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Local Moves Generating Writhe Polynomials of Virtual Knots

Takuji Nakamura (Osaka Electro-Communication University)
Yasutaka Nakanishi (Kobe University)
Shin Satoh (Kobe University)

1 Introduction

This paper is a résumé of the paper [8]. The odd writhe $J(K)$ is one of fundamental invariants of a virtual knot K (cf. [5]). In [10], Taniguchi and the third author introduced local moves called Ξ-moves and proved that Ξ-moves correspond to the odd writhe; namely, two virtual knots have the same odd writhe if and only if they are related by a finite sequence of Ξ-moves.

On the other hand, there is another invariant of a virtual knot K called the writhe polynomial $W_K(t)$ (cf. [1, 6, 10]). The writhe polynomial is stronger than the odd writhe in the sense that $J(K) = -W_K(-1)/2$ holds. Therefore, it is natural to ask which local moves correspond to the writhe polynomial.

In Section 2, we introduce local moves for virtual knots and links called shell moves. In Section 3, we prove that the writhe polynomial correspond to shell moves. In Section 4, we study which invariants of a 2-component virtual link correspond to shell moves.

2 Gauss diagrams

A *Gauss diagram* consists of a finite number of oriented circles and oriented, signed chords equipped with signs of endpoints as shown in Figure 1.

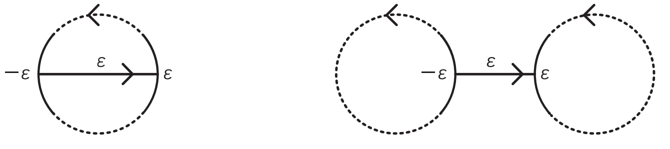

Figure 1: Self- and nonself-chords of a Gauss diagram

A μ-component *virtual link* is an equivalence class of Gauss diagrams with μ circles up to Reidemeister moves R1–R3 (cf. [4, 5]). See Figure 2. If $\mu = 1$, it is called a *virtual knot*.

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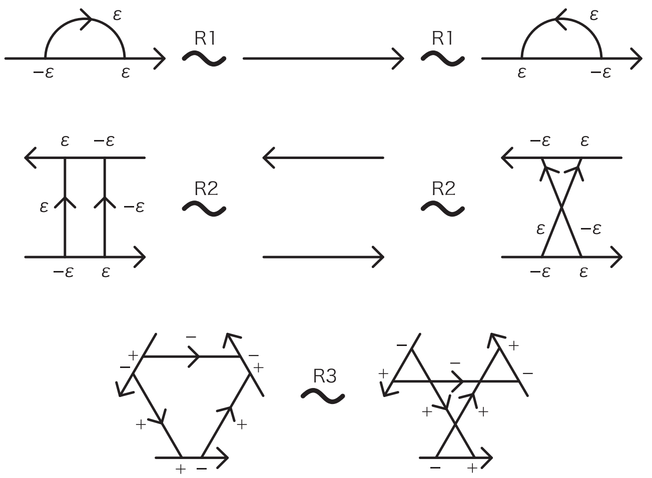

Figure 2: Reidemeister moves for Gauss diagrams

The *shells* for a chord γ are parallel self-chords which are oriented with respect to the sign of the endpoint of γ as shown in Figure 3. We remark that a shell in this paper is a special case of an anklet in [7].

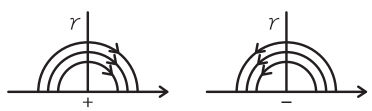

Figure 3: Shells for γ

The *shell moves* S1 and S2 are local moves defined by using Gauss diagrams as shown in Figure 4. Precisely, an S1-move slides a shell along a chord to the opposite side with the same sign, and an S2-move changes the position of the adjacent endpoints of chords with making a pair of shells with respect to the signs of the endpoints.

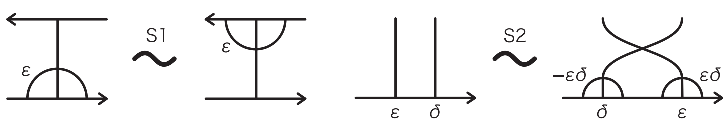

Figure 4: Shell moves S1 and S2

We say that two Gauss diagram are *S-equivalent* if they are related by a finite sequence of Reidemeister moves and shell moves, and two virtual links are *S-equivalent* if their Gauss diagrams are S-equivalent.

