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Hidden symmetries of hyperbolic links

Han Yoshida
National Institute of Technology, Nara College

1 Introduction

Hidden symmetry of a manifold M is a homeomorphism of finite degree covers of M that does not descend to an automorphism of M. In [7], W. Neumann and A. Reid conjectured that the figure-eight knot and the two dodecahedral knots are the only hyperbolic knots in S^3 admitting hidden symmetries. Many researchers concerned with this problem. M. Macasieb, and T. W. Mattman [5] showed that $(-2,3,n)$ pretzel knot $(n \in \mathbb{N})$ does not admit hidden symmety. By using computer, O. Goodman, D. Heard and C. Hodgson [3] have verified for hyperbolic knots with 12 or fewer crossings. A. Reid and G. S. Walsh [8] showed that non-arithmetic 2-bridge knot complements admit no hidden symmetry.

For two component links, E. Chesebro and J. DeBlois [1] constructed infinitely many two components non-arithmetic link complements admitting hidden symmetries. Let $C_i (i = 1, \cdots, 3)$ be links as in Figure 1. O. Goodman, D. Heard and C. Hodgson showed that $S^3 - C_2$ and $S^3 - C_3$ have non-trivial hidden symmetries by using computer. $S^3 - C_2$ is obtained by cutting along the colored two punctured disk of $S^3 - C_1$ and regluing it. Repeat this process about the colored two punctured disk of $S^3 - C_2$. We can obtain $S^3 - C_3$. J. S. Meyer, C. Millichap and R. Trapp [6] constructed $n(\geq 6)$ component link

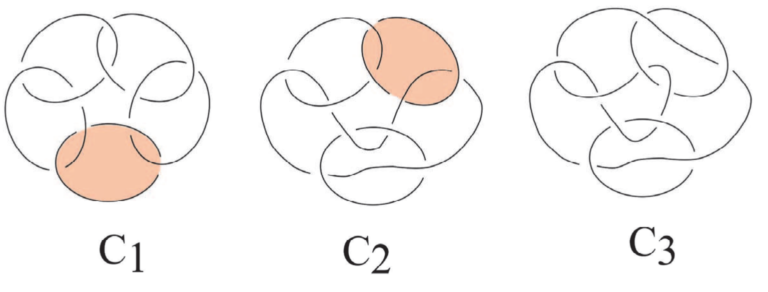

Figure 1: 5-link chain.

complements admitting hidden symmetries. They prove this by analyzing of the geometry of these link complements, including their cusp shapes and totally geodesic surfaces inside of these manifolds.

In this paper, we generalize the result of O. Goodman, D. Heard and C. Hodgson. Let L be an $n+1$-component alternating chain link as in the left side of Figure 2 $(n \geq 4)$.

Cut along the colored two punctured disk of $S^3 - L$ and reglue it. We name the resulting n-component link L_n.

Theorem 1 *$S^3 - L_n$ is non-arithmetic and admits a hidden symmetry ($n \geq 4$).*

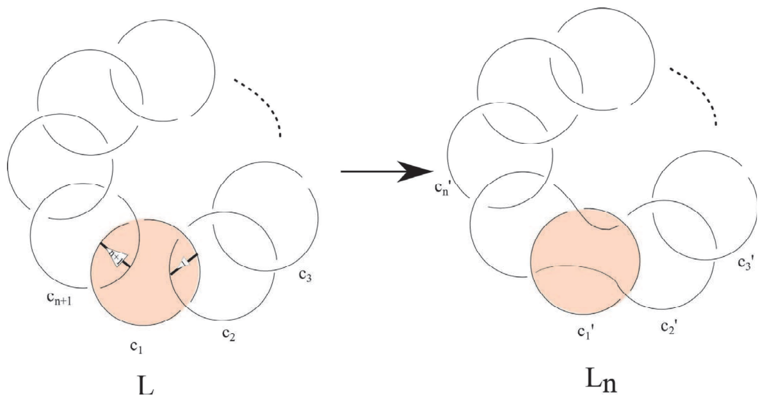

Figure 2: The link L_n.

We prove this theorem by using ideal polyhedral tessellation of $\mathbb{H}^3$.

2 Commensurator and normalizer

Two subgroups $G_1,\ G_2 < \mathrm{Isom}(\mathbb{H}^3)$ are said to be commensurable if and only if their intersection $G_1 \cap G_2$ has finite index in both G_1 and G_2. G_1 and G_2 are said to be commensurable in the wide sense if and only if there is a $h \in \mathrm{Isom}(\mathbb{H}^3)$ such that G_1 is commensurable with $h^{-1}G_2h$. The notion of commensurability can be directly transported to hyperbolic orbifolds by considering the respective fundamental groups. Then, commensurable hyperbolic orbifolds admit a finite-sheeted common covering orbifold. Commensurability is an equivalence relation.

For a Kleinian group Γ, the commensurator of Γ is defined by

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) : \mathrm{g}\Gamma\mathrm{g}^{-1} \text{ and } \Gamma \text{ are commensurable.}\}.$$

Clearly, $\mathrm{Comm}(\Gamma) > \Gamma$. Let Γ be a finitely generated Kleinian group of finite co-volume. It is well known that $\mathrm{Comm}(\Gamma)$ is a commensurabilty invariant (see [10]). $\mathrm{Comm}(\Gamma)$ contains every member of the commensurability class. G. Margulis [4] showed that $\mathrm{Comm}(\Gamma)$ is discrete if and only if Γ is non-arithmetic. For a non-arithmetic group Γ, $\mathrm{Comm}(\Gamma)$ contains every member of the commensurability class “in finite index”.

