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Diagrammatic Algebra

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The purpose of this article is to demonstrate diagrammatic techniques that are used to study multi-categories. The focus will be an example that is described in terms of generators and relations. However, the example is not group theoretic in nature.

A *(small) multi-category* is a category in which there is a set of objects, the collection of 1-arrows between a pair of objects is a category, there are rules for composition across categorical dimensions, and in general the k-arrows are objects in a category. The associative, unital, composition for $(k+1)$-arrows is globular, but that which is commonly known as horizontal composition is mitigated via a natural family of exchangers that are arrows in one more dimension.

Isotopy classes of smooth, properly embedded surfaces in $\mathbb{R}^2 \times [0,1]$ will be described by such a multi-category. There are k-arrows for $k = 0, 1, 2, 3, 4$ and exchangers in degrees through 5. In this paper, a multi-category will be described in abstract that will subsequently be shown to coincide with surfaces.

Here are some principles that guide this work and our approach to categorification in general. (1) Different things are not equal. (2) Arrows are used to compare things. (3) "Doing" and then "undoing" may or may not be the same as "not doing." (4) Simultaneity is illusory. (5) "Change" followed by "exchange" is comparable to "exchange" followed by "change" via an arrow in one more degree.

1 Preliminaries

A *small category* consists of a set of *objects* such that between any two objects a and b, there is a set of 1-*arrows* $\{a \stackrel{f}{\to} b\}$ so that the *source* of an arrow f is the object a, so $s(f) = a$, and the target of an arrow f is the object b, so $t(f) = b$. For any object a, there is an arrow $(a \overset{\mathsf{I}}{-\!\!-\!\!-} a)$ called the identity on a. When necessary, we will write I_a.

If $a \xrightarrow{f} b$ and $b \xrightarrow{g} c$ are arrows so that $s(g) = t(f)$, then their composition is an arrow $a \xrightarrow{f \circ g} c$ with source $s(f \circ g) = a$ and target $t(f \circ g) = c$. Compositions of arrows is associative:

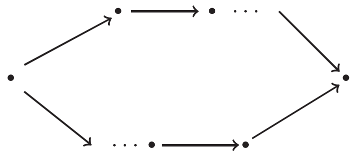
$$\left[a \xrightarrow{f} b \xrightarrow{g \circ h} d\right] = \left[a \xrightarrow{f \circ g} c \xrightarrow{h} d\right].$$

For any object a, the arrow $a \overset{\mathsf{I}}{-\!\!-\!\!-} a$ is an identity under compositions:

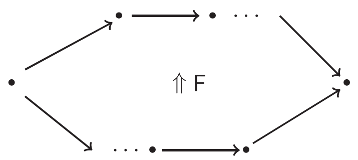
$$\left(a \overset{\mathsf{I}}{-\!\!-\!\!-} a \xrightarrow{f} b\right) = \left(a \xrightarrow{f} b\right) = \left(a \xrightarrow{f} b \overset{\mathsf{I}}{-\!\!-\!\!-} b\right).$$

Here the order of composition is as indicated, so if arrows are functions, then their domain variable is inserted upon the left of the expression.

Following the first and second principles, we often replace an equality between objects a and b with a pair of arrows between them $a \to b$, and $b \to a$. It is common, in a category, to assert that identities between compositions of arrows occur. In this case, we say that the diagram

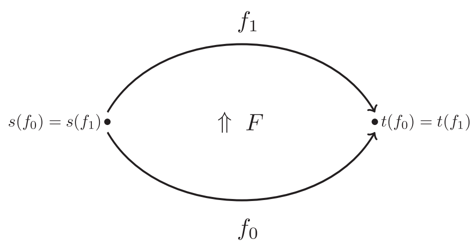


commutes. However, when we consider the second principle, they can be compared by means of a double arrow

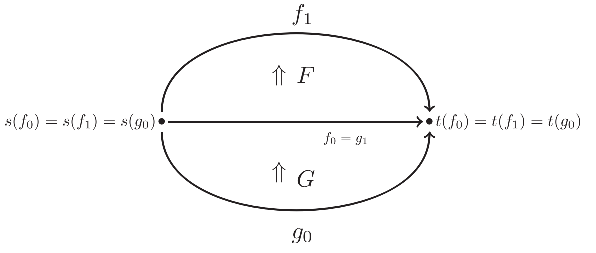


Often, since the double arrow was once thought of as an equality, the double arrow F pretends to be an isomorphism. In that case, its source and target are reversed to form a new double arrow Ⅎ which pretends to be an inverse. These are composed by gluing the polygonal diagrams together along a consistently oriented sequence of arrows in one of two possible ways. Either resulting composition is asserted to be related to an identity double arrow. The relationship with the identity double arrow is illustrative of the fourth categorical principle. If the double arrow F is thought of as a "doing something," then Ⅎ is though of as "undoing that thing." There are compositions $\left[\begin{array}{c} \mathsf{F} \\ Ⅎ \end{array}\right]$ and $\left[\begin{array}{c} Ⅎ \\ \mathsf{F} \end{array}\right]$, that, in the future, will be compared to doing nothing. That is the realm in which the third principle operates. First, let us explain the composition of double arrows.