Lemma 2.1. *If two Gauss diagrams are related by a deformation* (1) *or* (2) *as shown in* Figure 5*, then they are S-equivalent.* □

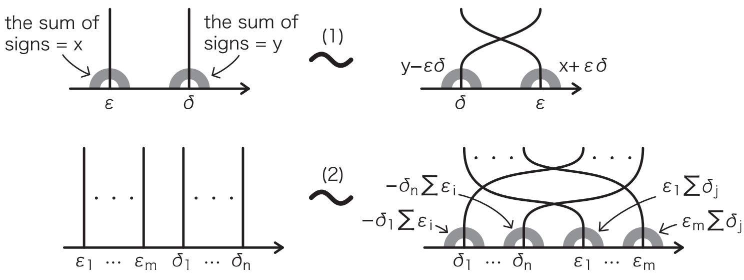

Figure 5: S-equivalent Gauss diagrams in Lemma 2.1

Lemma 2.2. *If two Gauss diagrams are related by a deformation* (1)–(4) *as shown in* Figure 6*, then they are S-equivalent. Here, P and Q are portions of whole chords.* □

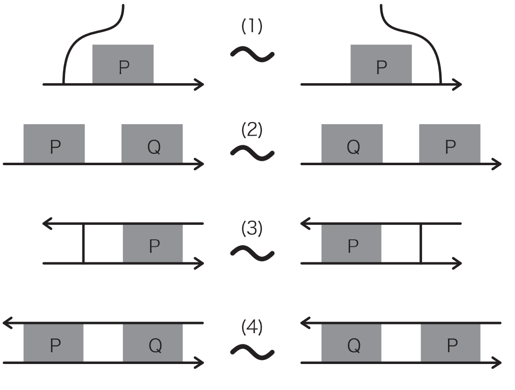

Figure 6: S-equivalent Gauss diagrams in Lemma 2.2

Let G be a Gauss diagram with μ circles $C_1, \ldots, C_\mu$. For $n \in \mathbb{Z}$ and $1 \leq i \neq j \leq \mu$, we define the *n-snail of type i* and *the n-snail of type (i, j)* to be the portion of chords as shown in Figure 7. Then we have the following.

Lemma 2.3. *If two Gauss diagrams are related by a deformation* (1)–(4) *as shown in* Figure 8*, then they are S-equivalent.* □

By using Lemmas 2.1–2.3, we have the following standard form of a Gauss diagram up to S-moves.

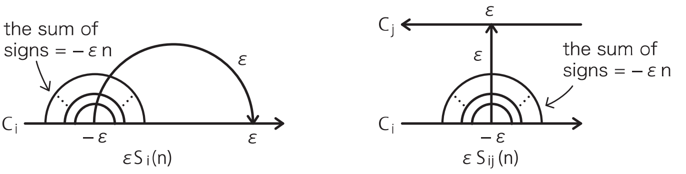

Figure 7: The n-snails $\varepsilon S_i(n)$ and $\varepsilon S_{ij}(n)$

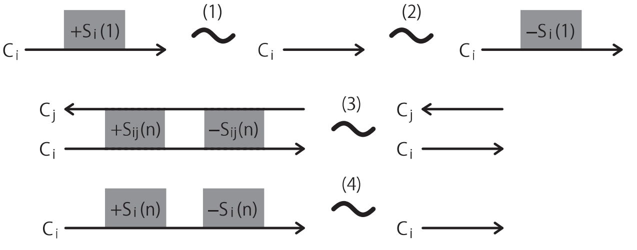

Figure 8: S-equivalent Gauss diagrams in Lemma 2.3

Proposition 2.4. *Any Gauss diagram of an oriented μ-component virtual link is S-equivalent to a Gauss diagram G with μ circles $C_1, \ldots, C_\mu$ which satisfies the following conditions.* Figure 9 *shows the case $\mu = 3$.*

(i) *The chords of G form a finite number of snails.*

(ii) *There is an arc α_i on each C_i such that all snails of type i spans α_i.*

(iii) *All snails of type (i, j) spans $(C_i \setminus \alpha_i) \cup (C_j \setminus \alpha_j)$ in parallel.*

(iv) *There is no snails $\pm S_i(0)$ or $\pm S_i(1)$ for any i.*

(v) *There is no pair of snails $+S_i(n)$ and $-S_i(n)$ for any i and n.*

(vi) *There is no pair of snails $+S_{ij}(n)$ and $-S_{ij}(n)$ for any $i \neq j$ and n.* □

3 The case $\mu = 1$

In this section, we consider an oriented virtual knot K and its Gauss diagram G with a circle C. For portions $P_1, \ldots, P_k$ of chords, we denote by $\left(\sum_{i=1}^{k} P_i\right)$ the Gauss diagram as shown in Figure 10. For integers $a, n \in \mathbb{Z}$, $aS(n)$ denotes the concatenation of $|a|$ copies of $\varepsilon S(n)$, where ε is the sign of a.