The normalizer of Γ is

$$N(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3)) : \mathrm{g}\Gamma\mathrm{g}^{-1} = \Gamma\}.$$

Clearly, $N(\Gamma) < \mathrm{Comm}(\Gamma)$. $N(\Gamma)/\Gamma \simeq \mathrm{Isom}(\mathbb{H}^3/\Gamma)$ and $N(\Gamma)$ is discrete.

If $N(\Gamma) \neq \mathrm{Comm}(\Gamma)$, we say Γ admits a hidden symmetry. For an arithmetic Kleinian group Γ, $\mathrm{Comm}(\Gamma)$ is not discrete. Thus arithmetic Kleinnian group always admits a hidden symmetry.

3 Proof of Main Theorem

Let Γ (resp. Γ_n) be a Kleinian group such that $S^3 - L = \mathbb{H}^3/\Gamma$ (resp. $S^3 - L_n = \mathbb{H}^3/\Gamma_n$). W. Neumann and A. Reid showed that $S^3 - L$ is non-arithmetic (see [7] Theorem 5.1). W. Thurston [9] showed $S^3 - L$ is obtained by glueing two ideal drums as in Figure 3. The side angles of this drum are $\arccos\left(\frac{\cos\pi/(n+1)}{\sqrt{2}}\right)$ and other angles are $\pi - 2\alpha$.

The colored two punctured disc corresponds to the colored ideal quadrilaterals as in Figure 3. $S^3 - L_n$ is obtained by cutting along the colored two punctured disk and reglueing it. Thus, $S^3 - L_n$ is obtained by glueing two ideal drums as the arrows are matched as in Figure 3. Lift the ideal polyhedral decompositions of $S^3 - L$ and $S^3 - L_n$. We can

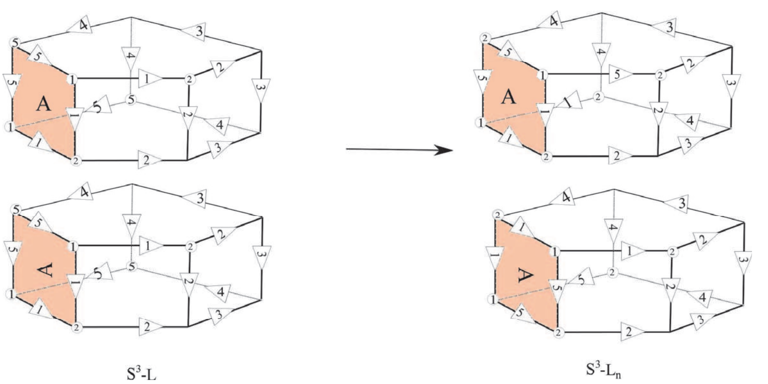

Figure 3: Ideal polyheral decompositions of $S^3 - L$ and $S^3 - L_n$ $(n = 4)$.

get the same ideal polyhedral tessellation of $\mathbb{H}^3$. Denote it by T. The symmetry group of T is discrete. Γ and Γ_n preserve the tessellation T. Thus Γ and Γ_n are commensurable with the symmetry group of T. As commensurability is an equivalence relation, Γ_n is commensurable with the non-arithmetic group Γ. Hence, $\mathrm{Comm}(\Gamma_n)$ =Comm (Γ) and Γ_n is non-arithmetic.

Let P be an ideal drum which is a lift of this ideal polyhedral decomposition. We consider the symmetry that rotates the chain L clockwise, taking each link into the next. This corresponds to $2\pi/(n+1)$-rotation about the geodesic which is perpendicular to the top and bottoms of P. Denote it by γ. As γ is a lift of a symmetry of $\mathbb{H}^3/\Gamma$, $\gamma \in N(\Gamma) < \mathrm{Comm}(\Gamma) = \mathrm{Comm}(\Gamma_{\mathrm{n}})$.

Let $c_1, \cdots, c_{n+1}$ (resp. $c'_1, \cdots, c'_n$) be the cusps of $S^3 - L$ (resp. $S^3 - L_n$) as in Figure 2. The cusp c_i corresponds to two ideal vertices of P. By cuttng and re-glueing along the colored twice punctured disk, c_2 and c_{n+1} correspond to the cusp c'_2. (See Figure 3.)

Let V_i be the ideal vertices of $\mathbb{H}^3$ which correspond to the cusp c_i'. We can see $\gamma(V_1) \neq V_i$ $(i = 1, \cdots, n-1)$. γ is not a lift of isometry of $\mathbb{H}^3/\Gamma_n$. Thus $\gamma \notin N(\Gamma_n)$.

We have $N(\Gamma_n) \neq \mathrm{Comm}(\Gamma_n)$. Hence Γ_n admits a hidden symmetry.

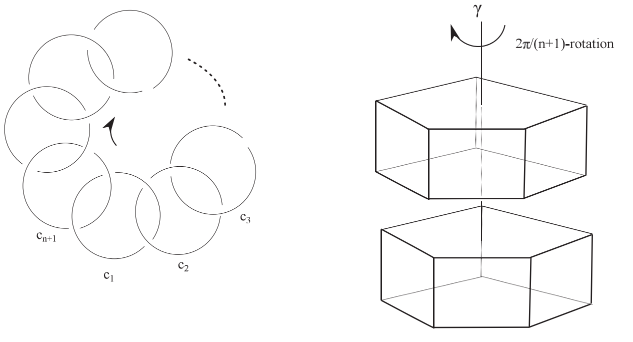

Figure 4: Rotation of $S^3 - L$ $(n = 4)$.

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National Institute of Technology, Nara College
22 Yatamachi, Yamatokoriyama Nara 639-1080
JAPAN
E-mail address: han@libe.nara-k.ac.jp

奈良工業高等専門学校　吉田 はん