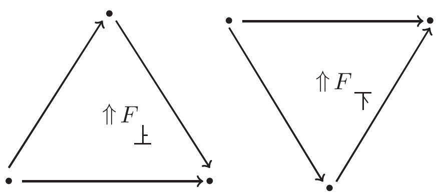
The collection of double arrows will form a category. Therefore a double arrow has a 1-arrow as a source and a 1-arrow as a target. In a schematic form, the structure of the double arrow is depicted here:



In the figure, $S(F) = f_0$, and $T(F) = f_1$. Suppose that double arrows F and G are given with the source 1-arrow of F coinciding with the target 1-arrow of G. The *globular composition* of double arrows F and G is indicated.

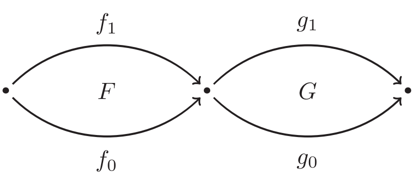


Often, the source or target of a double arrow can be written as the composition of more than one single arrow. By composing all the arrows in the source, and leaving two arrows in the target we obtain the configuration on the left. Alternatively, we can compose all the arrows in the target and leave two in the source to obtain the configuration on the right.



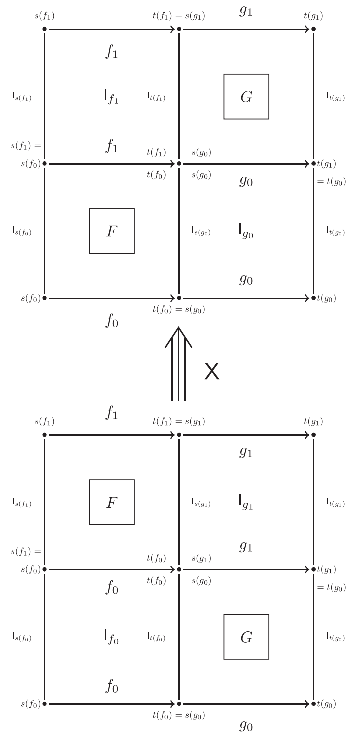
The triangles are labeled with up/down [上 , 下] to make the following indications. In the down [下] case, there is one target arrow, and in the up [上] case, there is one source arrow. The orientation of these *kanji* has the horizon of the character along the single arrow source or target.

Those who are familiar with 2-categories, might expect a horizontal composition as indicated.



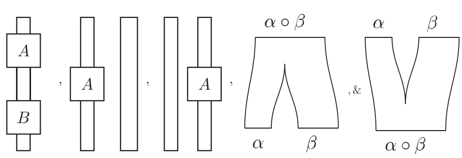
In our situation this is disallowed because it suggests that the application of the double arrows occurs simultaneously. Thus the fourth principle is violated. Furthermore, double arrows are required to have a portion of their 1-dimensional boundary coincident in order to be composed. So we replace this with the composition that is depicted as the source in the illustration below. That choice of resolution is, of course, arbitrary. We make that initial choice and then suppose that there is a higher order relation that connects it with the other possible choice. This higher relation is called an *exchanger* and it is denoted by X it is depicted upon the next page.

The exchanger depends upon the double arrows F and G. So there is a family of exchangers. We say that the exchangers form a *natural family of isomorphisms* in the sense that: (1) exchanging back and forth is equivalent to not doing an exchange; (2) the naturality condition is that change followed by exchange is the same as exchange followed by change. Exchangers exist in every dimension as a method to mitigate the notion of horizontal composition of k-arrows.



We will use the graphic 

to mean the identity double arrow on a 1-arrow α. Such a vertical band is analogous to a wire strip of jumper cables commonly found inside a computer. Here some schematics for double arrows in general. If A and B are double arrows with suitable sources and targets, then each of the diagrams here



is a double arrow. The two diagrams on the right are schematic methods for factoring the compositions of 1-arrows.

We will discuss triple and quadruple arrows, their compositions, and so forth within the context of our main example.

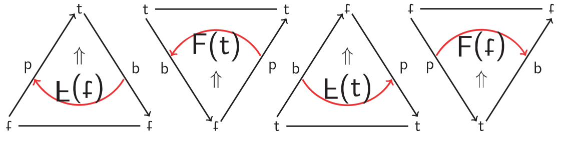
Suppose that $F \Uparrow$ and $G \Uparrow$ are double arrows whose source and target *objects* agree. Suppose that the 1-arrow sources $S(F)$ and $S(G)$ are equal as are the targets $T(F)$ and $T(G)$. A triple arrow between F and G is a globe whose boundary is decomposed into hemispherical disks. Say S and T lie close to the equator with their source s at the location of Equatorial Guinea and (for geographic convenience) their target t at the Solomon Islands. Say that F is the double arrow in the south and G is the double arrow in the north. Then a triple arrow between them lies in the interior of the globe.

