|\omega^{-1/2}1_{\{\omega>1\}}v_{g,\Lambda}\|^2+i)^{-1} - (H_0(v_{g,\Lambda},\omega) + \|\omega^{-1/2}1_{\{\omega>1\}}v_{g,\Lambda}\|^2+i)^{-1}$ *converges to 0 in norm as* Λ *tends to* ∞.

From this we conclude that if a self-energy renormalisation scheme exists then $H_\eta^{\mathrm{Ren}}(v_g,\omega)$ must be independent of η, which is not physically interesting. In other words, the contribution from the qubit disappears, as the ultraviolet cutoff is removed. This result is similar to the result in [4], where it is shown, that the mass-shell in a certain model becomes "almost flat" as the ultraviolet cutoff is removed. Thus the contribution from the matter particle vanishes as the ultraviolet cutoff is removed.

5 Sketch of proof of Theorem 4.1.

In this section we give the central ideas behind the proof of Theorem 4.1. From now on, ω will always denote an injective, non negative and selfadjoint operator on $\mathcal{H}$. As a simplifying assumption we have

$$m(\omega) > 0.$$

We will also assume $\{v_g\}_{g\in(0,\infty)}$ satisfies the assumptions of Theorem 4.1. It is easy to see that if they hold for some $\widetilde{m}$ then it will also hold for $\widetilde{m} = m$. Hence we will now assume that $\widetilde{m} = m$. Using Lemma 2.4 we see that

$$\widetilde{F}_{\eta,m}(v_g,\omega) = \eta W(2\omega^{-1}v_g,-1) + d\Gamma(\omega)$$

Hence it is enough to prove that if $\{h_g\}_{g\in(0,\infty)} \subset \mathcal{H}$ satisfies

$$\lim_{g\to\infty} \|h_g\| = \infty$$

then

$$T_\eta(h_g,\omega) := \eta W(h_g,-1) + d\Gamma(\omega)$$

converges to $d\Gamma(\omega)$ is norm resolvent sense as g tends to infinity. The following Lemma goes back to [5] and is the first fundamental observation.

Lemma 5.1. *Theorem 4.1 holds if* ω *has compact resolvents.*

Proof. First we see that $W(h_g,-1)$ converges to 0 weakly for g tending to infinity. By [10, Theorem 4.26] it is enough to check exponential vectors. We calculate

$$\begin{aligned}\langle \epsilon(g_1), W(v_g,-1)\epsilon(g_2)\rangle &= e^{-\|v_g\|^2/2+\langle v_g,g_2\rangle}\langle \epsilon(g_1),\epsilon(v_g-g_2)\rangle \\ &= e^{-\|v_g\|^2/2+\langle v_g,g_2\rangle+\langle g_1,v_g\rangle-\langle g_1,g_2\rangle},\end{aligned}$$

which converges to 0. We calculate

$$\begin{aligned}(T_\eta(v_g,\omega)-i)^{-1} - (d\Gamma(\omega)-i)^{-1} =& \eta(T_\eta(v_g,\omega)-i)^{-1}W(v_g,-1)(d\Gamma(\omega)-i)^{-1} \\ =& \eta^2(\widetilde{T}_\eta(v_g,\omega)-i)^{-1}W(v_g,-1)(d\Gamma(\omega)-i)^{-1}W(v_g,-1)(d\Gamma(\omega)-i)^{-1} \\ &+ \eta(d\Gamma(\omega)-i)^{-1}W(v_g,-1)(d\Gamma(\omega)-i)^{-1}.\end{aligned}$$

This implies

$$\|(\widetilde{F}_\eta(v_g,\omega)-i)^{-1} - (d\Gamma(\omega)-i)^{-1}\| \leq (|\eta|+1)|\eta|\|(d\Gamma(\omega)-i)^{-1}W(v_g,-1)(d\Gamma(\omega)-i)^{-1}\|$$

By Lemma 2.1 we see $(d\Gamma(\omega)-i)^{-1}$ is compact so the result is finished. □

The next Lemma is very technical. The full proof can be found in [2] and we only make a short sketch:

Lemma 5.2. *Assume ω is a selfadjoint, non negative and injective operator on $\mathcal{H}$. Let P_ω be the spectral measure of ω. Define the measurable function $f_k:\mathbb{R}\to\mathbb{R}$*

$$f_k(x)=\sum_{n=0}^{\infty}(n+1)2^{-k}1_{(n2^{-k},(n+1)2^{-k}]\cap(m,\infty)}(x).$$

along with $\omega_k=\int_{\mathbb{R}} f_k(\lambda)dP_\omega(\lambda)$. Then the following holds

