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An energy analysis of a fluttering flag

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Abstract

This work deals with the dynamics and stability of a flapping flag, with emphasis on the onset of flutter instability. The flag is considered as a thin plate. The fluid forces are evaluated by employing the theory of Theodorsen (1935). The aim of the work is to (i) obtain a physical understanding of the mechanism of the flutter oscillations, and (ii) to obtain a better understanding of dynamically varying fluid forces, expressed via the complex Theodorsen function C, as opposed to the quasisteady approximation, where $C = 1 - \mathrm{i}0$. An energy balance analysis shows that (i) flutter can occur only in a 'dragging' sort of motion, in other words, in the form of downstream travelling wave motion, and (ii) a small imaginary part of the Theodorsen function, $C = 1 - \mathrm{i}\epsilon$, $0 < \epsilon \ll 1$, has a destabilizing effect, in the sense that the critical flow speed is smaller than by the quasisteady approximation $C = 1 - \mathrm{i}0$. These predictions have been verified by numerical eigenvalue analyses. The result (ii) is in opposition to previously reported results. Furthermore, it is found that certain terms in the equation of motion of the flag that previously have been discarded, on the assumption that they are associated with very slow changes across the flag, have a significant effect on the stability of the flag.

1 Introduction

While the flapping of flags has been the subject of speculation since antiquity [12], the first serious theoretical investigation of the phenomenon is only of recent date (in the time-line of the history of mankind) and appears to be due to Lord Rayleigh [11]. However, after the appearance of this work, not much was done for nearly another century, probably due to the complexity of the problem. A second landmark paper, published by Taneda [15] in 1968, was based on careful experiments on the waving motion of flags in a wind tunnel. In the remainder of the 20st century the general scientific interest in the flag problem was not large but still, a number of noteworthy studies were published [12]. An experiment-based paper by Zhang et al. [17], who studied the dynamics of flexible filaments in a flowing soap film as a model for a one-dimensional flag in a two-dimensional flow, may be viewed as the third landmark study which, at this time, initiated not only a renewed interest in the problem but a veritable flood of new papers.

Among these, the work of Argentina and Mahadevan [1], and particularly their discussion of the mechanism of instability in terms of asymptotic limits, is interesting. The theory is based on that of Theodorsen [16], who described the influence of a wake model in terms of what now is known as the Theodorsen function. In [1] it is claimed that the critical flow speed is significantly lower (lower by a factor of three!) when the influence of the wake is ignored. It is claimed also that when the wake effect is taken into account, the flutter oscillations are dominated by the first (fundamental, free vibration) eigenmode, while coupled-mode flutter, with a coupling between the first and the second eigenmode, takes place when the wake effect is ignored . These results differ from classical results for flutter of cantilevered plates in an airstream (e.g. [10]), and the aim of the present paper is to clarify them though analytical energy considerations.

2 Equation of motion

A flag lies, in undisturbed condition, in the domain $0 \leq x \leq L$, $y = 0$, $0 \leq z \leq l$, where (x, y, z) is a usual right-hand cartesian coordinate system, with the z axis orthogonal to the x and y axes and running into the paper. A fluid of density ρ_f is moving with uniform velocity U in the positive x-direction. The flag is modelled as a thin plate and it is assumed that this plate vibrates only in beam modes, that is, at any point $x \in [0, L]$ the deflection is the same for any $z \in [0, l]$. Let $Y(x, t)$ be the deflection of the flag at position x and time t. Then the equation of motion is given by

$$m\frac{\partial^2 Y}{\partial t^2} + B^* \frac{\partial^5 Y}{\partial^4 x \partial t} + B\frac{\partial^4 Y}{\partial x^4} = l\Delta P. \tag{1}$$

Here $m = \rho_s hl$ is the mass per unit length of the flag, where ρ_s is the density of the flag material of thickness h, $B = Eh^3/12(1-\nu^2)$ is the flexural rigidity ,where E is Young's modulus and ν is Poisson's ratio, B^* is a parameter that represents internal damping in the flag material, and ΔP is the pressure difference across the flag due to the fluid flow.

