SURJECTIVE ISOMETRIES ON A BANACH SPACE OF ANALYTIC FUNCTIONS ON THE OPEN UNIT DISC

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1. Introduction

Let $(M, \|\cdot\|_M)$ and $(N, \|\cdot\|_N)$ be normed linear spaces, respectively. A mapping $T\colon (M, \|\cdot\|_M) \to (N, \|\cdot\|_N)$ is an isometry if and only if it preserves the distance of two points in M, that is,

$$\|T(a) - T(b)\|_N = \|a - b\|_M \qquad (a, b \in M).$$

Here, we assume that T is not necessarily complex linear. The Mazur-Ulam theorem [16] states that every surjective isometry T between two normed linear spaces is real linear provided $T(0) = 0$.

We mention the characterization of isometries on several normed linear spaces. Isometries were studied on various spaces by many researchers, as for example in [3, 12, 13, 21, 22]. In 1932, isometries are studied by Banach [1, Theorem 3 in Chapter XI] (see also [24, Theorem 83]). There have been numerous papers on isometries defined on Banach spaces of analytic functions; see [2, 4, 5, 8, 11, 14].

Among the basic problems in analytic function spaces, Novinger and Oberlin, in [20], characterized *complex linear* isometries on a normed space $\mathcal{S}^p$. The underlying space $\mathcal{S}^p$ is a normed space consisting of analytic functions f on the open unit disc $\mathbb{D}$ whose derivative f' belongs to the classical Hardy space $(H^p(\mathbb{D}), \|\cdot\|_p)$ for $1 \leq p < \infty$. They introduced the norm $|f(0)| + \|f'\|_p$ on the normed space $\mathcal{S}^p$.

In this talk, we study surjective isometries on the Banach space $\mathcal{S}_A$ of analytic functions f defined on $\mathbb{D}$ whose derivative can be extended to the closed unit disc $\bar{\mathbb{D}}$, and endowed with the norm $\|f\|_\sigma = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|$. We denote by $A(\bar{\mathbb{D}})$ the disc algebra, that is, the algebra of all analytic functions on $\mathbb{D}$ which can be extended to continuous functions on $\bar{\mathbb{D}}$.

2. Main result

Let $A(\bar{\mathbb{D}})$ be the Banach space of all analytic functions on the open unit disc $\mathbb{D}$ that can be continuously extended to the closed unit disc $\bar{\mathbb{D}}$ with the supremum norm on $\mathbb{D}$. For each $v \in A(\bar{\mathbb{D}})$, v' means the derivative of v on $\mathbb{D}$, that is,

$$v'(z) = \lim_{h \to 0} \frac{v(z+h) - v(z)}{h} \qquad (z \in \mathbb{D}).$$

We define $\mathcal{S}_A$ by the linear space of all analytic functions f on $\mathbb{D}$ whose derivative f' belongs to $A(\bar{\mathbb{D}})$. By [6, Theorem 3.11], we see that $\mathcal{S}_A \subset A(\bar{\mathbb{D}})$. By the definition of $\mathcal{S}_A$, f' is an analytic function on $\mathbb{D}$ which can be extended to a continuous function on $\bar{\mathbb{D}}$. Let $\widehat{v}$ be the unique continuous extension of $v \in A(\bar{\mathbb{D}})$ to $\bar{\mathbb{D}}$. In fact, such an extension is unique since $\mathbb{D}$ is dense in $\bar{\mathbb{D}}$. We define the norm $\|f\|_\sigma$ of $f \in \mathcal{S}_A$ by

$$\|f\|_\sigma = |f(0)| + \|\widehat{f'}\|_\infty \qquad (f \in \mathcal{S}_A), \tag{2.1}$$

where $\|\widehat{f'}\|_\infty = \sup\{|\widehat{f'}(z)| : z \in \bar{\mathbb{D}}\} = \sup\{|f'(z)| : z \in \mathbb{D}\}$. It is not difficult to check that $(\mathcal{S}_A, \|\cdot\|_\sigma)$ is a complex Banach space.

Theorem 1. *If $T\colon (\mathcal{S}_A, \|\cdot\|_\sigma) \to (\mathcal{S}_A, \|\cdot\|_\sigma)$ is a surjective, not necessarily complex linear, isometry, then one of the following four forms is occured;*

there exist constants $c_{1,1}, c_{1,2}, \lambda_1 \in \mathbb{T}$ and $a_1 \in \mathbb{D}$ such that

$$T(f)(z) = T(0)(z) + c_{1,1} f(0) + \int_{[0,z]} c_{1,2} f'(\rho(\zeta))\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

there exist constants $c_{2,1}, c_{2,2}, \lambda_2 \in \mathbb{T}$ and $a_2 \in \mathbb{D}$ such that

$$T(f)(z) = T(0)(z) + \overline{c_{2,1} f(0)} + \int_{[0,z]} c_{2,2} f'(\rho(\zeta))\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

there exist constants $c_{3,1}, c_{3,2}, \lambda_3 \in \mathbb{T}$ and $a_3 \in \mathbb{D}$ such that

$$T(f)(z) = T(0)(z) + c_{3,1} f(0) + \int_{[0,z]} \overline{c_{3,2} f'(\rho(\overline{\zeta}))}\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

there exist constants $c_{4,1}, c_{4,2}, \lambda_4 \in \mathbb{T}$ and $a_4 \in \mathbb{D}$ such that

$$T(f)(z) = T(0)(z) + \overline{c_{4,1} f(0)} + \int_{[0,z]} \overline{c_{4,2} f'(\rho(\overline{\zeta}))}\, d\zeta \quad (\forall f \in \mathcal{S}_A,\ \forall z \in \mathbb{D}),$$

where $\rho(z) = \lambda_j \dfrac{z - a_j}{\overline{a_j} z - 1}$ for all $z \in \bar{\mathbb{D}}$ and for $j = 1, 2, 3, 4$.

Conversely, each of the above forms is a surjective isometry on $\mathcal{S}_A$ with the norm $\|\cdot\|_\sigma$, where $T(0)$ is an arbitrary element of $\mathcal{S}_A$.

We start by defining an embedding of $\mathcal{S}_A$ into a subspace B consisting of complex valued continuous functions. Then using the Arens-Kelley theorem (see [10, Corollary 2.3.6 and Theorem 2.3.8]), we give a characterization of extreme points of the unit ball B_1^* of the dual space B^* of B. Then we construct some maps to describe extreme points of B_1^*.

We used an idea by Ellis for the characterization of surjective real linear isometries on uniform algebras (see [9]). An adjoint operator of a surjective real linear isometry on the dual space B^* preserves extreme points. The action of such adjoint operator on the set of extreme points gives a representation for the isometries on B. We show that the isometries of $\mathcal{S}_A$ are integral operators of weighted differential operators.

For the details of proof, refer to [18].

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