Composition of triple arrows is defined in a globular manner that mimics the globular composition of double arrows. A Θ-*sphere* is comprised of three disks: $\{(x,y,z) : z = \sqrt{1-x^2-y^2}\}$ — the northern disk, $\{(x,y,z) : z = -\sqrt{1-x^2-y^2}\}$ — the southern disk, and $\{(x,y,z) : x^2+y^2 \leq 1 \;\&\; z = 0\}$ — the equatorial disk. The equatorial disk is the target of one triple arrow and the source of the other. The *globular composition* of these triple arrows is the result of merging the globes that represent them along the equatorial disk. It is as if two dumplings in a pot are stuck together along a portion of the wrapper.

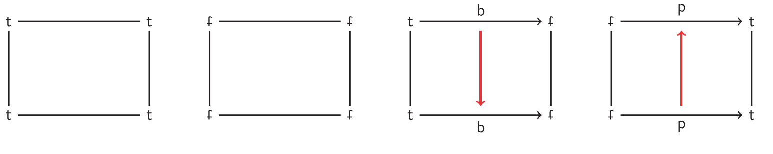
2 The multi-category $\mathcal{S}$

Let $\mathcal{S}$ denote the multi-category that has two objects ɟ, t. The letter ɟ is the vertical reflection of the letter t that is thought of as a poorly drawn f. Loosely, we call ɟ "from," and we call t "to." Suppose that there are generating arrows p : ɟ → t, called "positive," and b : t → ɟ, called "bad." There are also identity arrows ɟ —— ɟ and t —— t. We will say that p : ɟ → t and b : t → ɟ are *reverses* of each other. In general, a non-identity arrow is a finite sequence pbpb . . . b, pbpb . . . p, bpbp . . . b, or bpbp . . . p. In these expressions, instances of identity arrows have been contracted. We can also add identities to such a sequence as necessary.

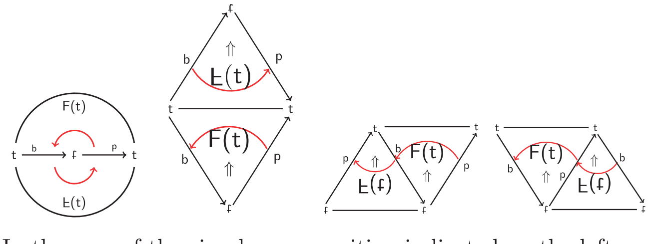
Compare the compositions bp and pb to the identities on t and ɟ, respectively. For example, we can think of pb as going from "from" to "to" and returning. So by principle (3), this may not be the same as staying in place. But by principle (2) we can construct double arrows to compare the composition to t —— t. The generating double arrows are depicted below. In each diagram there is a curve that points upwards from the arrow labeled p and downwards at the one labeled b when these 1-arrows are considered to be pointing towards the right. Such curves are a short hand notation for the double arrows, and they will provide a method for assigning a surface to a composition of triple arrows.



For convenience, the identities upon t —— t, ɟ —— ɟ, p, and b are indicated.



To compile the triple arrows, let us first restrict attention to the double arrow F(t). There are several ways that it can be composed with another double arrow. These are depicted here.



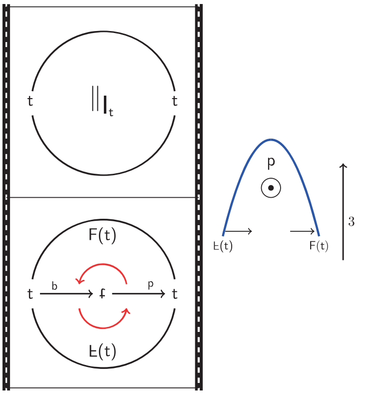
In the case of the circular composition indicated on the left, we can compare it to the identity double arrow that is the identity upon t. By principle (2), a comparison will be given by a pair of triple arrows. Furthermore, this composition may be interpreted by means of principle (3). The composition of these double arrows is read as a reverse pair of 1- arrows that has been created and immediately annihilated. Doing so is different than doing nothing. So we create a pair of reverse triple arrows to compare the creation/annihilation to a static phenomena. These triple arrows are called *birth* and *death*.