Lemma 3.1. *Any Gauss daigram of K is S-equivalent to $\left(\sum_{n \neq 0,1} a_n S(n)\right)$ for some $a_n \in \mathbb{Z}$.* □

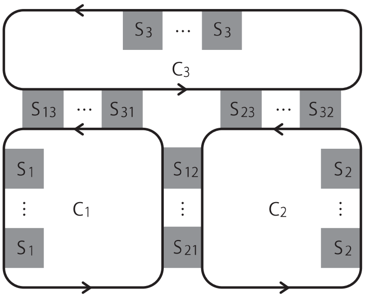

Figure 9: A Gauss diagram with three circles

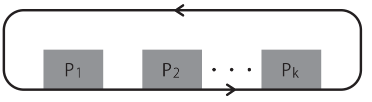

Figure 10: The Gauss diagram $\left(\sum_{i=1}^{k} P_i\right)$

A chord γ divides the circle C into two arcs. Let α be the one oriented from the initial to the terminal endpoint of γ. We define the *index* of γ to be the sum of signs of endpoints of chords on α. For each $n \neq 0$, the sum of signs of all chords whose index is equal to n defines an invariant of K. It is called the *n-writhe* of K and denoted by $J_n(K)$. The *writhe polynomial* is defined by

$$W_K(t) = \sum_{n\neq 0} J_n(K)(t^n - 1) \in \mathbb{Z}[t, t^{-1}].$$

Refer to [1, 2, 6, 10] for more details.

Lemma 3.2. *Let K be an oriented virtual knot.*

(i) *The writhe polynomial $W_K(t)$ is invariant under S-moves.*

(ii) *If K is presented by a Gauss diagram given in* Lemma 3.1, *then we have*

$$W_K(t) = \sum_{n\neq 0,1} a_n t^n - \left(\sum_{n\neq 0,1} n a_n\right) t + \sum_{n\neq 0,1} (n-1)a_n.$$

□

By Lemma 3.1 and Lemma 3.2(ii), we have the following.

Proposition 3.3. *Let K and K' be oriented virtual knots. If $W_K(t) = W_{K'}(t)$ holds, then K and K' are S-equivalent.* □

Therefore the following holds by Lemma 3.2(i) and Proposition 3.3.

Theorem 3.4. *For two oriented virtual knots K and K', the following are equivalent.*

(i) $W_K(t) = W_{K'}(t)$.

(ii) *K and K' are related by a finite sequence of shell moves.* □

4 The case $\mu = 2$

In this section, we consider an oriented 2-component virtual link $L = K_1 \cup K_2$ and its Gauss diagram G with a pair of circles C_1 and C_2. By Proposition 2.4, we have the following.

Lemma 4.1. *Any Gauss daigram of L is S-equivalent to a Gauss diagram*

$$\left(\sum_{n\neq 0,1} a_n S_1(n), \sum_{n\neq 0,1} b_n S_2(n); \sum_{m\in\mathbb{Z}} c_m S_{12}(m), \sum_{m\in\mathbb{Z}} d_m S_{21}(m)\right)$$

for some integers a_n, b_n $(n \neq 0,1)$ and c_m, d_m $(m \in \mathbb{Z})$ as shown in Figure 11*. Here, the entries present the concatenations of snails of type* 1, 2, (1, 2)*, and* (2, 1)*, respectively.* □

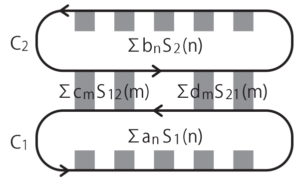

Figure 11: A Gauss diagram of an oriented 2-component virtual link

For $(i, j) = (1, 2)$ or $(2, 1)$, the (i, j)*-linking number* of L, denoted by $\mathrm{Lk}(K_i, K_j)$, is defined to be the sum of signs of all nonself-chords oriented from C_i to C_j. The *virtual linking number* of L is defined by $\lambda(L) = \mathrm{Lk}(K_1, K_2) - \mathrm{Lk}(K_2, K_1)$ [9] (cf. [3]). It is easy to see that $\mathrm{Lk}(K_1, K_2)$, $\mathrm{Lk}(K_2, K_1)$, and $\lambda(L)$ are invariant under S-moves.

