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RECENT PROGRESSES ON THE VOLUME CONJECTURES FOR CERTAIN QUANTUM INVARINATS

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1. Volume Conjectures

In [36], Witten interpreted values of the Jones polynomial using the Chern-Simons gauge theory, and constructed a sequence of complex valued 3-manifold invariants based on this idea. This approach was mathematically rigorously formalized by Reshetikhin and Turaev [26, 27] though the representation theory of quantum groups, where they generalized the Jones polynomial to a sequence of polynomial invariants of a link, later called the colored Jones polynomials of that link. They also defined a sequence of 3-manifold invariants corresponding to Witten's invariants. The Reshetikhin-Turaev construction of 3-manifold invariants starts from a surgery description [21] of the manifold, and evaluates the colored Jones polynomials of the surgery data at certain roots of unity. In [33], Turaev and Viro developed a different approach from a triangulation of a 3-manifold constructing real valued invariants of the manifold. These Turaev-Viro invariants turned out to be equal to the square of the norm of the Reshetikhin-Turaev invariants [28, 32, 35].

Using quantum dilogarithm functions, Kashaev [19, 20] defined for each integer n a complex valued link invariant. He observed in a few examples, and conjectured in the general case, that the absolute value of these invariants grow exponentially with n, and that the growth rate is given by the hyperbolic volume of the complement of the link. In [18], Murakami and Murakami showed that Kashaev's invariants coincide with the values of the colored Jones polynomials at a certain root of unity, and reformulated Kashaev's conjecture as follows.

Volume Conjecture ([20, 18])**.** *For a hyperbolic link L in S^3, let $J_n(L,q)$ be its n-th colored Jones polynomial. Then*

$$\lim_{n\to+\infty}\frac{2\pi}{n}\log\left|J_n(L,e^{\frac{2\pi\sqrt{-1}}{n}})\right| = \mathrm{Vol}(S^3\setminus L),$$

where $\mathrm{Vol}(S^3\setminus L)$ is the hyperbolic volume of the complement of L.

This conjecture has now been proved for a certain number of cases: the figure-eight knot [18], all hyperbolic knots with at most seven crossings [22, 23, 24], the Borromean rings [14], the twisted Whitehead links [37] and the Whitehead chains [34]. Various extensions of this conjecture have been proposed, and proved for certain cases.

In [6], Chen and the author investigated the asymptotic behavior of the Reshetikhin-Turaev and the Turaev-Viro invariants evaluated at the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$. Based on numerical computations, they made the following

Conjecture 1.1 ([6])**.** *For a hyperbolic 3-manifold M, let $\mathrm{TV}_r(M,q)$ be its Turaev-Viro invariant and let $\mathrm{Vol}(M)$ be its hyperbolic volume. Then for r running over all odd integers and for $q =$*

$e^{\frac{2\pi\sqrt{-1}}{r}}$,

$$\lim_{r\to+\infty}\frac{2\pi}{r}\log\big(\mathrm{TV}_r(M,q)\big)=\mathrm{Vol}(M).$$

Conjecture 1.2 ([6]). *Let M be a closed oriented hyperbolic 3-manifold and let $\mathrm{RT}_r(M,q)$ be its Reshetikhin-Turaev invariants. Then for $q=e^{\frac{2\pi\sqrt{-1}}{r}}$ with r odd and for a suitable choice of the arguments,*

$$\lim_{r\to+\infty}\frac{4\pi\sqrt{-1}}{r}\log\big(\mathrm{RT}_r(M,q)\big)=\mathrm{CS}(M)+\mathrm{Vol}(M)\sqrt{-1}\mod \pi^2\mathbb{Z},$$

where $\mathrm{CS}(M)$ denotes the Chern-Simons invariant of the hyperbolic metric of M multiplied by $2\pi^2$.

2. Recent progress

2.1. **Ohtsuki's result and method.** The first family of examples for which Conjecture 1.1 and 1.2 hold was given by Ohtsuki [25], where he also obtained a full asymptotic expansion of the Reshetikhin-Turaev invariants.

Theorem 2.1 ([25]). *Let p be an integer and let M_p be the 3-manifold obtained from the figure-8 knot by a p Dehn-filling. Then for $|p|>4$. the Reshetikhin-Turaev invariant $\mathrm{RT}_r(M_p,e^{\frac{2\pi i}{r}})$ is expanded as $r\to\infty$ in the following form,*

$$\mathrm{RT}_r(M_p,e^{\frac{2\pi i}{r}})=e^{\frac{r}{4\pi i}(\mathrm{CS}(M_p)+i\mathrm{Vol}(M_p))}\epsilon(p,r)\omega(M_p)r^{\frac{3}{2}}\Big(1+\sum_{k=1}^{d}\kappa_k(M_p)(\frac{4\pi i}{r})^k+O(\frac{1}{r^{d+1}})\Big),$$

where $\epsilon(p,r)$ is a concrete root of unity depending only on p and r, and $\omega(M_p)$ and $\kappa_k(M_p)$ are constants determined by M_p.