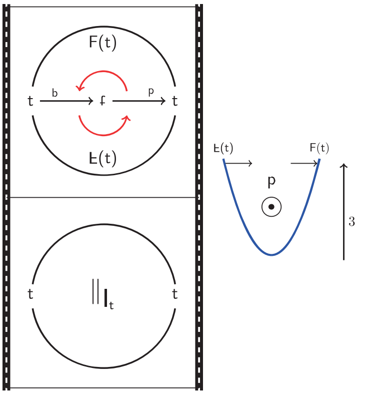
The $\substack{\triangle \\ \triangledown}$ stack of $\mathsf{F(t)}$ and its reverse Ⅎ(t) also can be compared to the identity upon bp. Those comparisons give rise to *saddle* and *fork*-type triple arrows. The parallelogram shaped double arrows on the right each give rise to *cusp*-type triple arrows. Observe here that the Δ or $\perp$ shaped double arrows in these compositions are literally the horizontal reflection of $\mathsf{F(t)}$.

Perform similar compositions with the remaining generating double arrows. A generating set of triple arrows is formed by making comparisons between possible two-fold compositions of double arrows and identity double arrows.

The movie expressions below represent the collection of triple arrows that encode these comparisons. These generate the triple arrows in the multi-category $\mathcal{S}$. They are of the form, birth, death, saddle, fork[1], and cusp. Here we name the triple arrows by the singularity to which they will map under the functor from the multi-category $\mathcal{S}$ to surfaces that are embedded in 3-space. A film strip indicates a source double arrow at its bottom and a target double arrow at its top. To the right of the strip is a glyph that names the triple arrow.

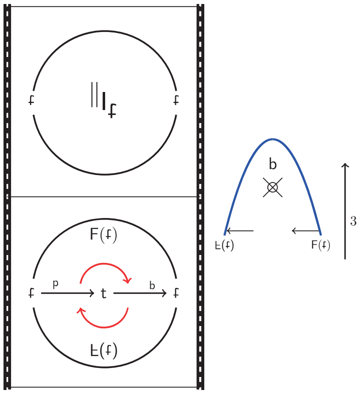
A *positive death* is indicated in the left movie; that on the right is its reflexive companion, a *positive birth*. Here $\|\mathsf{I}_{\mathsf{t}}$ indicates the identity double arrow on the identity arrow on the object t. The identity upon the identity on ɟ is denoted by $\|\mathsf{I}_{ɟ}$.

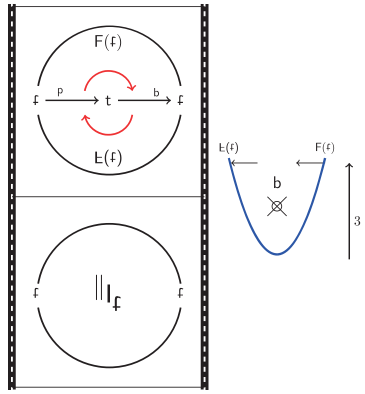




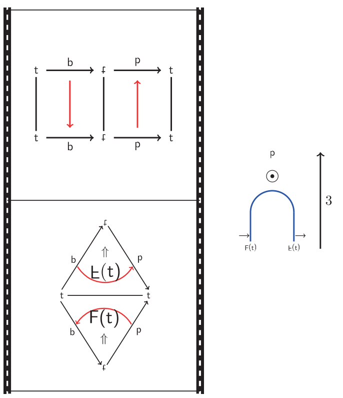
A *negative death* is indicated below by the film strip on the left; its reflective companion a *negative birth* is illustrated on the right.

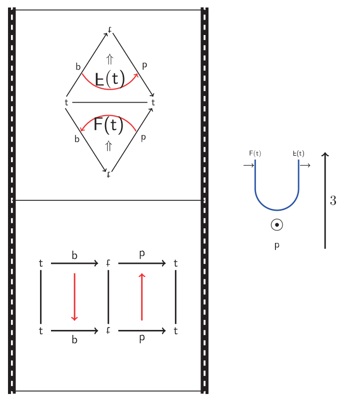
[1]We have learned that British tailors use the word "fork" to describe region in a pair of trousers that is negatively curved and at which the legs merge.



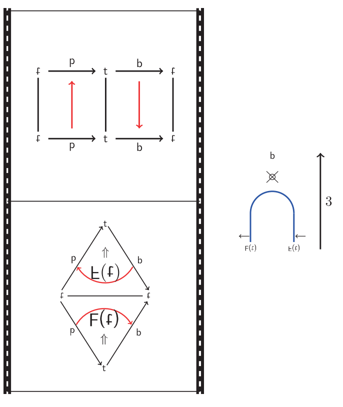


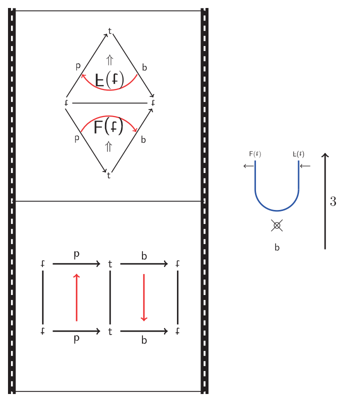
The triple arrow that is depicted below in the film strip on the left is called a *positive fork*; that on the right is its reflexive companion, a *positive saddle*.



