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A HOMFLY-PT type invariant for integral homology 3-spheres

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In this paper, we will introduce how to construct some invariants for integral homology 3-spheres, using skein algebras.

1 Skein algebras

Let Σ be a compact connected oriented surface with nonempty boundary. We fix base point sets J^-, J^+ and J of $\partial\Sigma$ such that $J^- \cap J^+ = \emptyset$ and $\sharp J^- = \sharp J^+$. We denote by $\mathcal{S}(\Sigma, J)$ the Kauffman bracket skein module, which is the quotient of the set of the free $\mathbb{Q}[[A+1]]$-module with basis the set of the tangles in $\Sigma \times I$ with basis J by the relation shown in Fig. 2. Here, we denote by I the closed interval between 0 and 1. We denote by $\mathcal{A}(\Sigma, J^-, J^+)$ the HOMFLY-PT skein module which is the quotient of the set of the free $\mathbb{Q}[\rho][[h]]$-module with basis the set of the tangles in $\Sigma \times I$ from J^- to J^+ by the relation shown in Fig. 1. See, for example, [7] p.21 Definition 3.2 and [10] p.9 Definition 3.2. We simply denote $\mathcal{S}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{S}(\Sigma, \emptyset)$ and $\mathcal{A}(\Sigma) \stackrel{\text{def.}}{=} \mathcal{A}(\Sigma, \emptyset)$. Let e_1 and e_2 be embeddings from $\Sigma \times I$ into $\Sigma \times I$ defined by $e_1(x,t) = (x, \frac{t+1}{2})$ and $e_2(x,t) = (x, \frac{t}{2})$. For $x, x' \in \mathfrak{g}$ and $z \in \mathfrak{g}_M$, xx', xy and yx is defined by $e_1(x) \cup e_2(x')$, $e_1(x) \cup e_2(y)$ and $e_1(y) \cup e_2(x)$ where $(\mathfrak{g}, \mathfrak{g}_M) = (\mathcal{S}(\Sigma), \mathcal{S}(\Sigma, J)), (\mathcal{A}(\Sigma), \mathcal{A}(\Sigma, J^-, J^+))$. We defined a Lie bracket $[x, x']$ and a Lie action $\sigma(x)(z)$ satisfying $\epsilon_{\mathfrak{g}}[x, x'] = xx' - x'x$ and $\epsilon_{\mathfrak{g}}\sigma(x)(z) = xz - xz$ where $\epsilon_{\mathcal{S}(\Sigma)} = -A + A^{-1}$ and $\epsilon_{\mathcal{A}(\Sigma)} = h$. For details, see [7] p.23 Definition 3.8. and [10] p.16 Definition 3.13. We can consider some skein algebras and there exists some Lie homomorphisms around skein algebras.

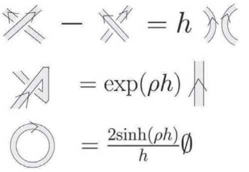

Fig. 1: $\mathcal{A}(\Sigma)$

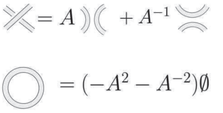

Fig. 2: $\mathcal{S}(\Sigma)$

We fix two base points $*_1, *_2 \in \partial\Sigma$. We simply denote $\mathcal{A}(\Sigma, *_1, *_2) \stackrel{\text{def.}}{=} \mathcal{A}(\Sigma, \{*_1\}, \{*_2\})$ and $\mathcal{S}(\Sigma, *_1, *_2) \stackrel{\text{def.}}{=} \mathcal{S}(\Sigma, \{*_1, *_2\})$.

