Structures and Their Cryptomorphic Manifestations: Searching for Inquiry Tools

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Abstract. The search for the concept of a structure independent from the particular context of its application has a long history going back to the attempt made by Bourbaki more than a half century ago, but there is still no answer to the general question "What is a structure?" It is no wonder that the parallel question about the definition of cryptomorphism (understood as much more general concept than Birkhoff's cryptoisomorphism) has not been answered. The answer to one question can open a way to the other, as the group of cryptomorphisms should produce the corresponding structure as its invariant. The present paper considers tools for the inquiry in both directions of a structure and of a cryptomorphism.

1. Introduction

This is a report on the continuation of author's research presented and published earlier on the subject of the general concept of structure [1-3]. Author's interest in this general concept was stimulated by his work on the structural study of information. In order to develop a methodology for structural analysis of information independent from the specifics of its implementation we have to establish first a general meaning of structure and the task is surprisingly difficult. [1] The earlier papers of the author followed the line of thinking about structures initiated by Klein's Program which became dominant in natural sciences (in particular in physics) and in mathematics. In this approach structure is identified as an invariant of some group of transformations (automorphisms). [2,3] We express this through the statements that structure is symmetric with respect to this group of transformations or that a given group is a symmetry group for some structure.

Symmetry is a very powerful tool. It is so powerful and effective that sometimes we forget to ask questions about the reason why we expect symmetry with respect to given group. For instance, it is easy to explain why in physics we demand symmetry with respect to Galilean group or Lorenz group (we simply want to have the structure of physical reality independent from the choice of observer or rather from the choice of reference frame), but there is no such easy answer (or just no answer at all) to the same question regarding so called internal symmetries producing classification of elementary particles. Internal symmetry groups produce classifications of particles well-fitting the expectations coming from empirical work.

In mathematics, we are facing different, but to some extent similar challenge. Symmetry associates the group acting on a set underlying the structure with that which does not change in this action. The hidden aspect of this association is in the selection of properties, features or characteristics which we want to have preserved. Our experience from the development of mathematical theories suggests that the consideration of one group of transformations may be insufficient for the study of a structure. For instance, in group theory itself information about the structure of a given group coming from the invariance with respect to group of automorphisms can be enriched by the knowledge of invariance with respect to the subgroup of inner automorphisms (i.e. automorphisms defined by group operation: $\varphi_y(x)= y^{-1}xy$) leading to the concept of normal subgroups. This suggests that the same collective object, in this case the same group can be considered to have different structures (or to be equipped with different structures) which depend on the choice of symmetry group. Since we associate the concept of a group with an instance of algebraic structure, we have a bizarre conclusion that a structure can have different structures.

The earlier papers of the author proposed a line of inquiry of structures based on the concept of symmetry in closure spaces. Symmetry in a given closure space is defined by the Galois connection between the lattice of subsets of the set of automorphisms of the lattice of closed subsets of a given closure space and the lattice of families of closed subsets of this closure space. [1-3] This approach has several advantages over simple determination of the symmetry group. First, we can avoid the problem mentioned above, as the Galois connection is between collectives of automorphisms and collectives of configurations, not particular group of automorphisms and one configuration. This is not a new idea, although frequently forgotten, as the concept of Galois connection is derived from the earliest work engaging groups of automorphisms and their invariants. More important is that the focus is not on automorphisms, but on lattices of collections of automorphisms, lattices of substructures and their relationship. This way we can avoid the use of coordinates to define transformations and their groups. The foundation of the approach is an arbitrary closure space and virtually all mathematical theories of structures can be formulated in terms of lattices and partial order. Thus the formalism has a high level of universality.