In particular, Theorem 2.1 implies that Conjecture 1.1 and 1.2 hold for M_p. Based on this, he further conjectured that $\mathrm{RT}_r(M,e^{\frac{2\pi i}{r}})$ of any closed hyperbolic 3-manifold M can be expanded in the same form as above, and the sub-leading coefficient $\omega(M)$ is closely related to the Reidemeister torsion of M. He also expected that $\kappa_k(M)$ define new invariants of M.

The method Ohtsuki used to get the result can be roughly summarized in the following three steps. In the first step he wrote the quantum invariants as the sum of values $f(\mathbf{n})$ of a complex holomorphic function f at the (multi)-integers. The function f comes from the quantum dilogarithm function which serves an integral representation of quantum factorials. In the second step, by using Poisson Summation Formula, he wrote the sum of $f(\mathbf{n})$ into the sum of the Fourier coefficients $\hat{f}(\mathbf{m})$ of f, each of which is an integral of complex holomorphic function. By carefully verifying a technical condition he posed, he can prove that among all these Fourier coefficients $\hat{f}(\mathbf{m})$, two of them dominate the asymptotics. Finally, by rigorously using the Saddle Point Method, he managed to analyze the growth rate of the two dominating Fourier coefficients, and hence obtained the growth rate of the invariants.

2.2. **Another approach.** By studying the Reshetikhin-Turaev and the Turaev-Viro TQFTs [3, 4, 33], Detcherry, Kalfagianni and the author [11] gave a formula relating the Turaev-Viro invariants of the complement of a link L in S^3 to the values of the colored Jones polynomials of L, and Belletti, Detcherry, Kalfagianni and the author [2] gave a formula relating the Turaev-Viro invariants the complement of a link L in an oriented closed 3-manifold M to the relative Reshetikhin-Turaev invariants of the pair (M,L). Using these relations, they verified Conjecture 1.1 for the figur-8 knot, the Borromean rings complement [11] and the fundamental shadow link complements [2].

For a multi-integer $\mathbf{n} = (n_1, \dots, n_l)$ of l components, we use the notation $1 \leqslant \mathbf{n} \leqslant m$ to describe all such multi-integers with $1 \leqslant n_k \leqslant m$ for each $k \in \{1, \dots, l\}$. Given a link L in S^3 with l components, let $J_{\mathbf{n}}(L,t)$ denote the $\mathbf{n}$-th colored Jones polynomial of L whose k-th component is colored by n_k. If all the components of L are colored by the same integer n, then we simply denote $J_{(n,\dots,n)}(L,t)$ by $J_n(L,t)$. If L is a knot, then $J_n(L,t)$ is the usual i-th colored Jones polynomial. The polynomials are indexed so that $J_1(L,t) = 1$ and $J_2(L,t)$ is the ordinary Jones polynomial, and are normalized so that

$$J_n(U,t) = [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

for the unknot U, where by convention $t = q^2$.

Theorem 2.2 ([11])**.** *Let L be a link in S^3 with l components.*

(1) *For an integer $r \geqslant 3$, we have*

$$\mathrm{TV}_r(S^3 \setminus L, e^{\frac{\pi i}{r}}) = \frac{2\sin^2\left(\frac{\pi i}{r}\right)}{r} \sum_{1 \leqslant \mathbf{n} \leqslant r-1} |J_{\mathbf{n}}(L,t)|^2.$$

(2) *For an odd integer $r \geqslant 3$, a primitive $2r$-th root of unity A and $q = A^2$, we have*

$$\mathrm{TV}_r(S^3 \setminus L, e^{\frac{2\pi i}{r}}) = \frac{2^{l+1}\sin^2\left(\frac{2\pi i}{r}\right)}{r} \sum_{1 \leqslant \mathbf{n} \leqslant \frac{r-1}{2}} |J_{\mathbf{n}}(L,t)|^2.$$

As a consequence, they got the following

Theorem 2.3 ([11])**.** *Let L be either the figure-eight knot or the Borromean rings, and let M be the complement of L in S^3. Then*

$$\lim_{r \to +\infty} \frac{2\pi}{r} \log \mathrm{TV}_r(M, e^{\frac{2\pi i}{r}}) = \lim_{m \to +\infty} \frac{4\pi}{2m+1} \log |J_m(L, e^{\frac{4\pi i}{2m+1}})| = \mathrm{Vol}(M),$$

where $r = 2m+1$ runs over all odd integers.