1.1 $\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1} : \mathcal{A}(\Sigma) \to S(\mathbb{Q}\hat{\pi}_1(\Sigma))$

Let $\hat{\pi}_1(\Sigma)$ be the set of the conjugacy classes of $\pi_1(\Sigma)$. The quotient map is denoted by $|\cdot| : \mathbb{Q}\pi_1(\Sigma) \to \mathbb{Q}\hat{\pi}_1(\Sigma)$. Goldman defined the Lie bracket $[\cdot,\cdot] : \mathbb{Q}\hat{\pi}_1(\Sigma)\times\mathbb{Q}\hat{\pi}_1(\Sigma) \to \mathbb{Q}\hat{\pi}_1(\Sigma)$ in [1]. Kawazumi-Kuno defined the Lie action $\sigma : \mathbb{Q}\hat{\pi}_1(\Sigma) \times \mathbb{Q}\pi_1(\Sigma,*_1,*_2) \to \mathbb{Q}\pi_1(\Sigma,*_1,*_2)$ in [3]. Let $S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \stackrel{\text{def.}}{=} \bigoplus_{i=0} S^i(\mathbb{Q}\hat{\pi}_1(\Sigma))$ be the symmetric tensor algebra of $\mathbb{Q}\hat{\pi}_1(\Sigma)$ over $\mathbb{Q}$. The Leibniz rule defines the Lie bracket $[\cdot,\cdot] : S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \times S(\mathbb{Q}\hat{\pi}_1(\Sigma)) \to S(\mathbb{Q}\hat{\pi}_1(\Sigma))$. The Lie action $\sigma(\cdot)(\cdot) : S(\mathbb{Q}\hat{\pi}_1(\Sigma))\times S(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes\mathbb{Q}\pi_1(\Sigma,*_1,*_2) \to S(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes \mathbb{Q}\pi_1(\Sigma,*_1,*_2)$ is defined by $\sigma(v_1\cdots v_j)(V_1\otimes\gamma) \stackrel{\text{def.}}{=} [v_1\cdots v_j, V_1]\otimes\gamma + V_1\cdot(\sum_{j'=1}^{j} v_1\cdots v_{j'-1}\cdot v_{j'+1}\cdot v_j\otimes\sigma(v'_j)(\gamma))$. Let L be an oriented framed link in $\Sigma\times I$ having connected components $l_1,\cdots,l_j$. Then, we put $\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(L) \stackrel{\text{def.}}{=} [l_1]\cdots[l_j] \in S^j(\mathbb{Q}\hat{\pi}_1(\Sigma))$. Let T be an oriented framed tangle from $*_1$ to $*_2$ having closed components $l_1,l_2,\cdots,l_j$ and non-closed components r. Then, we also put $\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(T) \stackrel{\text{def.}}{=} [l_1]\cdots[l_j]\otimes[r] \in S^j(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes\mathbb{Q}\pi_1(\Sigma,*_1,*_2)$. The maps $\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}$ induces the $\mathbb{Q}$-linear maps

$$\begin{aligned}\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1} &: \mathcal{A}(\Sigma) \to S(\mathbb{Q}\hat{\pi}_1(\Sigma))\\ \psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1} &: \mathcal{A}(\Sigma,*_1,*_2) \to S(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes\mathbb{Q}\pi_1(\Sigma,*_1,*_2)\end{aligned}$$

satisfying

$$\begin{aligned}&[(\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(x), \psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1})(x')] = \psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}([x,x']),\\ &\sigma(\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(x))(\psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(y)) = \psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(\sigma(x)(y))\end{aligned}$$

for any $x,x' \in \mathcal{A}(\Sigma)$ and $y \in \mathcal{A}(\Sigma,*_1,*_2)$.

1.2 $\mathcal{A}(\Sigma) = \bigoplus_{x\in H_1(\Sigma,\mathbb{Z})} \mathcal{A}_x(\Sigma)$

Let L_+, L_- and L_0 be three framed links which are differ only in a closed ball as shown in the first, second and third terms of the first equation. The homology classes of L_+, L_- and L_0 equal each other. Let L and L' be two framed links which are differ only in a closed ball as shown in the first and second terms of the second equation in Fig. 1. Then, the homology classes of L equals the one of L'. Hence, there exists a direct sum decomposition

$$\mathcal{A}(\Sigma) = \bigoplus_{x\in H_1(\Sigma,\mathbb{Z})} \mathcal{A}_x(\Sigma).$$

Then we have

$$\begin{aligned}\mathcal{A}_x(\Sigma)\cdot\mathcal{A}_y(\Sigma) &\subset \mathcal{A}_{x+y}(\Sigma)\\ [\mathcal{A}_x(\Sigma),\mathcal{A}_y(\Sigma)] &\subset \mathcal{A}_{x+y}(\Sigma)\end{aligned}$$

for any $x,y \in H_1(\Sigma,\mathbb{Z})$.