Another advantage is in avoiding an apparent fallacy of impredicative definition. If we want to introduce the concept of a structure as an invariant of the subgroup of the group of transformations (bijections) which satisfies some conditions, then we already refer to some structure. We can call this distinct group a global symmetry group. For instance, Klein considered symmetries of configurations of points on Euclidean plane studying subgroups of the (global) group of isometries (transformations preserving distance). The subgroups of this global symmetry group are associated with particular configurations (i.e. structures). An example can be the association of the Klein's four-group (*Vierergruppe*) with the group of symmetries of the structure of a non-square rectangle. However, this requires pre-established structure of a metric space. For Klein, the main subject of the study in his Erlangen Program was the unification of different geometries by their classification through different groups of global symmetry, not the classification of particular configurations (i.e. local structures with local symmetry determined by subgroups of the global symmetry group). We want to have tools for the study of any type of structures, no matter whether they serve as global or local structures in some contexts. How can we define the symmetry group of a structure without any reference to already existing structure? For other, non-geometric structures, we also have to use another structure restricting the group of transformations. The exception is combinatorics, where the foundation is set by the so called symmetric group of all bijections and every subgroup of this group is associated with some structure. However, this is due to the fact that in combinatorics we do not pre-define any particular structure and we consider all sets as potential structures.

In the approach proposed by the author in earlier papers there is one type of structure (closure space or alternatively a complete lattice of closed subsets) which is pre-defined, but it is only one and the same for all possible structures. Of course, there is one more pre-defined structure of a group (group of transformations). Thus, the concept of a general structure in the approach proposed in earlier papers of the author requires only two pre-defined structures of a complete lattice and of a group and Galois connection between them.

There are however negative sides of the approach. Less important is the fact that the formalism requires simultaneous handling the objects from the three levels of set theory. We start from a set then we consider subsets of this set (or its power set) and finally sets of these subsets (power sets of power sets). Thus, we have to use predicate calculus of the order higher than the first order, which makes some logicians uncomfortable. For instance, Quine objects higher order predicate calculus as a hidden import of a set theory. However, we can observe that we always start from a given set and whatever set theory is employed, the objects invoked, even if they are within powers of powers of sets, they have secured status of a set. Thus, we can safely stay within the limits of the naive set theory.

More problematic is the fact that the complete lattice of closed subsets used in the study of symmetry has the explicit role of substructure lattice. Thus far nobody could provide the reason why substructure relationship should be distinguished. Once again we have the situation that this works well when we apply this formalism to all earlier studies of symmetry. The complete lattice of substructures was in the center of interest for the decades and for the decades it is known that non-isomorphic structures (e.g. groups) may have identical lattice of substructures (subgroups), unless groups are noncyclic finite simple groups. Thus, the equivalence relation on the set of structures defined on a given set by the condition that structures are equivalent if and only if they have the same lattice of substructures is not an identity. Therefore, there is a legitimate question about the reason why substructure lattice has so distinguished role in the study of symmetry and general structures. [1]

We can see that there are some open questions which require consideration when we want to set foundations for the study of general structures through symmetry. This paper is in some sense a step back from the details of the description of symmetry in terms of closure spaces to get more general perspective on the issues arising when we ask the question: “What is a structure?”

2. Re-Statement of the Problem

The question “What is a structure?” was considered in many different contexts without a definite answer. Even if we restrict our attention to mathematical structures, the definitions are typically restricted to a narrow domain of study. Most typical is a restriction to relational structures, i.e. a set with a number of n-ary relations without much concern for the issues related to differences in n-arties of relations and following these differences the impossibility to compare such structures using familiar mathematical tools such as concepts of homomorphism, isomorphism, or automorphism. Some attempts to clarify the meaning of the term “structure” required modifications of set theory and introduction of the idiosyncratic concepts, such as for instance the conceptual framework of “named sets” used by Mark Burgin. [4] In this paper the attempt will be made not to go beyond standard mathematical and logical framework.

The earliest and most comprehensive attempt to define the concept of a general mathematical structure can be credited to Bourbaki. [5,6] Saunders MacLane presented a very concise description of this idea on the occasion of promoting category theory as a way to avoid complexity of the study of structures.