Remark 2.4. The asymptotic behavior of the values of $J_m(L,t)$ at $t = e^{\frac{2\pi i}{m+\frac{1}{2}}}$ is not predicted either by the original volume conjecture [20, 18] or by its generalizations [15, 17]. Theorem 2.3 seems to suggest that these values grow exponentially in m with growth rate the hyperbolic volume. This is somewhat surprising because as noted in [14] that for any positive integer k, $J_m(L, e^{\frac{2\pi i}{m+k}})$ grows only polynomially in m. They ask the following question.

Question 2.5. *Is it true that for any hyperbolic link L in S^3, we have*

$$\lim_{m \to +\infty} \frac{2\pi}{m} \log |J_m(L, e^{\frac{2\pi i}{m+\frac{1}{2}}})| = \mathrm{Vol}(S^3 \setminus L)?$$

Given a link L in closed oriented 3-manifold M, let $\mathrm{RT}_r(M, L, \mathbf{n})$ denote the relative Reshetikhin-Turaev invariants [3] of the pair (M, L) with the k-th component of L colored by n_k.

Theorem 2.6 ([2])**.** *Let L be a link in a closed oriented 3-manifold M with l components.*

$$\mathrm{TV}_r(M \setminus L) = \sum_{0 \leqslant \mathbf{n} \leqslant r-2} |\mathrm{RT}_r(M, L, \mathbf{n})|^2,$$

where the sum is over multi-integers with l components.

As a consequence, they got the following

Theorem 2.7 ([2]). *Conjecture 1.1 holds for the complements of all fundamental shadow links.*

Remark 2.8. As proved by Costantino and Thurston [8], the fundamental shadow link complements form a universal class in the sense that any orientable 3-manifold with empty or toroidal boundary is obtained from a complement of a fundamental shadow by Dehn filling. Therefore, if Ohtsuki's method mentioned in Section 2.1 could be made to work in this situation, one would be able to solve Conjectures 1.1 and 1.2 for all closed and cusped hyperbolic 3-manifolds.

A key observation in their proof of Theorem 2.7 besides Theorem 2.6 is below a sharp upper bound on the growth of the quantum 6j-symbol.

Proposition 2.9 ([2]). (1) *For any sequence of r-admissible 6-tuples $\{(n_1^{(r)}, \dots, n_6^{(r)})\}$, we have*

$$\limsup_{r\to+\infty} \frac{2\pi}{r} \log \left| \begin{matrix} n_1^{(r)} & n_2^{(r)} & n_3^{(r)} \\ n_4^{(r)} & n_5^{(r)} & n_6^{(r)} \end{matrix} \right|_{q=e^{\frac{2\pi i}{r}}} \leqslant v_8,$$

where v_8 is the volume of the regular ideal hyperbolic octahedron.

(2) *If $n_i^{(r)} = \frac{r\pm1}{r}$ for each $i \in \{1, \dots, 6\}$, we have*

$$\lim_{r\to+\infty} \frac{2\pi}{r} \log \left| \begin{matrix} n_1^{(r)} & n_2^{(r)} & n_3^{(r)} \\ n_4^{(r)} & n_5^{(r)} & n_6^{(r)} \end{matrix} \right|_{q=e^{\frac{2\pi i}{r}}} = v_8.$$

Part (2) of Proposition 2.9 is actually a special case of a result originally due to Costantino [7] at the root $q = e^{\frac{\pi i}{r}}$ and recaptured independently by Chen and Murakami [5] and Detcherry and the author [12] at the root $q = e^{\frac{2\pi i}{r}}$ for odd r.

Theorem 2.10 ([7, 5, 12]). *Let $\{(n_1^{(r)}, \dots, n_6^{(r)})\}_r$ be a sequence of r-admissible 6-tuples with*

$$\alpha_i = \lim_{r\to\infty} \frac{2\pi n_i^{(r)}}{r},$$

and let $\theta_i = |\pi - \alpha_i|$. If $(\theta_1 \dots, \theta_6)$ are the dihedral angles of a hyperideal hyperbolic tetrahedral Δ, then

$$\lim_{r\to+\infty} \frac{2\pi}{r} \log \left| \begin{matrix} n_1^{(r)} & n_2^{(r)} & n_3^{(r)} \\ n_4^{(r)} & n_5^{(r)} & n_6^{(r)} \end{matrix} \right|_{q=e^{\frac{2\pi i}{r}}} = \mathrm{Vol}(\Delta).$$

It worth mentioning that in [5], Chen and Murakami also computed the second leading term of the asymptotic expansion of quantum $6j$-symbols, which is closely related to the Gram matrix of Δ.

Remark 2.11. Since quantum $6j$-symbols are the main building blocks of the Turaev-Viro invariants and hyperbolic tetrahedral are building blocks of the hyperbolic structure, Theorem 2.10 essentially says that Conjecture 1.1 is ture at least at the level of building blocks.