1.3 $\psi_{\mathcal{A}\to\mathcal{S}} : \mathcal{A}_0(\Sigma) \to \mathcal{S}(\Sigma)$

We put $\psi'_{\mathcal{A}\to\mathcal{S}}$ by the $\mathbb{Q}$-module homomorphism from $\mathcal{A}(\Sigma,J^-,J^+)$ to $\mathcal{S}(\Sigma,J^-\cup J^+)$ defined by $T \mapsto (-A)^{w(T)}T$, $h \mapsto -A^2 + A^{-2}$ and $\exp(\rho h) = A^4$. Here $w(T) \stackrel{\text{def.}}{=}$

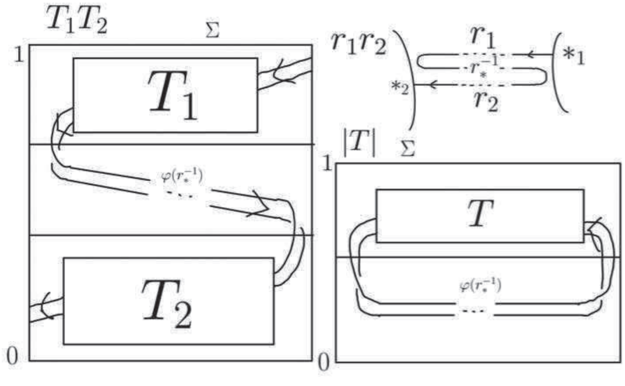

Fig. 3: T_1T_2, $|T|$, r_1r_2

$\sharp\{$positive crossings of $T\} - \sharp\{$negative crossings of $T\}$ is the writhe number of T. By [10] Proposition 7.15, $\psi'_{\mathcal{A}\to\mathcal{S}}$ is well-defined. We remark that $\psi'_{\mathcal{A}\to\mathcal{S}}$ is not a Lie algebra homomorphism. The $\mathbb{Q}$-module homomorphism $\psi_{\mathcal{A}\to\mathcal{S}}$ is defined by $\psi_{\mathcal{A}\to\mathcal{S}} \stackrel{\text{def.}}{=} \frac{1}{A+A^{-1}}\psi'_{\mathcal{A}\to\mathcal{S}}$. Then, $\psi_{\mathcal{A}\to\mathcal{S}} : \mathcal{A}_0(\Sigma) \to \mathcal{S}(\Sigma)$ is a Lie algebra homomorphism.

2 Formulas for Dehn twists

We denote by $\pi_1^+(\Sigma, *_1, *_2)$ the set of the regular homotopy classes of free immersed paths from $*_1$ to $*_2$. We fix a simple path $r_* \in \pi_1^+(\Sigma, *_1, *_2)$. A composite $\cdot : \pi_1^+(\Sigma, *_1, *_2) \times \pi_1^+(\Sigma, *_1, *_2) \to \pi_1^+(\Sigma, *_1, *_2)$ is defined by Fig. 3. We have $r \cdot r_* = r_* \cdot r = r$ for any $r \in \pi_1^+(\Sigma, *_1, *_2)$. For an immersed path $\varsigma : I \to \Sigma$, a framed oriented tangle $\varphi(\varsigma)$ is defined by $\varphi(\varsigma) : I \times I \to \Sigma \times I, (t, s) \mapsto (\varsigma(t), \frac{2-t+\epsilon s}{3})$ where $\epsilon > 0$ is a number small enough. This φ induces a $\mathbb{Q}[\rho][[h]]$-module homomorphism $\varphi : \mathcal{P}(\Sigma, *_1, *_2) \to \mathcal{A}(\Sigma, *_1, *_2)$ where $\mathcal{P}(\Sigma, *_1, *_2)$ is the quotient of the free $\mathbb{Q}[\rho][[h]]$-module with basis $\pi_1^+(\Sigma, *_1, *_2)$ by the relation which is the second equation of Fig. 1. A composit $\mathcal{A}(\Sigma, *_1, *_2) \times \mathcal{A}(\Sigma, *_1, *_2) \to \mathcal{A}(\Sigma, *_1, *_2)$ is defined by Fig. 3. Then, the $\mathbb{Q}[\rho][[h]]$-module homomorphism is a $\mathbb{Q}[\rho][[h]]$-algebra homomorphism. We define the closure $\mathcal{A}(\Sigma, *_1, *_2) \to \mathcal{A}(\Sigma)$ by Fig. 3.