Assuming that the flag/plate is clamped at $x = 0$ and free at $x = L$, the four boundary conditions are given by

$$Y(0,t) = 0, \quad \left[\frac{\partial Y}{\partial x}\right]_{x=0} = 0, \quad \left[\frac{\partial^2 Y}{\partial x^2}\right]_{x=L} = 0, \quad \left[\frac{\partial^3 Y}{\partial x^3}\right]_{x=L} = 0. \tag{2}$$

Argentina and Mahadevan [1] have carried out an analysis of the fluid forces acting on the flag, based on the classical work of Theodorsen [16]. In their analysis they evaluate a non-circulatory velocity potential ϕ_{nc} to account for the flow along the flag and a circulatory velocity potential ϕ_{c} to account for the vortex wake shed from the trailing edge of the flag, in order to satisfy the Kutta-Joukowski condition as well as Kelvin's theorem [7]. Making use of the linearized Bernoulli equation

$$p(x,y,t) = -\rho_f \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right] \phi(x,y,t), \tag{3}$$

where p is the pressure and $\phi = \phi_{\text{nc}} + \phi_{\text{c}}$, the pressure difference across the flag is evaluated as

$$\Delta P = -\rho_f U C(\kappa)\, f\left(\frac{x}{L}\right) \left\{\frac{\partial Y}{\partial t} + U\frac{\partial Y}{\partial x}\right\} - \rho_f L\, g\left(\frac{x}{L}\right) \left\{\frac{\partial^2 Y}{\partial t^2} + 2U\frac{\partial^2 Y}{\partial t \partial x} + U^2\frac{\partial^2 Y}{\partial x^2}\right\}. \tag{4}$$

Here

$$f(\xi) = 2\sqrt{\frac{1-\xi}{\xi}}, \quad g(\xi) = 2\sqrt{(1-\xi)\xi}, \tag{5}$$

and $C(\kappa)$ is the Theodorsen function (to be specified a little later). It is noted that Argentina and Mahadevan [1] discard the last two terms in (4) on the ground that they are small in comparison with the first term of the second line. It will become evident that this is not the case.

We will make use of nondimensional versions of (1) and (2) which can be obtained by introducing the nondimensional parameters

$$\xi = \frac{x}{L}, \quad \tau = \frac{t}{L}U_B = \frac{t}{L}\frac{1}{L}\sqrt{\frac{B}{m}}, \quad \eta = \frac{Y}{L}, \tag{6}$$
$$\rho = \frac{\rho_f Ll}{m} = \frac{\rho_f}{\rho_s}\frac{L}{h}, \quad u = \frac{U}{U_B} = UL\sqrt{\frac{m}{B}}, \quad \sigma^* = \frac{B^*}{B}U_B,$$

where U_B is the elastic wave propagation speed. Equations (1) and (2) then take the forms

$$\{1+\rho g(\xi)\}\frac{\partial^2\eta}{\partial\tau^2} + \sigma^*\frac{\partial^5\eta}{\partial\xi^4\partial\tau} + \frac{\partial^4\eta}{\partial\xi^4} + \rho u C(\kappa)f(\xi)\left\{\frac{\partial\eta}{\partial\tau} + u\frac{\partial\eta}{\partial\xi}\right\} + \rho u g(\xi)\left\{2\frac{\partial^2\eta}{\partial\xi\partial\tau} + u\frac{\partial^2\eta}{\partial\xi^2}\right\} = 0, \tag{7}$$

and

$$\eta(0,\tau) = 0, \quad \left[\frac{\partial\eta}{\partial\xi}\right]_{\xi=0} = 0, \quad \left[\frac{\partial^2\eta}{\partial\xi^2}\right]_{\xi=1} = 0, \quad \left[\frac{\partial^3\eta}{\partial\xi^3}\right]_{\xi=1} = 0. \tag{8}$$

The first term in (7) represents inertia forces, with the term multiplying unity (in the first pair of curly brackets) being the inertia forces of the flag itself, and the term multiplying $\rho g(\xi)$ the inertia forces due to the added fluid mass. The second term represents dissipative forces due to internal damping of the flag material. The third term represents the elastic forces of the flag. The fourth term, proportional to ρu, represents dissipative forces due to fluid damping. The fifth and final term in the first line, proportional to ρu^2, represents circulatory fluid forces. The two terms in the second line are fluid force terms as well, originating from the last two terms in (4). The first term (in the second line) represents Coriolis forces, while the second term represents a 'centrifugal force' [9].