“Their [Bourbaki] massive and widely used multivolume treatment of the ‘Eléments de Mathématique’, with a first part entitled 'Les structures fondamentales de l'analysé’ began with volume 1, 'Theorie des ensembles, Fascicule de resultats'. In this volume, Bourbaki carefully describes what he means by a structure of some specific type T. We do not really need to use this description, but we will now present it, chiefly to show both that one can indeed define 'structure' and that the explicit definition does not really matter. It uses three familiar operations on sets: the product E x F of two sets E and F, consisting of all the ordered pairs (e, f) of their elements, the power set

P(E), consisting of all the subsets S of E, and the function set E^F consisting of all the functions mapping F into E. For example, a topological structure on E is given by an element T of P(P(E)) satisfying suitable axioms—it is just the set consisting of all open sets, that is, the set T of all those sets U in P(E) which are open in the intended topology. Similarly, a group structure on a set G can be viewed as an element $M \in G^{GxG}$ with the usual properties of a group multiplication. With these examples in mind, one may arrive at Bourbaki's definition of a structure, say one built on three given sets E, F, and G. Adjoin to these sets any product set such as E x F, any function set such as E^G and any power set such as P(G). Continue to iterate this process to get the whole scale (échelle) of sets M successively so built up from E, F, and G. On one or more of the resulting sets M impose specified axioms on a relevant element (or elements) m. These axioms then define what Bourbaki calls a 'type' T of 'structure' on the given sets. This clearly includes algebraic structures like groups, topological structures, and combined cases such as topological groups (the definition also includes many bizarre examples of no known utility). Actually, I have here modified Bourbaki's account in an inconsequential way; he did not use function sets such as E^F. This modification does not matter; as best I can determine, he never really made actual use of his definition, and I will not make any use here of my variant. It is here only to show that it is indeed possible to define precisely 'type of structure' in a way that covers all the common examples.” [7]

If the only deficiency of Bourbaki’s approach was its complexity, there will be no need for the present paper. The approach is clearly most universal of all attempts, but its fatal feature which prevented its use in mathematics is not just complexity, but the fact that it does not take into account the fact that structures commonly considered in mathematics as identical become not only different, but incomparable. Bourbaki considered a generalization of the concept of isomorphism called polymorphism, but polymorphism required a bijective correspondence between n-arities of operations for polymorphic algebras, which excluded polymorphism between algebras with different numbers of operations or different n-arities of operations.

The fact that in mathematics structures can be, and actually frequently are defined in very different way which does not allow for the use of the standard tool of structure identification through isomorphism or even polymorphism was explicitly stated by Garrett Birkhoff. [8] He observed that even most frequently used mathematical structures are defined as algebras with different number of operations and/or with different n-arities of operations. Birkhoff referred to the example of the concept of a group, which can be define as a universal algebra with different signatures. Signature of the algebra is a sequence of natural numbers indicating n-arities of all its operations. Thus, the constant, such as a neutral element for other operations is a nullary operation; we have also unary, binary, and possibly higher order operations. Birkhoff observed that a group can be defined as an algebra with signatures (1,2), (0,1,2), (2,2), (2). [8:154]

In the first case we have the (unary) inverse operation x^{-1} and the group binary operation xy satisfying the two identities: $(xy)z = x(yz)$ (associativity) and $(x^{-1}x)y = y(x^{-1}x) = y$.

An alternative, second definition involves additional nullary operation “which picks a constant ‘identity element’ e” subject to the identities: $(xy)z = x(yz)$ and $x^{-1}x = x^{-1}x = e$ and $ey = ye = y$.

The third alternative is a pair of binary operations x/y and x\y ($x/y = xy^{-1}$ and $x\backslash y = x^{-1}y$) subject to the identities such as $x/x = y\backslash y$ and $y/(y\backslash x) = x$ and $x\backslash(y/z) = (x\backslash y)/z$, etc.

Finally, in the fourth case we have a single binary operation in which the equations xa=b and ay=b can always be solved for x and y.

Birkhoff introduced his concept of cryptoisomorphism to make these four non-polymorphic algebras essentially identical, since in the mathematical practice nobody would pay attention to their differences. His solution was to claim that the two algebras defined on the same set are cryptoisomorphic if the polynomial equations defining operations in the first algebra are equivalent to appropriate sets of arbitrary polynomial equations in the other (i.e. they have the same solution sets) and the other way around when we exchange the roles of algebras. This can be rephrased that algebras are cryptoisomorphic if we can define operations of one in terms of polynomial equations of the other.