2.1 Filtrations

In this subsection, we will introduce some filtrations of skein algebras and skein modules. An augmentation map $\mathrm{aug} : \mathcal{P}(\Sigma, *_1, *_2) \to \mathbb{Q}[\rho]$ is defined by $\mathrm{aug}(r) = 1$ and $\mathrm{aug}(h) = 0$ for $r \in \pi_1^+(\Sigma, *_1, *_2)$. Let $F^n\mathcal{P}(\Sigma, *_1, *_2)$ be a $\mathbb{Q}$-linear subspace generated by $\sum_{2i+j\geq n} h^i(\ker \mathrm{aug})^j$. For any $n \in \mathbb{Z}_{\geq 0}$, the filtrations $\{F^n\mathcal{A}(\Sigma, *_1, *_2)\}_{n\geq 0}$ and $\{F^n\mathcal{A}(\Sigma)\}_{n\geq 0}$ are defined by

$$F^n\mathcal{A}(\Sigma, *_1, *_2) \stackrel{\text{def.}}{=} \sum_{i_1+i_2+\cdots\geq n} \varphi(F^{i_1}\mathcal{P}(\Sigma, *_1, *_2))|\varphi(F^{i_2}\mathcal{P}(\Sigma, *_1, *_2))|\cdots,$$

$$F^n\mathcal{A}(\Sigma) \stackrel{\text{def.}}{=} \sum_{i_1+\cdots\geq n} |\varphi(F^{i_1}\mathcal{P}(\Sigma, *_1, *_2))||\varphi(F^{i_2}\mathcal{P}(\Sigma, *_1, *_2))|\cdots.$$

These filtrations $\{F^n\mathcal{A}(\Sigma,*_1,*_2)\}_{n\geq 0}$ and $\{F^n\mathcal{A}(\Sigma)\}_{n\geq 0}$ induces the filtrations $\{F^nS(\mathbb{Q}\hat{\pi}_1(\Sigma))\}_{n\geq 0}$, $\{F^nS(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes\mathbb{Q}\pi_1(\Sigma,*_1,*_2)\}_{n\geq 0}$, $\{F^n\mathbb{Q}\hat{\pi}_1(\Sigma)\}_{n\geq 0}$, $\{F^n\mathbb{Q}\pi_1(\Sigma,*_1,*_2)\}_{n\geq 0}$, $\{F^n\mathcal{S}(\Sigma)\}_{n\geq 0}$ and $\{F^n\mathcal{S}(\Sigma,*_1,*_2)\}_{n\geq 0}$ by

$$F^n\mathcal{S}(\Sigma) \stackrel{\text{def.}}{=} \psi_{\mathcal{A}\to\mathcal{S}}(F^n\mathcal{A}(\Sigma)), F^n\mathcal{S}(\Sigma,*_1,*_2) \stackrel{\text{def.}}{=} \psi_{\mathcal{A}\to\mathcal{S}}(F^n\mathcal{A}(\Sigma,*_1,*_2)),$$
$$F^nS(\mathbb{Q}\hat{\pi}_1(\Sigma)) \stackrel{\text{def.}}{=} \psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(F^n\mathcal{A}(\Sigma)), F^n((S(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes\mathbb{Q}\pi_1(\Sigma,*_1,*_2))) \stackrel{\text{def.}}{=} \psi_{\mathcal{A}\to\mathbb{Q}\hat{\pi}_1}(F^n\mathcal{A}(\Sigma,*_1,*_2)),$$
$$F^n\mathbb{Q}\hat{\pi}_1(\Sigma) \stackrel{\text{def.}}{=} \mathbb{Q}\hat{\pi}_1(\Sigma)\cap S(\mathbb{Q}\hat{\pi}_1(\Sigma)),$$
$$F^n\mathbb{Q}\pi_1(\Sigma,*_1,*_2) \stackrel{\text{def.}}{=} \mathbb{Q}\pi_1(\Sigma,*_1,*_2)\cap F^n(S(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes\mathbb{Q}\pi_1(\Sigma,*_1,*_2)).$$