It is noted that $\partial^2 Y/\partial t^2 = (U_B^2/L)\partial^2\eta/\partial\tau^2$, $2U\partial^2 Y/\partial t\partial x = (U_B^2/L)2u\partial^2\eta/\partial\tau\partial\xi$, and $U^2\partial^2 Y/\partial x^2 = (U_B^2/L)u^2\partial^2\eta/\partial\xi^2$, that is to say, the three terms in the last line of (4) are of the same order of magnitude; the last two terms cannot be neglected. It follows from (7) that the fluid force terms in the second line are of same order of magnitude as the last two terms in the first line; again, they *cannot* be neglected.

The Theodorsen function $C(\kappa)$ is defined by

$$C(\kappa) = \frac{\displaystyle\int_1^\infty \frac{\tilde\xi}{\sqrt{\tilde\xi^2-1}}\mathrm{e}^{-\mathrm{i}\kappa\tilde\xi}d\tilde\xi}{\displaystyle\int_1^\infty \sqrt{\frac{\tilde\xi+1}{\tilde\xi-1}}\mathrm{e}^{-\mathrm{i}\kappa\tilde\xi}d\tilde\xi} = \frac{H_1^{(2)}(\kappa)}{H_1^{(2)}(\kappa) + \mathrm{i}H_0^{(2)}(\kappa)}, \quad \kappa \in \mathbb{R}_+, \tag{9}$$

where $H_n^{(2)}$ is the Hankel function of second kind and order n, and

$$\kappa = \frac{\omega}{2u} \tag{10}$$

is a nondimensional, real, positive wavenumber, with ω being the nondimensional frequency of the flag oscillations. The function C represents the vortex wake shed from the trailing edge $\xi = 1$, modelled by the mentioned non-circulatory potential ϕ_{nc}, and as indicated by (9), it involves evaluation of integrals over the shed vorticity in the domain $\tilde\xi \in [1,\infty)$ [1, 2, 4, 16]. Thus κ, defined by (10), is the wavenumber for the oscillating vortex sheet representing the wake from the flag.

3 Energy considerations

3.1 General equations

Multiplication of the equation of motion (7) by the lateral flag velocity $\partial\eta/\partial\tau$, followed by integration over the flag surface, $0 \leq \xi \leq 1$, gives a power (rate of work) balance equation. Integrating this equation over time, say from $\tau = \tau_1$ to τ_2, gives an energy balance equation. Writing (7) in the operator form $\mathfrak{L}(\eta) = 0$ we have

$$\int_{\tau_1}^{\tau_2} \int_0^1 \frac{\partial \eta}{\partial \tau} \mathfrak{L}(\eta) d\xi d\tau = 0. \tag{11}$$

Inserting (7), this equation can be rewritten as

$$\begin{aligned} &[\Delta E]_{\tau_1}^{\tau_2} = [\Delta T + \Delta V]_{\tau_1}^{\tau_2} = \\ &\int_{\tau_1}^{\tau_2} \left[-\int_0^1 \sigma^* \left(\frac{\partial^3 \eta}{\partial \xi^2 \partial \tau} \right)^2 d\xi - \rho u \int_0^1 C(\kappa) f(\xi) \left\{ \left(\frac{\partial \eta}{\partial \tau} \right)^2 + u \frac{\partial \eta}{\partial \tau} \frac{\partial \eta}{\partial \xi} \right\} d\xi \right] d\tau \\ &- \int_{\tau_1}^{\tau_2} \int_0^1 \rho u g(\xi) \frac{\partial \eta}{\partial \tau} \left\{ 2 \frac{\partial^2 \eta}{\partial \tau \partial \xi} + u \frac{\partial^2 \eta}{\partial \xi^2} \right\} d\xi d\tau \end{aligned} \tag{12}$$