Birkhoff was satisfied with this definition of cryptoisomorphism, because it had resolved the most urgent need for the unification of such concepts as Boolean lattices and Boolean algebras. However, his solution did not help in so basic ramifications of concepts such as that of a lattice and of partially ordered sets with the suprema and infima for all their pairs of elements. It did not help in the unification of the dozens of different definitions for topological spaces or in the formulation of the answer to the question about the choice of generalizations of concepts such as topological space which retain the characteristics qualifying them as topological.

In the section devoted to the discussion of cryptoisomorphism, Birkhoff was using this term and its shorter form cryptomorphism. Later the latter term cryptomorphism entered mathematical folklore vernacular, but without clear definition and with much wider scope of understanding. Cryptomorphic structures typically mean structures which are apparently different, but their differences are of secondary importance. The main criterion for cryptomorphism is that we can reconstruct concepts describing one structure in the description of the other, but the other criteria are vague and admit some level of tolerance for differences. For instance, the lattice of subgroups of a group defined as in the second case above (with the identity element) is always non-isomorphic with the same lattice of subgroups of a group defined as in the first case, but this is considered not important.

We can conclude that one of our ultimate goals (clearly beyond the scope of this paper) is to formulate a definition of cryptomorphism to be used for the purpose of achieving another ultimate goal of answering the question: “What is a structure?” Of course, two particular objects have the same structure if they are cryptomorphic. If it happens that there is a conventional isomorphism between two objects, we can claim cryptomorphism between them. So cryptomorphism is a generalization of the concept of isomorphism. This generalization is an extension to the situation when the concept of isomorphism cannot be applied due to restrictions in the way how isomorphism is understood. This is not trivial, as this means that for two objects for which the conditions of isomorphism are meaningful, but not true, we can claim that they are non-cryptomorphic. The difference between cryptomorphism and isomorphism is when they are considered in the context of conceptually incompatible structures. For instance, two structures defined on sets of different cardinality are clearly non-cryptomorphic and non-isomorphic, because there is no bijection between the sets.

Lifting our study to a higher level of abstraction, we can say that we try to formulate the definition of an abstract concept of a structure. We expect that symmetry, i.e. invariance with respect to a group of transformations (cryptomorphisms) can be useful in this task. Finally we can observe that structures are collectives, and therefore we have to compare objects which are at least sets of other sets. Thus we have to consider relations not just within a given set S, but within its power set 2^S.

3. General Structure and Abstraction

Typical way to form abstract concepts is based on equivalence relations. We replace individual objects of our study forming a set S by the classes of abstraction which are elements of the partition of the set S associated with the equivalence relation. Each class of abstraction becomes an abstract object of our study. Although this is a commonly accepted procedure, there are many arguments that the actual process requires more general relation than equivalence. The most famous among the opponents to the standard procedure of abstraction was Ludwig Wittgenstein. [9,10]

Obviously, isomorphisms of mathematical structures define an equivalence relation on structures. It is not obvious that we have to require that cryptomorphisms have to define equally strong relation. The alternative is similarity relation known in mathematics as a tolerance relation generalizing equivalence which meets expectations of Wittgensein. [10]

Let's review the theory of such relations in terms of the algebra of binary relations. A binary relation on a set S is a subset of the direct product S×S. If we have any predicate for two variables R(x,y) with variables assuming values in the set S, we can associate it with the relation $\boldsymbol{R}$={(x,y): ((x,y)∈S×S & R(x,y)}. As the set $\mathscr{R}$(S) of all binary relations on S is a set of subsets of S×S, and therefore a set, it can be partially ordered by inclusion. This partial order can be defined in $\mathscr{R}$(S): $\boldsymbol{R} \leq \boldsymbol{T}$ *iff* x$\boldsymbol{R}$y ⇒ x$\boldsymbol{T}$y. We can consider a Boolean algebra structure on $\mathscr{R}$(S) by importing set theoretical operations from S×S. Boolean operations distinguish the *empty relation* **Ø** and the *full or universal relation* S×S. We can also define a *complementary relation* $\boldsymbol{R}^c$ for relation $\boldsymbol{R}$ in $\mathscr{R}$(S) by:∀x,y∈S: x$\boldsymbol{R}^c$y *iff* not x$\boldsymbol{R}$y, or in other words:∀x,y∈S: x$\boldsymbol{R}^c$y *iff* (x,y)∉$\boldsymbol{R}$.