These filtrations satisfy

$$[F^i\mathfrak{g},F^j\mathfrak{g}]\subset F^{i+j-2}\mathfrak{g}, \sigma(F^i\mathfrak{g})(F^j\mathfrak{g}_M)\subset F^{i+j-2}\mathfrak{g}_M$$

for $(\mathfrak{g},\mathfrak{g}_M)=(\mathcal{A}(\Sigma),\mathcal{A}(\Sigma,*_1,*_2))$, $(\mathcal{S}(\Sigma),\mathcal{S}(\Sigma,*_1,*_2))$, $(S(\mathbb{Q}\hat{\pi}_1(\Sigma)),S(\mathbb{Q}\hat{\pi}_1(\Sigma))\otimes\mathbb{Q}\pi_1(\Sigma,*_1,*_2))$, $(\mathbb{Q}\hat{\pi}_1(\Sigma),\mathbb{Q}\pi_1(\Sigma,*_1,*_2))$. For $\mathfrak{g}=\mathcal{P}$, $\mathcal{A}$, $\mathcal{S}$, $\mathbb{Q}\hat{\pi}_1$, $\mathbb{Q}\pi_1$, Their completion is denoted by

$$\widehat{\mathfrak{g}}(\Sigma,*_1,*_2) \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty}\mathfrak{g}(\Sigma,*_1,*_2)/F^i\mathfrak{g}(\Sigma,*_1,*_2),$$
$$\widehat{\mathfrak{g}}(\Sigma) \stackrel{\text{def.}}{=} \varprojlim_{i\to\infty}\mathfrak{g}(\Sigma)/F^i\mathfrak{g}(\Sigma).$$

2.2 A formula for Dehn twists on $\mathbb{Q}\hat{\pi}_1(\Sigma)$

Kawazumi-Kuno and Massuyeau-Turaev give a formula for the action of Dehn twists on $\widehat{\mathbb{Q}\pi_1}(\Sigma,*_1,*_2)$. For a simple closed curve c, they put $L_{\mathbb{Q}\hat{\pi}_1}(c) \stackrel{\text{def.}}{=} \frac{1}{2}|(\log(r_c))^2|$ where r_c is an element of $\pi_1(\Sigma)$ such that $|r_c|=c$.

Theorem 2.1 ([2] [3][5])**.** *For a Dehn twist t_c along c, we obtain*

$$t_c(\cdot)=\exp(\sigma(L_{\mathbb{Q}\hat{\pi}_1}(c)))(\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty}\frac{1}{i!}(\sigma(L_{\mathbb{Q}\hat{\pi}_1}(c)))^i(\cdot):\widehat{\mathbb{Q}\pi_1}(\Sigma,*_1,*_2)\to\widehat{\mathbb{Q}\pi_1}(\Sigma,*_1,*_2).$$

2.3 A formula for Dehn twists on $\mathcal{A}(\Sigma)$

We denote by φ_n the $\mathbb{Q}$-module homomorphism from $(\hat{\mathcal{P}}(\Sigma,*_1,*_2))^{\hat{\otimes}n}$ to $\widehat{\mathcal{A}}(\Sigma,*_1,*_2)$ defined by $\varphi_n(r_1\otimes\cdots\otimes r_n)=|\varphi(r_1)|\cdots|\varphi(r_n)|$. By induction, we define $\lambda^{[n]}(X_1,\cdots,X_n)$ by $\lambda^{[1]}(X_1)=\frac{1}{2X_1}(\log(X_1))^2$ and

$$\lambda^{[n+1]}(X_1,\cdots,X_{n+1}) \stackrel{\text{def.}}{=} \frac{\lambda^{[n]}(X_1,\cdots,X_n)-\lambda^{[n]}(X_2,\cdots,X_{n+1})}{X_1-X_{n+1}}.$$