where

$$T = \tfrac{1}{2} \int_0^1 \{1 + \rho g(\xi)\} \left(\frac{\partial \eta}{\partial \tau} \right)^2 d\xi \tag{13}$$

is the kinetic energy of the flag, including the extra contribution due to the added fluid mass, and

$$V = \tfrac{1}{2} \int_0^1 \left(\frac{\partial^2 \eta}{\partial \xi^2} \right)^2 dx \tag{14}$$

is the potential energy of the flag. $[\Delta E]_{\tau_1}^{\tau_2}$ thus indicates the change in mechanical energy from time $\tau = \tau_1$ to time $\tau = \tau_2$.

In the following we will assume that the time interval $[\tau_1, \tau_2]$ is coinciding with one period of oscillation, say, $\tau_1 = 0$ and $\tau_2 = 2\pi/\omega$, such that the deflection of the flag at time $\tau = \tau_2$ coincides with the deflection at $\tau = \tau_1$.

The first term in the right hand side of (12) represents the energy dissipated by internal (flag material) damping. Since the integrand is positive definite and there is a minus in front of the term, it is clear that this term reduces the mechanical energy in each period of oscillation.

First, we ignore the terms in the last line of (12) and consider Argentina and Mahadevan's [1] example for the case where $\rho \to 0$ and $u \to \infty$ in such a way that the fluid force multiplier ρu^2 remains finite. Equation (10) then gives that $\kappa \to 0$ and the Theodorsen function

$$C(\kappa) \sim 1 - (\pi/2)\kappa - \mathrm{i}(\ln 2 + 1 - \gamma)\kappa \tag{15}$$

in this limit. (Here γ is Euler's constant, $\gamma = 0.5771\ldots$, and $\ln 2 + 1 - \gamma \approx 1.1160$.) We will here first consider the limit case $\lim_{\kappa \to 0} C(\kappa) = 1 - \mathrm{i}\,0$, the quasi-steady approximation, which is precisely satisfied in the case of steady flow [1]. Since the function $f(\xi) > 0$ for all $\xi \in [0, 1]$, cf. (5), the second term on the right hand side of (12) (proportional to ρu) thus acts as a dissipative term too.

Flutter motion at the threshold of instability is characterized by $[\Delta E]_0^{2\pi/\omega} = 0$ and unstable flutter oscillations by $[\Delta E]_0^{2\pi/\omega} > 0$. Both of these conditions can be satisfied only if the third term on the right hand side of (12) (proportional to ρu^2) is positive. This is realized only if $\partial\eta/\partial\tau$ and $\partial\eta/\partial\xi$ have opposite signs over most part of the domain $\xi \in [0,1]$, and over most part of the vibrational period $0 \le \tau \le \pi/\omega$, cf. Subsection 3.2.

Consider next the influence of the Coriolis force term and the centrifugal force term, that is, the two terms represented by the last line of (12). Integration by parts of the Coriolis force term (the first term) gives, with use of the boundary conditions and the fact that $g(0) = g(1) = 0$,

$$2\int_0^1 g(\xi)\frac{\partial\eta}{\partial\tau}\frac{\partial^2\eta}{\partial\tau\partial\xi}d\xi = -\int_0^1 \frac{dg(\xi)}{d\xi}\left(\frac{\partial\eta}{\partial\tau}\right)^2 d\xi. \tag{16}$$

It will be shown in the following subsection that the integral on the left hand side actually is positive (or equivalently, that the one on the right hand side is negative) and thus, that the Coriolis force term acts as a dissipative term.