The only nontrivial operations giving $\mathscr{R}$(S) its rich structure going beyond Boolean algebra are composition and converse operations. The *composition operation* is defined for any ordered pair of relations $\boldsymbol{R}$, $\boldsymbol{T}$ by: ∀x,y∈S: x$\boldsymbol{RT}$y *iff* ∃z∈S: x$\boldsymbol{R}$z and z$\boldsymbol{T}$y. The *equality relation* $\boldsymbol{E}$ = {(x,y): x = y} is compatible with the order and gives $\mathscr{R}$(S) the structure of an *partially ordered monoid*. The other specific relation algebraic unary operation on $\mathscr{R}$(S) is *converse* $\boldsymbol{R} \rightarrow \boldsymbol{R}^*$ defined by ∀x,y∈S: x$\boldsymbol{R}^*$y *iff* y$\boldsymbol{R}$x.

Binary relations are defined on the set S, but they generate binary relations on $\boldsymbol{2^S}$, the *power set of S* (set of all subsets of S): $\boldsymbol{2^S}$ = {A: A ⊆ S}. For instance, we can consider relations $\boldsymbol{R^a}$ and $\boldsymbol{R^e}$ on $\boldsymbol{2^S}$ defined by: ∀A ⊆ S: $\boldsymbol{R^a}$(A) = {y∈S: ∀x∈A: x$\boldsymbol{R}$y}, ∀A ⊆ S: $\boldsymbol{R^e}$(A) = {y∈S: ∃y∈A: x$\boldsymbol{R}$y}.

The definitions can be expressed in words that the subset R^a(A) of S consists of all elements in relation $\boldsymbol{R}$ with all elements of A, while the subset $\boldsymbol{R^e}$(A) of S consists of all elements in relation $\boldsymbol{R}$ with at least one of elements of A (this explains letters "a" and "e" in symbols $\boldsymbol{R^a}$(A) and $\boldsymbol{R^e}$(A), since "a" stands for "all", "e" for "exists").

For one-element sets the two corresponding sets coincide, so we can simplify our notation for single element subsets: $\boldsymbol{R}$(x) = $\boldsymbol{R^a}$({x}) = $\boldsymbol{R^e}$({x}).

Obviously: $\boldsymbol{R^a}$(A) = ∩{$\boldsymbol{R}$(x): x∈A} and $\boldsymbol{R^e}$(A) = ∪{$\boldsymbol{R}$(x): x∈A}.

Now we can distinguish the following classes of binary relations of special interest for us defined by conditions:

- $\boldsymbol{R}$ is *symmetric* if $\boldsymbol{R} = \boldsymbol{R}^*$,
- $\boldsymbol{R}$ is *reflexive* if $\boldsymbol{E} \leq \boldsymbol{R}$,
- $\boldsymbol{R}$ is *transitive* if $R^2 = \boldsymbol{RR} \leq \boldsymbol{R}$,
- $\boldsymbol{R}$ is *weakly reflexive* if ∀x∈S: (x$\boldsymbol{R}^c$x ⇒ ∀y∈S: x$\boldsymbol{R}^c$y),
- $\boldsymbol{R}$ is a *function* if ∀x∈S ∃y∈S: x$\boldsymbol{R}$y & ∀x∈S∀y_1,y_2∈S: {y_1, y_2} ⊆ $\boldsymbol{R}$(x) ⇒ $y_1 = y_2$,

In the following part of the paper we will refer to relations not only on a given set S, but also to relations on its power set $\boldsymbol{2^S}$ = {A: A ⊆ S}. Since we consider both the sets of objects, associated with elements of S, as well as the sets of properties characterizing objects associated with subsets of S, this interest in the interdependence of relations at the two levels of set theoretical hierarchy is natural.