If $\lambda(L) < 0$, then by switching the roles of K_1 and K_2, the case reduces to $\lambda(L) > 0$. In what follows, we may assume that $\lambda(L) \geq 0$. We denote $\lambda(L)$ by λ for simplicity. The following propositions give standard forms of L up to S-equivalence.

Proposition 4.2. *Let G be a Gauss diagram of L.*

(i) *If $\lambda \geq 1$, then*

$$G \sim \left(\sum_{n\neq 0,1,-\lambda,-\lambda+1} a_n S_1(n), \sum_{n\neq 0,1,\lambda,\lambda+1} b_n S_2(n);\right.$$

$$\left. \sum_{m=0}^{\lambda-1} c_m S_{12}(p+m), \sum_{m=0}^{\lambda-1} d_m S_{21}(-p-m) \right)$$

for some integers a_n $(n \neq 0, 1, -\lambda, -\lambda+1)$, b_n $(n \neq 0, 1, \lambda, \lambda+1)$, c_m, d_m $(0 \leq m \leq \lambda - 1)$, *and* p.

(ii) *In particular, if* $\lambda = 1$, *then*

$$G \sim \left(\sum_{n\neq 0,1,-1} a_n S_1(n), \sum_{n\neq 0,1,2} b_n S_2(n); c_0 S_{12}(0), d_0 S_{21}(0) \right)$$

for some integers a_n $(n \neq 0, 1, -1)$, b_n $(n \neq 0, 1, 2)$, c_0, *and* d_0. □

Proposition 4.3. *We have the following S-equivalent Gauss diagrams.*

(i) *If* $\lambda = 0$, *then*

$$\left(P, Q; \sum_{m\in\mathbb{Z}} c_m S_{12}(m), \sum_{m\in\mathbb{Z}} d_m S_{21}(m)\right)$$
$$\sim \left(P, Q; \sum_{m\in\mathbb{Z}} c_m S_{12}(m+k), \sum_{m\in\mathbb{Z}} d_m S_{21}(m-k)\right)$$

for any $k \in \mathbb{Z}$.

(ii) *If* $\lambda \geq 2$, *then*

$$\left(P, Q; \sum_{m=0}^{\lambda-1} c_m S_{12}(p+m), \sum_{m=0}^{\lambda-1} d_m S_{21}(-p-m)\right)$$
$$\sim \left(P, Q; \sum_{m=0}^{\lambda-1} c'_m S_{12}(p'+m), \sum_{m=0}^{\lambda-1} d'_m S_{21}(-p'-m)\right),$$

where

$$\begin{cases} (c'_0, \ldots, c'_{\lambda-k-1}, c'_{\lambda-k}, \ldots, c'_{\lambda-1}) = (c_k, \ldots, c_{\lambda-1}, c_0, \ldots, c_{k-1}), \\ (d'_0, \ldots, d'_{\lambda-k-1}, d'_{\lambda-k}, \ldots, d'_{\lambda-1}) = (d_k, \ldots, d_{\lambda-1}, d_0, \ldots, d_{k-1}), \end{cases}$$

and $p' = p + k - \sum_{i=0}^{k-1}(c_i - d_i)$ *for any* k *with* $1 \leq k \leq \lambda - 1$. □

A self-chord γ spanning C_i divides C_i into two arcs. Let α be the one oriented from the initial to the terminal endpoint of γ in C_i. we define the *index of* γ *in* G to be the sum of signs of endpoints of self- and nonself-chords on α. For $n \in \mathbb{Z}$ and $i = 1, 2$, the sum of signs of all self-chords whose indices are equal to n defines the invariants $J_n(K_1; L)$ for $n \neq 0, -\lambda$ and $J_n(K_2; L)$ for $n \neq 0, \lambda$. These are called the *n-writhes of* K_1 *and* K_2 *in* L, respectively.