Considering now the last term, the centrifugal force term, it is seen that is order for this term to act as an energy source, which sends energy from the flow into the fluttering flag, it is necessary that $\partial\eta/\partial\tau$ and $\partial^2\eta/\partial\xi^2$ have opposite signs over most part of the domain $\xi \in [0,1]$, and over most part of the vibrational period $0 \le \tau \le \pi/\omega$. It will be shown in the following subsection that this is actually not the case, and thus that it is the last term in the second line of (12) that acts as the main energy source.

3.2 Influence of the phase angle distribution on the energy balance

In order to get a more detailed understanding of how flutter is realized by a flapping flag, we will consider vibrations just at the threshold of flutter instability, at the critical flow speed $u = u_c$, where steady-state oscillations occur, with $[\Delta E]_0^{2\pi/\omega} = 0$. These vibrations can be expressed as [13, 14]

$$\eta(\xi,\tau) = A(\xi)\cos(\omega_c\tau + \phi(\xi)), \tag{17}$$

where $A(\xi)$ is an amplitude function, $\phi(\xi)$ a is phase angle function, and ω_c is the flutter frequency.

Consider first the case $C(\kappa) \equiv 1 - \mathrm{i}0$. Inserting (17) into (12) and evaluating the time integrals, we obtain

$$\begin{aligned}[\Delta E]_0^{2\pi/\omega_c} = &-\pi\omega_c\int_0^1 \sigma^*\left[\left\{\frac{d^2A}{d\xi^2} - A\frac{d^2\phi}{d\xi^2}\right\}^2 + \left\{2\frac{dA}{d\xi}\frac{d\phi}{d\xi} + A\frac{d^2\phi}{d\xi^2}\right\}^2\right]d\xi \\ &-\rho u_c\pi\left[\omega_c\int_0^1 f(\xi)A^2(\xi)d\xi + u_c\int_0^1 f(\xi)A^2(\xi)\frac{d\phi(\xi)}{d\xi}d\xi\right] \\ &-\rho u_c\pi\left[2\omega_c\int_0^1 g(\xi)A(\xi)\frac{dA(\xi)}{d\xi}d\xi + u_c\int_0^1 g(\xi)\left\{2A(\xi)\frac{dA(\xi)}{d\xi}\frac{d\phi(\xi)}{d\xi} + A^2(\xi)\frac{d^2\phi(\xi)}{d\xi^2}\right\}d\xi\right]\end{aligned} \tag{18}$$

Here the terms in the last line correspond to the terms in the last line of (7). Assuming first that these terms are zero, then, in order to satisfy $[\Delta E]_0^{2\pi/\omega} = 0$, it is necessary that the integral

$$\int_0^1 f(\xi)A^2(\xi)\frac{d\phi(\xi)}{d\xi}d\xi < 0, \tag{19}$$

since all the proceeding integrals are positive definite (and there is a minus in front of any term). Since $f(\xi)$ is positive definite, the phase angle gradient $\partial\phi(\xi)/\partial\xi$ needs to be negative over the most part of $\xi \in [0,1]$.

Consider next the influence of the Coriolis force term and the centrifugal force term, that is, the 'new' terms represented by the last line of (18). Integration by parts of the Coriolis force term gives, with use of the boundary conditions and the fact that $g(0) = g(1) = 0$,

$$2\int_0^1 g(\xi)A(\xi)\frac{dA(\xi)}{d\xi}d\xi = -\int_0^1 \frac{dg(\xi)}{d\xi}A^2(\xi)d\xi. \tag{20}$$

It is noted that $dg(\xi)/d\xi > 0$ for $0 < \xi < 1/2$ and $dg(\xi)/d\xi < 0$ for $1/2 < \xi < 1$, cf. (5). Noting the symmetry of $g(\xi)$, it follows that the integral on the left hand side of (20) is positive if $\int_{1/2}^1 A^2(\xi)d\xi > \int_0^{1/2} A^2(\xi)d\xi$, and negative otherwise. With the clamped-free boundary conditions, this will be so, that is, (20) will be positive. Thus the Coriolis force term will act as a dissipative term.