For a simple closed curve c, let r_c be an element of $\pi_1^+(\Sigma,*_1,*_2)$ such that $|\varphi(r_c)|$ is an element of $\mathcal{A}(\Sigma)$ represented by a knot presented by a diagram c. We put

$$L_{\mathcal{A}}(c) \stackrel{\text{def.}}{=} (\frac{h/2}{\operatorname{arcsinh}(h/2)})^2(\sum_{n=1}^{\infty}\frac{h^{n-1}\exp(n\rho h)}{n}\varphi_n(r_{1,n}\cdots r_{n,n}\lambda^{[n]}(r_{1,n},\cdots,r_{n,n}))-\frac{1}{3}\rho^3h^2),$$

where $r_{i,n} \stackrel{\text{def.}}{=} {r_*}^{\otimes(i-1)} \otimes \exp(-\rho h) r_c \otimes {r_*}^{\otimes n-i}$. Here, for $F(X_1, X_2, \cdots, X_n) = \sum a_{i_1,i_2,\cdots,i_n}(X_1 - 1)^{i_1}(X_2-1)^{i_2}\cdots(X_n-1)^{i_n} \in \mathbb{Q}[[X_1-1, X_2-1, \cdots, X_n-1]]$ and $x_1, x_2, \cdots, x_n \in r_*^{\otimes n} + (\ker \mathrm{aug})^{\otimes n}$, we define

$$F(x_1, x_2, \cdots, x_n) = \sum a_{i_1,i_2,\cdots,i_n}(x_1 - r_*^{\otimes n})^{i_1}(x_2 - r_*^{\otimes n})^{i_2}\cdots(x_n - r_*^{\otimes n})^{i_n}.$$

Here $r_*^{\otimes n}$ is a unit of $\hat{\mathcal{P}}(\Sigma, *_1, *_2)^{\hat{\otimes} n}$.

Theorem 2.2 ([10] p.29 Theorem 5.2)**.** *The element $L_{\mathcal{A}}(c)$ is well-defined.*

Theorem 2.3 ([10] p.28 Theorem 5.1)**.** *For a Dehn twist t_c along c, we obtain*

$$t_c(\cdot) = \exp(\sigma(L_{\mathcal{A}}(c)))(\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!}(\sigma(L_{\mathcal{A}}(c)))^i(\cdot) : \widehat{\mathcal{A}}(\Sigma, *_1, *_2) \to \widehat{\mathcal{A}}(\Sigma, *_1, *_2).$$

2.4 A formula for Dehn twists on $\mathcal{S}(\Sigma)$

For a simple closed curve c, we denote by

$$L_{\mathcal{S}}(c) \stackrel{\text{def.}}{=} \frac{-A + A^{-1}}{4\log(-A)}(\operatorname{arccosh}(\frac{-c}{2}))^2 - (-A + A^{-1})\log(-A).$$

Here we also denote by c an element of $\mathcal{S}(\Sigma)$ represented by a knot presented by a diagram c.

Theorem 2.4 ([7] p.26 Theorem 4.1)**.** *For a Dehn twist t_c along c, we obtain*

$$t_c(\cdot) = \exp(\sigma(L_{\mathcal{S}}(c)))(\cdot) \stackrel{\text{def.}}{=} \sum_{i=0}^{\infty} \frac{1}{i!}(\sigma(L_{\mathcal{S}}(c)))^i(\cdot) : \widehat{\mathcal{S}}(\Sigma, *_1, *_2) \to \widehat{\mathcal{S}}(\Sigma, *_1, *_2).$$

3 Embeddings of the Torelli group into $(F^3\widehat{\mathfrak{g}}(\Sigma_{g,1}), \mathrm{bch})$ for $\mathfrak{g} = \mathbb{Q}\hat{\pi}_1, \mathcal{S}, \mathcal{A}$