3. Relationship to the AMU Conjecture

According to Nielsen-Thurston's classification of the elements of the mapping class group of surfaces, every irreducible orientation preserving self-homeomorphism of a surface of finite type is either periodic (of finite order) or pseudo-Anosov (preserving two transverse measure laminations). Here a self-homeomorphism being irreducible means that it does not restrict of a proper subsurface. In [1], Andersen, Masbaum and Ueno made the following

Conjecture 3.1 ([1]). *Let Σ be a orientable surface of finite type, let ϕ be a pseudo-Anosov self-homeomorphism of Σ, and let $\{\rho_r\}_r$ be the sequence of the Turaev-Viro representations of the mapping class group of Σ. Then for r sufficiently large, $\rho_r([\phi])$ is a linear transformation of infinite order.*

Combined with the fact that the image of a finite order element under any group representation is of finite order, the AMU Conjecture essentially claims that the sequence of Turaev-Viro representations of the mapping class groups respects the Nielsen-Thurston classification. The similar conjecture can be made for the Reshetikhin-Turaev representations, which are a sequence of projective representations of mapping class group of surfaces. The AMU conjecture is known to be true for punctured spheres [1, 13] and the once-punctured torus [29]. Recently, Marché and Santharoubane [16] related the Turaev-Viro representations to representations of the fundamental group of surfaces, and provide an efficient algorithm of determining whether an element of the fundamental group can be represented by a simple closed curve on the surface, assuming that the AMU Conjecture is true.

Observed by Santharoubane [30] (see also Detcherry and Kalfagianni [10]), the AMU Conjecture is a consequence of the following a weaker version of Conjecture 1.1.

Conjecture 3.2. *Let M be a hyperbolic 3-manifold with finite volume, and let $\mathrm{TV}_r(M,q)$ be its r-th Turaev-Viro invariant at the root of unity q. Then for r running over all the odd integers,*

$$\liminf_{r\to+\infty}\frac{1}{r}\ln \mathrm{TV}_r(M,e^{\frac{2\pi i}{r}})>0.$$

The relationship between the two conjectures mentioned above is given by the underlying TQFTs. Roughly speaking, the mapping cylinder MC_ϕ of ϕ can be considered as a cobordism from Σ to itself. Hence for each r, the Turaev-Viro TQFT assigns MC_ϕ a linear map, which is exactly $\rho_r([\phi])$ by the construction of the Turaev-Viro representation. By the TQFT axioms, the trace of $\rho_r([\phi])$ equals to the Turaev-Viro invariant $\mathrm{TV}_r(M_\phi)$ of the mapping torus M_ϕ of ϕ. Since ϕ is pseudo-Anosov, Thurston's result shows that M_ϕ is hyperbolic. Then Conjecture 3.2 implies that $\mathrm{TV}_r(M)$ grows at least exponentially at particular roots of unity. On the other hand, if $\rho_r([\phi])$ was of finite order, the each of its eigenvalues should be a root of unity. As a consequence, the trace of $\rho_r([\phi])$ is at most the dimension of the TQFT vector space of Σ, which by the Verlinde formula is only a polynomial in r. That is a contradiction.

In a recent work [9], Detcherry and Kalfagianni showed that the behavior of the Turaev-Viro invariant is "similar to" that of the hyperbolic volume, in the sense that it does not increase under Dehn-fillings.

Theorem 3.3 ([9]). *Let M and M' be compact, oriented 3-manifolds with empty or toroidal boundary. Then we have the following.*

(1) *If T is an embedded torus in M and M' is the manifold obtained by cutting M along T, then*

$$\mathrm{TV}_r(M)\leqslant\Big(\frac{r-1}{2}\Big)\mathrm{TV}_r(M').$$

(2) *Suppose that $\partial M'\neq 0$. If M is obtained from M' by Dehn-filling some components of $\partial M'$, then*

$$\mathrm{TV}_r(M)\leqslant \mathrm{TV}_r(M').$$

Therefore, if one could prove Conjecture 3.2 for a 3-manifold M, then Conjecture 3.2 is automatically true for all the 3-manifolds obtained from M by removing a link inside it. By the work of Ohtsuki [25] and Belletti, Detcherry, Kalfagianni and the author [2] mentioned in Section 2, Conjecture 3.2 hold for all the 3-manifolds obtained from the examples mentioned above by remove a link inside them, and the AMU Conjecture holds for the fibered ones obtained from those examples by doing the same operation.

Corollary 3.4 ([10, 2])**.** *Let M_p be the 3-manifold obtained by doing p Dehn-filling along the figure-8 knot, and let M be a fibered 3-manifold obtained from M_p or from a fundamental shadow link complement by removing a link. Then Conjecture* [1] *holds for the holonomy of any fiber structure of M.*

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