Integration by parts of the first of the two centrifugal force terms gives, with use of the boundary conditions (and $g(0) = g(1) = 0$),

$$2\int_0^1 g(\xi)A(\xi)\frac{\partial A(\xi)}{\partial\xi}\frac{\partial\phi(\xi)}{\partial\xi}d\xi = -\int_0^1 g(\xi)A^2(\xi)\frac{d^2\phi(\xi)}{d\xi^2}d\xi - \int_0^1 \frac{dg(\xi)}{d\xi}A^2(\xi)\frac{d\phi(\xi)}{d\xi}d\xi. \tag{21}$$

Adding the very last term of (18) to this result, only the last term in the right hand side of (21) remains. For this term it is noted again that $dg(\xi)/d\xi < 0$ for $1/2 < \xi < 1$, and in this domain $A(\xi)$ is largest. Since $d\phi/d\xi < 0$ for all $\xi \in [0, 1]$ it follows that this terms will not contribute significantly to the flutter instability.

The conclusion is thus that it is mainly the term (19) that feeds energy to the fluttering flag; and this happens, again, if the phase angle gradient $\partial\phi(\xi)/\partial\xi$ is negative over the most part of $\xi \in [0, 1]$. In this relation, it is noted that the bending wavenumber k for the 'waving' flag (plate) is defined as 'minus the phase change per unit increase in distance' [3, p. 3], i.e. $k(\xi) = -\partial\phi/\partial\xi$ [6, p. 310]. The bending wave speed $c(\xi)$ is, at the flutter frequency ω_c, defined as

$$c(\xi) = \frac{\omega_c}{k(\xi)}. \tag{22}$$

This expression shows that the flutter motion is a travelling wave motion, travelling in the direction of positive ξ.

3.3 Influence of the Theodorsen function on the energy balance

In order to understand the apparent 'destabilizing effect' of the Theodorsen function we investigate in the following how its inclusion affects the energy balance (12). We return again to the representation (17) with a continuous phase angle function. The Theodorsen function C can be written as

$$C(\kappa) = F(\kappa) - \mathrm{i}\bar{G}(\kappa), \tag{23}$$

where F and $\bar{G}$ are real, positive definite functions. Rather than just inserting (23) into (18), it is useful to reconsider (12) in order to keep the final result in real form. Since multiplication with $-\mathrm{i}$ corresponds to a phase shift of $-\pi/2$ we can write

$$-\mathrm{i}\bar{\mathrm{G}}\frac{\partial\eta}{\partial\tau}\frac{\partial\eta}{\partial\xi} = \bar{\mathrm{G}}\frac{\partial\eta}{\partial\tau}\frac{\partial\tilde{\eta}}{\partial\xi}, \tag{24}$$

where

$$\tilde{\eta}(\xi,\tau) = A(\xi)\cos(\omega_c\tau - \pi/2 + \phi(\xi)) = A(\xi)\sin(\omega_c\tau + \phi(\xi)). \tag{25}$$

Inserting these expressions into (12) and evaluating again the time integrals, we obtain

$$[\Delta E]_0^{2\pi/\omega_c} = -\pi\omega_c \int_0^1 \sigma^* \left[\left\{ \frac{d^2 A}{d\xi^2} - A\frac{d^2\phi}{d\xi^2} \right\}^2 + \left\{ 2\frac{dA}{d\xi}\frac{d\phi}{d\xi} + A\frac{d^2\phi}{d\xi^2} \right\}^2 \right] d\xi \quad (26)$$

$$-\rho u_c \pi \left[\omega_c F(\kappa) \int_0^1 f(\xi) A^2(\xi) d\xi + u_c F(\kappa) \int_0^1 f(\xi) A^2(\xi) \frac{d}{d\xi}\phi(\xi) d\xi \right]$$

$$+\rho u_c^2 \pi \bar{G}(\kappa) \int_0^1 f(\xi) A(\xi) \frac{d}{d\xi} A(\xi) d\xi$$

$$-\rho u_c \pi \left[2\omega_c \int_0^1 g(\xi) A(\xi) \frac{dA(\xi)}{d\xi} d\xi + u_c \int_0^1 g(\xi) \left\{ 2A(\xi)\frac{dA(\xi)}{d\xi}\frac{d\phi(\xi)}{d\xi} + A^2(\xi)\frac{d\phi(\xi)}{d\xi} \right\} d\xi \right]$$