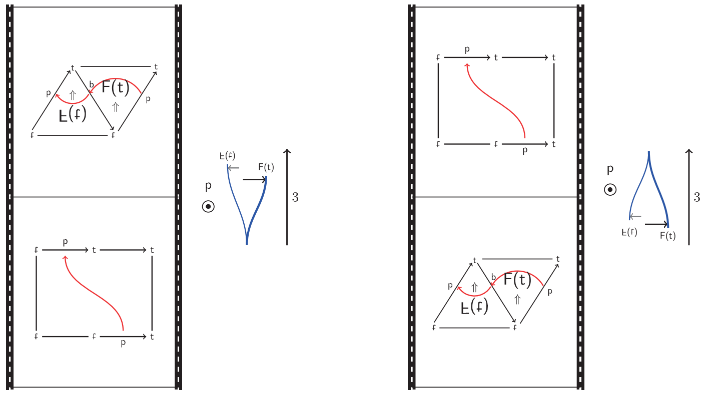


The triple arrow that appears below in the left film strip is called a *negative fork*; that on the right is its reflexive companion, a *negative saddle*.

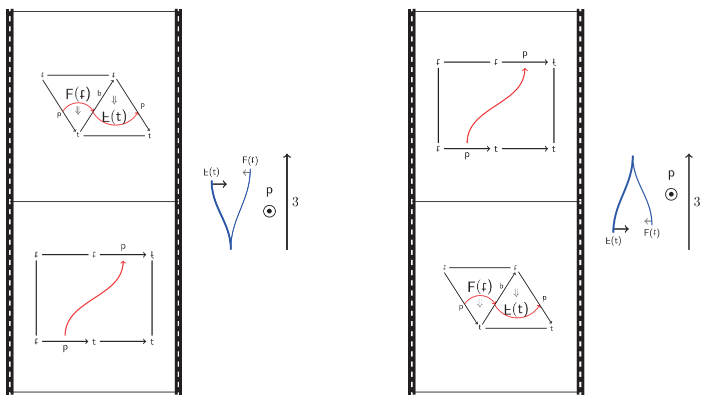




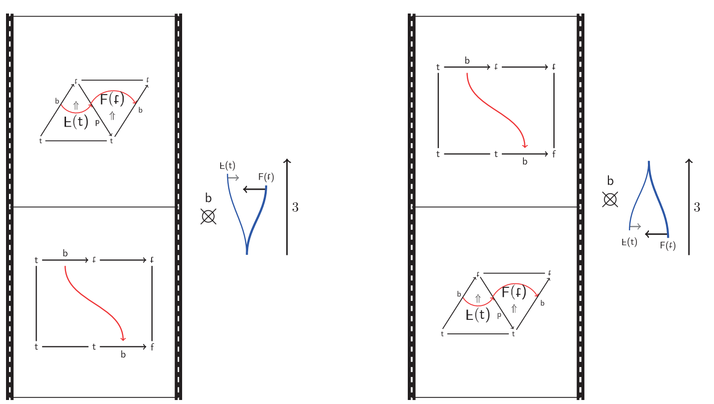
The triple arrow that is depicted below in the film strip on the left is called a *positive right down cusp*; that on the right is its reflexive companion, a *positive right up cusp*.



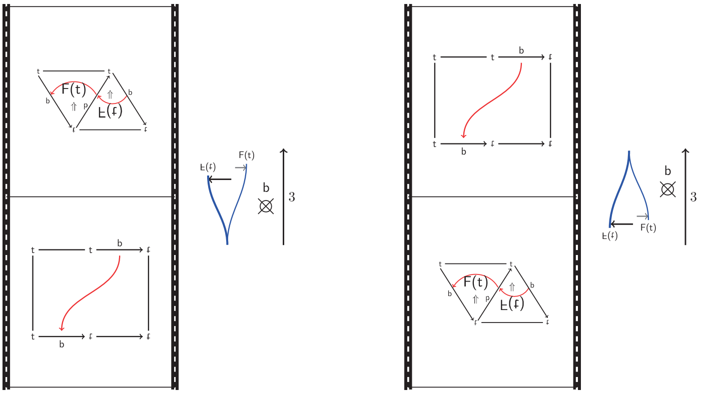
The triple arrow that is depicted below in the film strip on the left is called a *positive left down cusp*; that on the right is its reflexive companion, a *positive left up cusp*.



The triple arrow that is depicted below in the film strip on the left is called a *negative right down cusp*; that on the right is its reflexive companion, a *negative right up cusp*.