On the other hand, for nonself-chords γ and γ_0, the *relative index of* γ *with respect to* γ_0 to be the index of γ in the Gauss diagram obtained from G by surgery along γ_0. For $n \in \mathbb{Z}$ and $(i,j) = (1,2),(2,1)$, let $J_n^{ij}(G;\gamma_0)$ denote the sum of signs of nonself-chords γ oriented from C_i to C_j whose relative index with respect to γ_0 are equal to n. Put

$$F_{ij}(t;\gamma_0) = \sum_{n\in\mathbb{Z}} J_n^{ij}(G;\gamma_0)t^n.$$

For an integer $s \geq 0$, let Λ_s denote the Laurent polynomial ring $\mathbb{Z}[t,t^{-1}]/(t^s-1)$. In particular, we have $\Lambda_0 = \mathbb{Z}[t,t^{-1}]$ and $\Lambda_1 = \mathbb{Z}$. We consider an equivalence relation on $\Lambda_s \times \Lambda_s$ such that $(f_1(t), g_1(t))$ and $(f_2(t), g_2(t))$ are equivalent if there is an integer k with

$$f_2(t) = t^k f_1(t) \text{ and } g_2(t) = t^{-k} g_1(t).$$

We denote by $[f(t), g(t)]$ the equivalence class represented by $(f(t), g(t))$, and by $\Gamma(s)$ the set of such equivalence classes.

Then the equivalence class $[F_{12}(t;\gamma_0), F_{21}(t;\gamma_0)] \in \Gamma(\lambda)$ defines the invariant of L (cf. [2]). We call it the *linking class* of L and denote it by $F(L)$. In particular, $F(L) = (\mathrm{Lk}(K_1,K_2), \mathrm{Lk}(K_2,K_1)) \in \mathbb{Z}\times\mathbb{Z}$ for $\lambda = 1$. Then by Propositions 4.2 and 4.3, we have the following.

Theorem 4.4. *Let $L = K_1 \cup K_2$ and $L' = K_1' \cup K_2'$ be oriented 2-component virtual links with $\lambda = \lambda' = 0$. Then L and L' are related by a finite sequence of shell moves if and only if*

(i) $J_n(K_1;L) = J_n(K_1';L')$ *for any* $n \neq 0, 1$,

(ii) $J_n(K_2;L) = J_n(K_2';L')$ *for any* $n \neq 0, 1$, *and*

(iii) $F(L) = F(L')$.

□

Theorem 4.5. *Let $L = K_1 \cup K_2$ and $L' = K_1' \cup K_2'$ be oriented 2-component virtual links with $\lambda = \lambda' = 1$. Then L and L' are related by a finite sequence of shell moves if and only if*

(i) $J_n(K_1;L) = J_n(K_1';L')$ *for any* $n \neq 0, 1, -1$,

(ii) $J_n(K_2;L) = J_n(K_2';L')$ *for any* $n \neq 0, 1, 2$, *and*

(iii) $F(L) = F(L')$.

□

Theorem 4.6. *Let $L = K_1 \cup K_2$ and $L' = K_1' \cup K_2'$ be oriented 2-component virtual links with $\lambda = \lambda' \geq 2$. Then L and L' are related by a finite sequence of shell moves if and only if*

(i) $J_n(K_1; L) = J_n(K_1'; L')$ *for any* $n \neq 0, 1, -\lambda, -\lambda+1$,

(ii) $J_n(K_2; L) = J_n(K_2'; L')$ *for any* $n \neq 0, 1, \lambda, \lambda+1$,

(iii) $F(L) = F(L')$, *and*

(iv) $J_1(K_1; L) + J_{-\lambda+1}(K_1; L) + J_1(K_2; L) + J_{\lambda+1}(K_2; L)$
$$= J_1(K_1'; L') + J_{-\lambda+1}(K_1'; L') + J_1(K_2'; L') + J_{\lambda+1}(K_2'; L').$$

□

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Department of Engineering Science
Osaka Electro-Communication University
Osaka 572-8530
JAPAN
E-mail address: `n-takuji@osakac.ac.jp`

大阪電気通信大学・大学院工学研究科　中村 拓司

Department of Mathematics
Kobe University
Kobe 657-8501
JAPAN
E-mail address: nakanisi@math.kobe-u.ac.jp, shin@math.kobe-u.ac.jp

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