Now we can focus our attention on the relations that are subject of this study. We already have distinguished our equality relation $\boldsymbol{E} = \{(x,y): x=y\}$. *Equivalence relations* are defined as those which are reflexive, symmetric and transitive, conditions which combined can be written: $\boldsymbol{E} \leq \boldsymbol{R}^* = \boldsymbol{R} = \boldsymbol{R}^2$. Of course $\mathbf{E} \leq \boldsymbol{E}^* = \boldsymbol{E} = \boldsymbol{E}^2$, so equality is a special case of equivalence relation (the least equivalence relation on S).

It is a very elementary fact that equivalence relations correspond in a bijective manner to partitions of the set S on which they are defined. Subsets belonging to such partition $\mathfrak{C} \subseteq \mathbf{2}^S$ (i.e. family $\mathfrak{C}$ which satisfies the conditions $\cup\mathfrak{C} = S$ and $\forall A,B\in\mathfrak{C}: A\cap B = \emptyset$) are called classes of equivalence (or classes of abstraction) for the corresponding relation. If we start from a partition $\mathfrak{C}$, its corresponding equivalence relation is defined by the condition that the elements x and y are related, i.e. $x\boldsymbol{R}y$ if they both belong to one of the subsets of the partition ($x\boldsymbol{R}y$ *iff* $\exists A\in\mathfrak{C}: \{x,y\} \subseteq A$). If we start from the equivalence $\boldsymbol{R}$, the partition is uniquely determined by the condition $A\in\mathfrak{C}$ iff $A=\boldsymbol{R}^a(A)$.

Tolerance relations are more general, because they do not have to be transitive, i.e. they are defined by $\boldsymbol{E} \leq \boldsymbol{T}^* = \boldsymbol{T}$. For the reason which soon will become clear it is worth considering one small step in generalization to *weak tolerance relations* which are simply symmetric ($\boldsymbol{T}^* = \boldsymbol{T}$) and which are *weakly reflexive* ($\forall x\in S$: ($x\boldsymbol{T}^c x \Rightarrow \forall y\in S: x\boldsymbol{T}^c y$)). Originally the latter condition of weak reflexivity appeared in this theory because there are important, but irrelevant for our study mathematical structures which are not reflexive, but which satisfy it [11].

It turns out that an arbitrary covering of the set S (family of subsets $\mathscr{H} \subseteq \mathbf{2}^S$ which satisfies the condition $\cup\mathscr{H} = S$) defines a tolerance relation on S the same way as partitions defined equivalence relations, i.e. by: $x\boldsymbol{T}y$ *iff* $\exists A\in\mathscr{H}: \{x,y\} \subseteq A$. However, we do not have bijective correspondence as before. Different coverings can define the same tolerance relation and the relation between coverings and tolerance relations is highly nontrivial in comparison to the special case of equivalence relations.

Suppose we have a tolerance relation $\boldsymbol{T}$ on S. We can define a family of subsets $\mathscr{H}_T = \{A\subseteq S: \forall x,y\in S: \{x,y\}\subseteq A \Rightarrow x\boldsymbol{T}y\}$. This class will be called the family of all *pre-classes of tolerance* $\boldsymbol{T}$. Of course, $\forall x,y\in S$: $x\boldsymbol{T}y$ *iff* $\exists A\in\mathscr{H}_T : \{x,y\}\subseteq A$, but it is clear that this family is redundant. If $\boldsymbol{T}$ is an equivalence relation, then $\mathscr{H}_T$ in addition to all members of the family of equivalence classes $\mathfrak{C}$ includes all their subsets. Therefore, we want to reduce $\mathscr{H}_T$ as much as possible. Using Zorn's lemma, we can infer that in $\mathscr{H}_T$ every pre-class A can be extended to a maximal pre-class, which we will call a *class of tolerance relation*. The subfamily $\mathfrak{C}_T$ of all classes of tolerance is sufficient for the reconstruction of tolerance $\boldsymbol{T}$: $\forall x,y\in S$: $x\boldsymbol{T}y$ *iff* $\exists A\in\mathfrak{C}_T : \{x,y\}\subseteq A$. So, we have an efficient way to represent given tolerance relation by its family of tolerance classes.