The last two generating triple arrows are shown below. That in the film strip on the left is called a *negative left down cusp*; that on the right is its reflexive companion, a *negative left up cusp*.

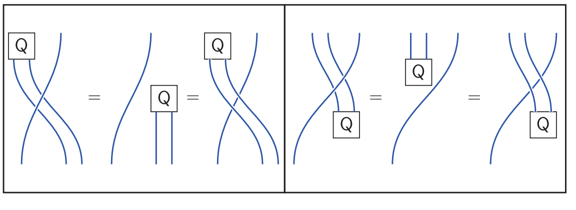


The multi-category $\mathcal{S}$ that is being described here has the following properties:

1. It has two objects ɟ and t.
2. There are generating arrows p:ɟ→t — "positive" and b:t→ɟ — "bad."
3. The objects ɟ and t, identity arrows upon them, and the arrows p and b generate a category.
4. There are two reverse pairs of generating double arrows Ⅎ(ɟ),F(ɟ) and Ⅎ(t),F(t).
5. Double arrows are composed globularly, but horizontal compositions are mitigated by a natural family of triple arrow exchangers.
6. There are (±)-birth, (±)-death, (±)-saddle, (±)-fork, and eight cusp-like triple arrows.
7. The quadruple arrows that are described in this section generate the last level of the multi-category. They are deemed to be equalities.

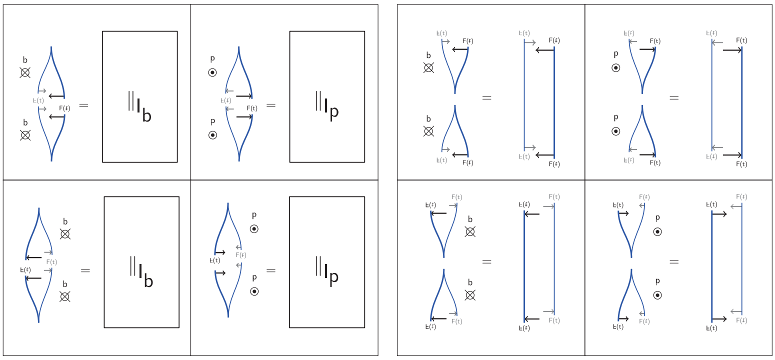
The generating set of quadruple arrows will be completely catalogued by using glyphs and their graphical compositions to depict them. There are six families of generating quadruple arrows: (1) naturality isomorphisms for exchangers, (2) lips, (3) beak-to-beak, (4) critical cancelation, (5) horizontal cusps, and (6) swallow-tails. All of these will be illustrated using the graphical language in which triple arrows are expressed, and the last five will be exemplified by movie moves.

The naturality isomorphism for exchangers is depicted:

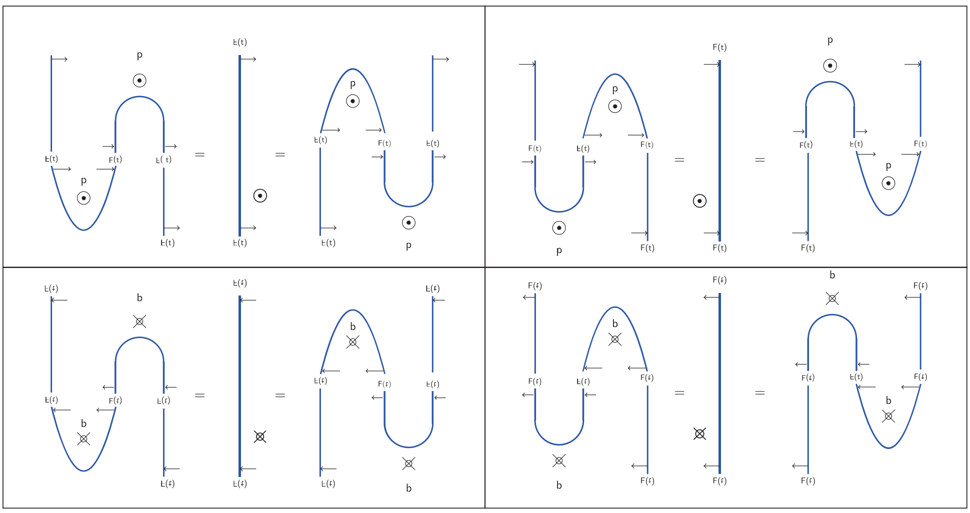


Here Q indicates any possible triple arrow as represented by its glyph.