We call the kernel of the action the mapping class group of $\Sigma_{g,1}$ on the homology class of $\Sigma_{g,1}$ the Torelli group. The Torelli group $\mathcal{I}(\Sigma_{g,1})$ is generated by $t_{c_1}{t_{c_2}}^{-1}$ for a pair of disjoint simple closed curves (c_1, c_2) bounding a surface, which is called by a bounding pair. We can consider $F^3\widehat{\mathfrak{g}}(\Sigma_{g,1})$ as the group where the composition is the Baker-Campbell-Hausdorff series, which is simply denote by $\mathrm{bch}(\cdot,\cdot)$. We can consider $F^3\widehat{\mathfrak{g}}(\Sigma_{g,1})$ as the group whose composition is the Baker-Campbell-Hausdorff series, which is simply denote by $\mathrm{bch}(\cdot,\cdot)$.

Theorem 3.1 ($\mathfrak{g} = \mathbb{Q}\hat{\pi}_1$: [2] [3], $\mathfrak{g} = \mathcal{S}$: [8] p.15 Theorem 3.13, p.16 Corollary 3.15, $\mathfrak{g} = \mathcal{A}$: [10] p.54 Theorem 7.13, Corollary 7.14.)**.** *The group homomorphism $\zeta_{\mathfrak{g}}$ from $\mathcal{I}(\Sigma_{g,1})$ to $(F^3\mathfrak{g}(\Sigma_{g,1}), \mathrm{bch})$ defined by $\zeta(t_{c_1}{t_{c_2}}^{-1}) = L_{\mathfrak{g}}(c_1) - L_{\mathfrak{g}}(c_2)$ for any bounding pair (c_1, c_2) is well-defined and injective. Furthermore, for $\xi \in \mathcal{I}(\Sigma_{g,1})$, we have $\xi(\cdot) = \exp(\sigma(\zeta_{\mathfrak{g}}(\xi)))(\cdot) : \widehat{\mathfrak{g}}(\Sigma, *_1, *_2) \to \widehat{\mathfrak{g}}(\Sigma, *_1, *_2)$.*

Let $\{\mathcal{I}^{(n)}(\Sigma_{g,1})\}_{n\geq 1}$ be the lower central series, defined by $\mathcal{I}^{(1)}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} \mathcal{I}(\Sigma_{g,1})$ and $\mathcal{I}^{(n+1)}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} [\mathcal{I}(\Sigma_{g,1}), \mathcal{I}^{(n)}(\Sigma_{g,1})]$. Using the embedding $\zeta_{\mathfrak{g}} : \mathcal{I}(\Sigma_{g,1}) \to F^3\mathfrak{g}(\Sigma_{g,1})$, we can define a filtration $\{\mathcal{M}^{(n)}_{\mathfrak{g}}\}_{n\geq 1}$ of $\mathcal{I}(\Sigma_{g,1})$ by $\mathcal{M}^{(n)}_{\mathfrak{g}}(\Sigma_{g,1}) \stackrel{\text{def.}}{=} \zeta_{\mathfrak{g}}^{-1}(F^{n+2}\widehat{\mathfrak{g}}(\Sigma_{g,1}))$ for $\mathfrak{g} = \mathbb{Q}\hat{\pi}_1, \mathcal{S}, \mathcal{A}$. We call $\{\mathcal{M}^{(n)}_{\mathbb{Q}\hat{\pi}_1}\}_{n\geq 1}$ the Johnson filtration. They satisfy

$$\mathcal{I}^{(n)}(\Sigma_{g,1}) \subset \mathcal{M}^{(n)}_{\mathcal{A}}(\Sigma_{g,1}) \subset \mathcal{M}^{(n)}_{\mathcal{S}}(\Sigma_{g,1}) \cap \mathcal{M}^{(n)}_{\mathbb{Q}\hat{\pi}_1}(\Sigma_{g,1}).$$

4 Invariants for integral homology 3-spheres

In this subsection, the symbol $\mathfrak{g}$ is a symbol $\mathcal{S}$ or a symbol $\mathcal{A}$. Using the embedding $\zeta_{\mathfrak{g}} : \mathcal{I} \to F^3\widehat{\mathfrak{g}}(\Sigma_{g,1})$ we obtain an invariant for integral homology 3-spheres.