It is noted that the real part $F(\kappa) \in [\frac{1}{2}, 1]$, with $F \to 1$ for $\kappa \to 0$, and that $F \to \frac{1}{2}$ for $\kappa \to \infty$. However, $F(\kappa)$ acts as a multiplier to both of the two 'competing' fluid force terms in the second line; in other words, they are both reduced by the same factor (relative to $\max_\kappa F = 1$). Equation (26) thus shows that, in the absence of internal damping ($\sigma^* = 0$), and if $\bar{G}(\kappa) = 0$ as well, a change in the magnitude of $F(\kappa)$ will have no effect on the critical flow speed.

Consider now the term proportional to $\bar{G}$. Equivalent to (20), integration by parts and use of the boundary conditions, and the fact that $f(1) = 0$, gives

$$\int_0^1 f(\xi) A(\xi) \frac{d}{d\xi} A(\xi) d\xi = -\frac{1}{2} \int_0^1 \frac{df(\xi)}{d\xi} A^2(\xi) d\xi. \quad (27)$$

It is easy to show that $df(\xi)/d\xi < 0$ for all $\xi \in [0, 1]$. Thus the term proportional to $\bar{G}$ is positive definite. This means that, a small positive value of $\bar{G}$ will *lower* the critical flow speed u_c relative to its value for $\bar{G} = 0$. It is noted that

$$C(\kappa) \sim 1 - \frac{\pi}{2}\kappa - \mathrm{i}(\ln 2 + 1 - \gamma)\kappa, \text{ that is}, F(\kappa) \sim 1 - \frac{\pi}{2}\kappa, \quad \bar{G}(\kappa) \sim (\ln 2 + 1 - \gamma)\kappa \text{ for } \kappa \to 0_+. \quad (28)$$

Thus, according to (26) and (28), increasing κ (starting from $\kappa = 0$) will lower the critical flow speed. It is noted, finally, that the Coriolis force term and the centrifugal force term, that is, the terms in the last line of (26), do not depend on the Theodorsen function.

4 Conclusion

In the present work we have considered the dynamics of a flapping flag, employing the mathematical model of Argentina and Mahadevan [1], with the aim of clarifying (i) the physical mechanism of the flutter instability, and (ii) the effect of the complex Theodorsen function on the stability (flutter) bound. As to point (i), it was found through an energy balance analysis that a necessary condition for flutter to occur is that the gradient of the phase angle distribution function, $\partial\phi(\xi)/\partial\xi$, is negative over most part of the flag domain, $0 \leq \xi \leq 1$. This means that the flutter motion is a downstream travelling wave motion, with a 'dragging' appearance. As to point (ii), the same energy balance analysis showed that a small imaginary part in the Theodorsen function, $C = 1 - \mathrm{i}\,\epsilon$, $0 < \epsilon \ll 1$, has a destabilizing effect, i.e., the critical flow speed u_c is smaller by this approximation than by the quasi-steady approximation $C = 1 - \mathrm{i}\,0$. This prediction has been verified by careful numerical eigenvalue analyses [5]. The conclusion of point (ii) is in opposition to earlier investigations which have considered critical flow speed curves (stability diagrams) only, not the distribution of the eigenvalues for lower and higher flow speeds than the critical one. The presentation [5] emphasizes the importance of tracing the eigenvalue branches (characteristic curves) for $0 < u < u_c$. Finally, it was found that the last two terms in (4), which correspond to a Coriolis force term and a centrifugal force term, respectively, have a significant effect on the stability of the flag and thus cannot be neglected.

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