1. *$\widetilde{F}_\eta(v,\omega_k)$ converges to $\widetilde{F}_\eta(v,\omega)$ in norm resolvent sense uniformly in v.*
2. *Let $\{h_g\}_{g\in(0,\infty)}$ be a collection of elements in $\mathcal{H}$. For each $k\in\mathbb{N}$, there are Hilbert spaces $\mathcal{H}_{1,k},\mathcal{H}_{2,k}$, selfadjoint operators $\omega_{1,k},\omega_{2,k}\geq 0$, a collection of elements $\{\widetilde{h}_{g,k}\}_{g\in(0,\infty)}\subset\mathcal{H}_{1,k}$ and a collection of unitary maps $\{\mathcal{U}_{g,k}\}_{g\in(0,\infty)}$ such that*

$$\mathcal{U}_{g,k}:\mathcal{F}_b(\mathcal{H})\to\mathcal{F}_b(\mathcal{H}_{1,k})\oplus\left(\bigoplus_{n=1}^{\infty}\mathcal{F}_b(\mathcal{H}_{1,k})\otimes S_n((\mathcal{H}_{2,k})^{\otimes n})\right),$$

$\omega_{1,k}\geq 2^{-k}$ has compact resolvents, $\|h_g\|=\|\widetilde{h}_{k,g}\|$ for all $g>0$ and

$$\mathcal{U}_{g,k}\widetilde{F}_\eta(h_g,\omega_k)\mathcal{U}_{g,k}^{*}=\widetilde{F}_\eta(\widetilde{h}_{g,k},\omega_{1,k})\oplus\bigoplus_{n=1}^{\infty}\left(\widetilde{F}_{(-1)^n\eta}(\widetilde{h}_{g,k},\omega_{1,k})\otimes 1+1\otimes d\Gamma^{(n)}(\omega_{2,k})\right)$$

for all $\eta\in\mathbb{R}$.

Proof. Part (1) can easily be derived from the fact that

$$\|(\widetilde{F}_\eta(v,\omega)-\widetilde{F}_\eta(v,\omega_k))\psi\|=\|(d\Gamma(\omega)-d\Gamma(\omega_k))\psi\|\leq\frac{2^{-k}}{m}\|d\Gamma(\omega)\psi\|.$$

for all $\psi\in\mathcal{D}(d\Gamma(\omega))$. Standard resolvent formulas then finishes the proof. In part (2), one constructs Hilbert spaces $\mathcal{H}_{1,k}$ and $\mathcal{H}_{2,k}$ and a unitary map $U_{g,k}:\mathcal{H}\to\mathcal{H}_{1,k}\oplus\mathcal{H}_{2,k}$ such that

$$U_{g,k}\omega_k U_{g,k}^{*}=\omega_{1,k}\oplus\omega_{2,k}$$

where $\omega_{1,k}\geq 2^{-k}$ has compact resolvents and $\widetilde{h}_{k,g}=U_{g,k}h_g\in\mathcal{H}_{1,k}$ for all $g\in(0,\infty)$. One now uses Lemmas 2.2, 2.5 and 2.6 to construct $\mathcal{U}_{g,k}$. □

We can now prove that Theorem 4.1 is true. From Lemma 5.2 part (2) and Lemma 5.1 we see that the theorem holds if $\omega=\omega_k$ for some k. Lemma 5.2 part (1) then finishes the proof.

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Bibliography

[1] V. Betz, F. Hiroshima and J. Lorinczi. Feynman-Kac-Type Theorems and Gibbs Measures on Path Space, with applications to rigorous Quantum Field Theory, 2011 Walter De Gruyter GmbH and Co. KG, Berlin/Boston.

[2] Dam, T. N.,Møller J. S.: Spin Boson Type Models Analysed Through Symmetries. arXiv:1803.05812 [math-ph].

[3] Dam, T. N.,Møller J. S.: Asymptotics in Spin-Boson type models. arXiv:1808.00085 [math-ph].

[4] Deckert D. A., Pizzo A.: Ultraviolet Properties of the Spinless, One-Particle Yukawa Model. Commun. Math. Phys. **327**, 887–920 (2014).

[5] M. Hirokawa, J. S. Møller and I. Sasaki. A Mathematical Analysis of Dressed Photon in Ground State of Generalized Quantum Rabi Model Using Pair Theory, Journal of Physics A: Mathematical and Theoretical 50(18): 184003, 2017.

[6] Oliver Matte and Thomas Norman Dam. SPECTRAL THEORY OF NON-RELATIVISTIC QED with Supplements.

[7] J. S. Møller. The translation invariant massive Nelson model: I. The bottom of the spectrum, Ann. Henri Poincaré 6 (2005), 1091–1135.

[8] E. NELSON. Interaction of nonrelativistic particles with a quantized scalar field, J. Math. Phys. 5 (1964), 1190-1197.

[9] K.R. Parthasarathy. An Introduction to Quantum Stochastic Calculus, Monographs in Mathematics, vol. 85, Birkhauser, Basel, 1992.

[10] J. Weidmann. Linear Operators in Hilbert Spaces. Springer, New York (1980).