Theorem 4.1 ($\mathfrak{g} = \mathcal{S}$: [9] p.1 Theorem 1.1, $\mathfrak{g} = \mathcal{A}$: [10] p.60 Theorem 9.1.)**.** *Fix a Heegaard splitting of $S^3 = H^+_g \cup_\iota H^-_g$ where H^+_g and H^-_g are handle bodies of genus g and ι is a diffeomorphism from ∂H^+_g to ∂H^-_g. We consider $\Sigma_{g,1}$ as the closure of ∂H^+_g except for a closed disk. Let $e : \Sigma_{g,1} \to S^3$ be a tubular neighborhood inducing $e_* : \widehat{\mathfrak{g}}(\Sigma_{g,1}) \to R_{\mathfrak{g}}$ where $R_{\mathcal{S}} \stackrel{\text{def.}}{=} \mathbb{Q}[[A+1]]$ and $R_{\mathcal{A}} \stackrel{\text{def.}}{=} \mathbb{Q}[\rho][[h]]$. Then, the map $Z_{\mathfrak{g}} : \mathcal{I}(\Sigma_{g,1}) \to R_{\mathfrak{g}}$ defined by $Z_{\mathfrak{g}}(\xi) \stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \frac{1}{i!\epsilon_{\mathfrak{g}}^i} e*((\zeta_{\mathfrak{g}}(\xi))^i)$ induces a map $z_{\mathfrak{g}} : \mathcal{H} \stackrel{\text{def.}}{=} \{\text{integral homology 3 spheres}\} \to R_{\mathfrak{g}}, H^+_g \cup_{\iota\circ\xi} H^-_g \mapsto Z(\xi)$.*

We denote

$$z^{\mathrm{sl}_2}(M) = 1 + z^{\mathrm{sl}_2}_1(M)(q-1) + z^{\mathrm{sl}_2}_2(M)(q-1)^2 + \cdots$$

the invariant of an integral homology 3-sphere M defined by T. Ohtsuki [6]. We also denote

$$z^{\mathrm{sl}_N}(M) = 1 + z^{\mathrm{sl}_N}_1(M)(q-1) + z^{\mathrm{sl}_N}_2(M)(q-1)^2 + \cdots$$

using the sl_N-quantum group in [4].

The theorem will appear in our paper in preparation.

Theorem 4.2 ([11])**.** *We obtain $z^{\mathrm{sl}_2}(\cdot) = z_{\mathcal{S}}(\cdot)_{|A^4=q} = z_{\mathcal{A}}(\cdot)_{|\exp(\rho h)=q, h=-q^{\frac{1}{2}}+q^{-\frac{1}{2}}}$ and $z^{\mathrm{sl}_N}(\cdot) = z_{\mathcal{A}}(\cdot)_{|\exp(\rho h)=q^{N/2}, h=-q^{\frac{1}{2}}+q^{-\frac{1}{2}}}$.*

Since $e_*(F^{2n-1}\mathfrak{g}) \subset (\epsilon^n_{\mathfrak{g}})$, we have

$$z_{\mathfrak{g}}(H^+_g \cup_{\iota\circ\xi} H^-_g) \in 1 + (\epsilon^n_{\mathfrak{g}})$$

for $\xi \in \mathcal{M}^{(2n-1)}_{\mathfrak{g}}$. Hence, if $\xi \in \mathcal{M}^{(2n-1)}_{\mathcal{S}}$, we have

$$z^{\mathrm{sl}_2}(H^+_g \cup_{\iota\circ\xi} H^-_g) \in 1 + ((q-1)^n).$$

Furthermore, if $\xi \in \mathcal{M}^{(2n-1)}_{\mathcal{A}}$, we have

$$z^{\mathrm{sl}_N}(H^+_g \cup_{\iota\circ\xi} H^-_g) \in 1 + ((q-1)^n)$$

for any N.

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