Another topic is the analysis of tolerance relation from the point of view of deviation from equivalence relation. For this purpose we can consider the nucleus N_T of T defined as an equivalence relation: $\forall x,y\in S$: $x\ N_T\ y$ *iff* $T(x) = T(y)$. Of course if T itself is an equivalence relation, then its nucleus is identical with itself, i.e. $N_T = T$. Otherwise the nucleus partitions S into subsets in which all elements are in relation T with each other. If all equivalence classes of nucleus N_T consist of only one element, i.e. $N_T = E$ (equality relation), the tolerance is *non-nuclear*.

Nucleus of a tolerance relation is a good candidate for the formation of the concept of a structure. While tolerance relation admits some level of diversity and restricts comparisons to similarity, its nucleus is an equivalence relation and can be used for the formal definition of a structure.

We can use the theory of tolerance and weak tolerance relations to relate more extensive class of relations with those two by a process of "symmetrization". For every binary relation $\boldsymbol{R}$ on a given set S, we can define a relation $\boldsymbol{T_R}$ as follows: $\boldsymbol{T_R} = \boldsymbol{RR^*}$. Then we have [11]:

(i) $\boldsymbol{T_R}$ is a symmetric relation on S.

(ii) $\forall x \in S \forall y \in S$: x $\boldsymbol{T_R}$ y *iff* $R(x) \cap R(y) \neq \varnothing$.

(iii) $\boldsymbol{T_R}$ is a tolerance *iff* $\boldsymbol{R}$ is defined everywhere (equivalent to $\boldsymbol{E} \leq \boldsymbol{RR^*}$)

(iv) $\boldsymbol{T_R}$ is a weak tolerance *iff* $\boldsymbol{R}$ is *weakly reflexive*, i.e. $\forall x \in S$: $(x\boldsymbol{R^c}x \Rightarrow \forall y \in S$: $x\boldsymbol{R^c}y)$

(v) $\boldsymbol{R}$ is a function $\Rightarrow$ $\boldsymbol{T_R}$ is an equivalence relation, but the reverse implication is not necessarily true.

(vi) $\boldsymbol{T}$ is an equivalence relation *iff* there exists a relation $\boldsymbol{R}$ which is a function and $\boldsymbol{T} = \boldsymbol{T_R}$.

(vii) Let $\boldsymbol{E} \leq \boldsymbol{T}$. Then T is an equivalence relation *iff* $\boldsymbol{T} = \boldsymbol{T_T}$.

This proposition links together the four types of relations: equality, equivalence, tolerance, and weak tolerance with each other and with the very general class of weakly reflexive relations. We can observe that the class of equivalence relations is here associated with functions, which in turn are the most typical instruments of the mathematical formalization of theories across all disciplines.

The overall picture of different levels of similarity is as follows:

At every level we have symmetric relation, i.e. $R=R^*$, where $R^*=\{(x,y): yRx\}$ (converse)

- Identity (Example: 2=2 identity)
- Equality (Example: 2=1+1 equality, but not identity)
- Equivalence $E \leq R$ (reflexive) & $R^2 \leq R$ (transitive)
- Tolerance $E \leq R$ (reflexive)
- Weak Tolerance $\forall x \in S$: $(xR^cx \Rightarrow \forall y \in S$: $xR^cy)$.