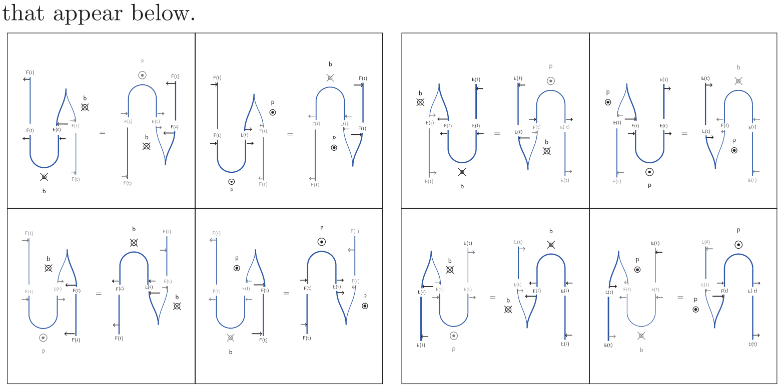
The *lips* and *beak-to-beak* quadruple arrows are depicted below. These and the swallow-tail quadruple arrows arise following principle (3). But in these cases, doing something and then undoing it, is the same as doing nothing.



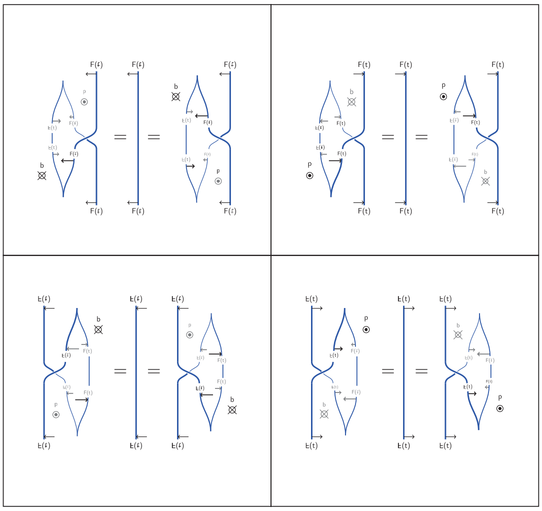
Critical cancelations occur in two types. Birth and fork triple arrows cancel as do death and saddle triple arrows. They are depicted below.



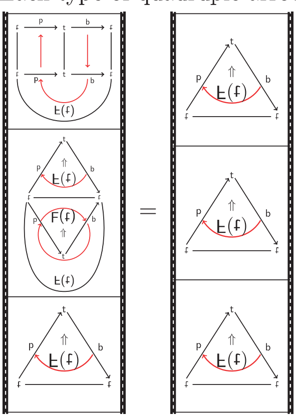
There are eight different *horizontal cusp* quadruple arrows that are indicated in the two tables that appear below.



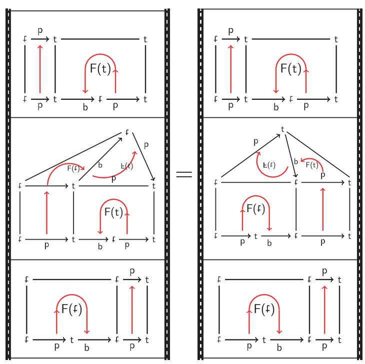
The eight possible *swallow-tail* quadruple arrows appear below.



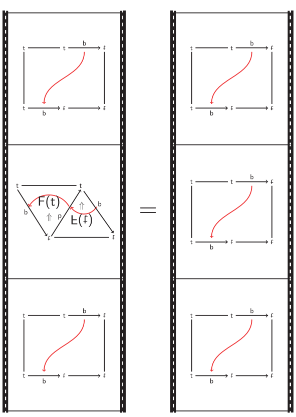
Each type of quadruple arrow is exemplified as a movie move.



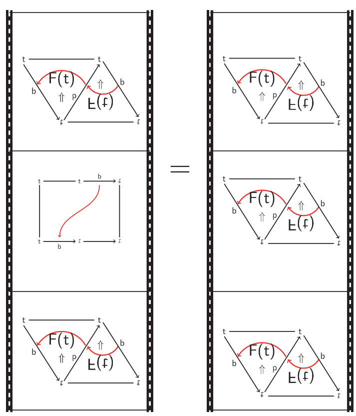
Critical cancelation



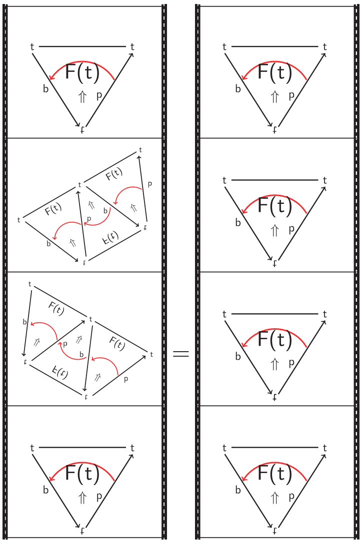
Horizontal cusp



Lips



Beak-to-beak



Swallow-tail

While the names of the triple and quadruple arrows were chosen to coincide with their counterparts as surface singularities, we demonstrated, for the case of triple arrows, that these can arise by considering the different ways that a given double arrow can be composed with other generating double arrows. In the case of quadruple arrows, there are ways of augmenting the composites of two double arrows to three double arrows. Then it turns out that in each of these cases, there are two possible sequences of triple arrows that can be applied, and by doing so sequentially a relation among relations (or movie move) would arise in each case. This is to say that the imposition of these quadruple arrows as relations in the multi-category $\mathcal{S}$ can be seen as natural conditions from the categorical perspective.

3 Surfaces embedded in 3-space

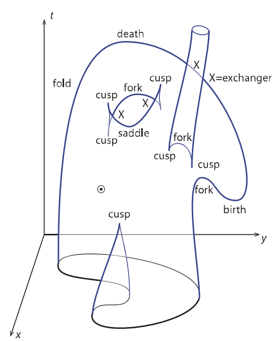


Figure 1: A properly embedded surface and its chart description

Consider the illustration given in Fig. 1. This is an example of a smoothly and properly embedded surface in $\mathbb{R}^2 \times [0,1]$. In addition, the drawing is, as it must be, an illustration of a generic projection of that surface into the (y, t)-plane. The critical points and surface singularities of such a map are labeled. This illustration and the discussion above should be sufficient to understand the theorem that appears below.

Let a smooth surface M and a proper embedding $g : M \to \mathbb{R}^2 \times [0,1]$ be given. Two such embeddings g_a and g_b of M are said to be *properly isotopic* if there is an isotopy between them that is constant on a neighborhood of the boundary ∂M. A *homogeneous triple arrow* in $\mathcal{S}$ is one for which its source and target double arrows have source and target 1-arrows that are the identities upon ʇ or t. Note that if the triple arrow is a composition, then the source and target 1-arrows of any intermediate double arrow are also identity 1-arrows. Triple arrows that are equal up to an application of quadruple arrows will be said to be *equivalent.*

Theorem 3.1. *A generic properly and smoothly embedding $g : M \hookrightarrow \mathbb{R}^2 \times [0,1]$ of a surface M, that has boundary $\partial M = (\partial M)_0 \sqcup (\partial M)_1$, corresponds to a globular composition of homogenous triple arrows in $\mathcal{S}$.*

Properly isotopic surfaces correspond to equivalent homogenous triple arrows in $\mathcal{S}$. Moreover, given a globular composition of homogeneous triple arrows in $\mathcal{S}$ a representative proper embedding can be constructed. Equivalent triple arrows give rise to properly isotopic surfaces.

This result is an initial step in the establishment of the cobordism hypothesis [BD95]. In some sense, it is a special case of the movie-move theorem [CS98]. The point here is that there is a correspondence between the quadruple arrows in $\mathcal{S}$ and the codimension 2 singularities of maps between surfaces. Smoothly and properly embedded surfaces can be given the structure of a multi-category by considering a higher categorical description of the fundamental groupoid in 3-space. There the composition of triple arrows is tantamount to stacking bricks.

4 Other Applications

We mapped the multi-category $\mathcal{S}$ to embedded surfaces in 3-space, by way of identifying t and ʇ with regions in the complement of the surface. The generating arrows p and b then correspond to arcs that intersect an embedded surface once. On the other hand, p and b are *weak inverses* in the sense that there are double arrows F(t), F(ʇ), Ⅎ(t), and Ⅎ(ʇ) that relate their composition to identity arrows. The triple arrows that were constructed are also higher order relations for compositions of p and b that are natural in the sense that they arise upon considering all possible 3-fold compositions of double arrows. In this sense $\mathcal{S}$ is the most free multi-category upon the weakly invertible arrows p and b.

In Table 1, a number of situations in which $\mathcal{S}$ applies are tabulated. The case in which t is a single point and ʇ is empty coincides with the case of surfaces in 3-space. The arrows p and b correspond to crossing the threshold of the surface, the double arrows are critical points of curves, and so on.

The case in which t is two points and ʇ is one corresponds to considerations about surfaces with trivalent seams. A neighborhood of a point on a seam is of the form $\mathsf{Y} \times \mathsf{I}$. The double arrows correspond to the seams with F(t) and so forth corresponding to maximal and minimal points along the seams. The cusp-like triple arrows are cancelations of critical points on the seams. The birth, death, saddle, and fork triple arrows correspond to splitting and gluing the surface with seams.

In the next two cases within Table 1, the triple arrows of the original situation become single arrows in the new situation. It is instructive to imagine the variety of higher arrows in terms of critical points of higher dimensional manifolds. In fact, handles, handle cancelations and handle equivalences are all represented within this categorical frame work. Thus the category replaces the Cerf-theoretic [Cer70] description of handles.

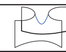
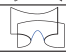
t	•	••	••	□	⊃ ⊂
ɟ		•		⧇	=
p		Y	∪		
b		⅄	∩		

Table 1: Some possible values

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