Since the concept of a structure outside of mathematics is most frequently associated rather with difference than similarity, we can distinguish different levels of differences by considering the complementary relation: $R^c=\{(x,y)$: xRy is not true$\}$ (complementary relation)

At every level we have symmetric relation, i.e. $R=R^*$, where $R^*=\{(x,y): yRx\}$ (converse)

- Difference/Orthogonality is a relation complementary to similarity $R^c=\{(x,y)$: NOT $xRy\}$
- Non-identity
- Non-equality
- Non-equivalence $E \cap R = \varnothing$ & (anti-transitive) i.e. $\forall x,y \in S$: $(xRy \Rightarrow \forall z \in S$: xRz or $yRz)$
- Orthogonality $E \cap R = \varnothing$ (non-reflexive)
- Weak-orthogonality $\forall x \in S$: $(xRx \Rightarrow \forall y \in S$: $xRy)$

It is interesting that in spite of the dominating role of the equivalence relation in mathematics, its complement, non-equivalence relation was never a subject of more systematic study.

Thus far, we considered the process generating the relations describing different levels of similarity from the weakly reflexive relations on a given set S. Now we will consider induction of these types of relations on the power set $\mathbf{2^S}$ of S (the set of all subsets of S) by the relations on S introduced already by E.C. Zeeman who introduced the concept of tolerance relation. [12] This is of special importance for us, as we want to seek the relation aggregating not individual objects, but collectives equipped with some structures.

Let $\boldsymbol{R}$ be a relation on S. Then we define a relation $\boldsymbol{R}^S$ on $\mathbf{2}^S$ as follows:

$$\forall A \subseteq S \forall B \subseteq S: A\ \boldsymbol{R}^S\ B \textit{ iff } B \subseteq \boldsymbol{R}^e(A) \text{ and } A \subseteq \boldsymbol{R}^e(B).$$

Then, if $\boldsymbol{T}$ is a tolerance relation on S, then $\boldsymbol{T}^S$ is a tolerance relation on $\mathbf{2}^S$. It is also easy to show that a weak tolerance relation $\boldsymbol{T}$ induces a weak tolerance relation $\boldsymbol{T}^S$ on $\mathbf{2}^S$ and an equivalence induces equivalence. Thus the similarity relation defined on a given set induces similarity of sets of objects. We can consider the reversed process of rather trivial "downward induction" from the power set of a set S to S when we consider the definition of $\boldsymbol{R}^S$ on $\mathbf{2}^S$ restricted to one-element sets. Obviously, if we start from the induction and proceed to the downward induction, we return to the original tolerance relation.

4. Abstraction and Groups of Transformations

Thus far we have a tool for the analysis of concept formation necessary to answer the question: "What is a structure?" But we lost from our sight the role of symmetry. However, the connection between abstraction and groups of transformations has been already considered by Bas Van Fraasen in his Laws and Symmetry [13]. We will use terminology of the action of a group on set in our study of the relationship between equivalence relations on a set S and groups acting on this set.

It is one of the most basic facts about group actions on a set that every group corresponds to the unique equivalence relation defined by the partition of the set into orbits, i.e.

$\forall x,y \in S$: xRy *iff* $\exists g \in G$: $y = g(x)$

However, it turns out that for every equivalence relation R on S there exist a group G whose orbits are equivalence classes of R. This can be seen as follows. Let Sym(S) is the symmetric group of S (i.e. the group of all bijections on S). Now, let $\{X_i: i \in I\}$be the partition corresponding to the equivalence relation R on S & let $G_i=\{g \in Sym(S): \exists h \in Sym(X_i): g(x)=h(x) \text{ if } x \in X_i \ \& \ g(x) = x \text{ otherwise}\}$. Obviously each G_i is a subgroup of Sym(S). Define G to be the subgroup of Sym(S) generated by the union $\cup\{G_i: i \in I\}$. Then G is a group acting on S with $\{X_i: i \in I\}$ as its orbits.

From this construction we cannot claim that G is unique, but we know that G is the greatest subgroup of Sym(S) with $\{X_i: i \in I\}$ as its orbits.

5. Conclusion

This paper presented tools for the study of the general concept of structure. We may consider formation of the abstract concept of a structure on a set S either as based on equivalence relation (as it is done in the much more narrow context of isomorphisms) or on similarity relation (not necessarily transitive). Moreover, there is a correspondence between equivalence relations on a set and groups acting on this set. Here, we have the link between abstraction and symmetry. With these tools, further work can continue to establish an answer to the question: "What is a structure?

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