TOPOLOGY OF THE SPACE ON WHICH CELLULAR AUTOMATA WORKS

HIROYUKI ISHIBASHI

Abstract. We investigate :

(A) Topology of the product of discrete topological spaces

and

(B) Homeomorphism of two topologies ($\mathrm{T_p}$) and ($\mathrm{T_m}$) of cell space $C = S^{\mathbb{Z}^n}$ of a cellular automaton $\mathcal{A} = (\mathbb{Z}, S, N, f)$.

[1]

1. Product of Discrete Topolpgical Spaces

Definition.

- X_i **: discrete topological space for** i **in** I

  $\mathcal{O}_i$ **: the system of open sets of** X_i **,**
- $X = \prod_{i \in I} X_i$ **: the product of** X_i **of weak topology**

  $\mathcal{O}$ **: the system of open sets of** X**,**

  and so
- $\mathcal{O}_0 = \{\prod_{j \in J} O_j \times \prod_{i \in I \setminus J} X_i \mid O_j \in \mathcal{O}_j,\ J \subseteq I,\ |J| < \infty\}$

  : an open basis of $\mathcal{O}$**.**

1991 *Mathematics Subject Classification.* 37B15, 68Q80.

Key words and phrases. cellular automaton, configuration set, discrete topology, product of spaces.

[1]This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

Now we state our theorem.

Theorem A. *For X_i's descrete the following are equivalent:*

(a) $X = \prod_{i \in I} X_i$ *is discrete.*

(b) $| \{ i \in I \mid |X_i| \geq 2 \} | < \infty$,

that is, $|X_i| = 1$ for almost all i in I.

Corollary. *The product of discrete topological spaces is discrete if and only if it is homeomorphic to a product of a finite number of discrete topological spaces. In particular, if S is a discrete topological space containing at least two elements, then its infinite product is not discrete.*

Proof.

(a) $\Rightarrow$ $X \ni {}^\forall x$ **: open,** **(since X is discrete)**

$\Rightarrow$ $X \ni {}^\forall x = \cup_{\lambda \in \Lambda} O_\lambda$ **for some** O_λ**'s** $\in \mathcal{O}_0$,

(for $\mathcal{O}_0$ is an open basis of $\mathcal{O}$)

$= O_\lambda$ **for any λ in Λ,**

(since $|\{x\}| = 1$)

$\Rightarrow$ **setting**

$O_\lambda = \prod_{j \in J} x_j \times \prod_{i \in I \setminus J} X_i$ **for some** J, $|J| < \infty$

and

$x = \prod_{j \in J} x_j \times \prod_{i \in I \setminus J} x_i$,

we have

$\Rightarrow$ $X_i = x_i$ **for** i **in** $I \setminus J$ **with** $|J| < \infty$,

$\Rightarrow$ **(b).**

(b) $\Rightarrow$ **For**

$$J = \{ i \in I \mid |X_i| \geq 2 \}$$

we have

$$|J| < \infty$$

$\Rightarrow$ **Choose**

$$X \ni {}^\forall x = \prod_{j \in J} x_j \times \prod_{i \in I \setminus J} x_i, \quad x_i \in X_i, \quad x_j \in X_j,$$

Then by the definition of J

$${}^\forall i \in I \setminus J, \quad x_i = X_i$$

and so

$$X \ni {}^{\forall}x = \prod_{j\in J} x_j \ \times \prod_{i\in I\setminus J} X_i \in \mathcal{O}_0, \quad x_j \in X_j$$

$\Rightarrow$ **(a)**

Q.E.D.

2. The topology of the configuration set C of a cellular automaton $\mathcal{A}$

- $\mathcal{A} = (\mathbb{Z}, S, N, f)$ **: a cellular automaton**

  $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ **: rational integers,**
- $\mathcal{Z}^n = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ **: cell space**
- $S = \{s_1, s_2, \cdots, s_m\}$ **: states**
- $f : S^l \to S$ **: local map**

Further we define

- $C = S^{\mathbb{Z}^n} = \mathrm{Map}(\mathbb{Z}^n, \mathrm{S})$ **: configurations**

where we give

- S **: the discrete topology**

Then $C = S^{\mathbb{Z}^n}$ **has two topologies** $(\mathbf{T}_p)$ **and** $(\mathbf{T}_m)$ **following :**

$(\mathbf{T}_m)$

For

${}^{\forall}c \in C = S^{\mathbb{Z}^n}$,

${}^{\forall}\epsilon = 2^{-\lambda}, \quad \lambda \in \mathbb{N}$

define

$U_\epsilon(c) = \{c' \in C \mid d(c, c') < \epsilon\}$

: the ϵ**-neighbourhood of** c

where

$$d(c,c') = 2^{-min\{\delta(0,i) \ | \ c(i) \neq c'(i), \ i \in I\}} \quad for \quad c, c' \quad in \quad C,$$

$$\delta(0,i) = \sqrt{{i_1}^2 + {i_2}^2 + \cdots + {i_n}^2} \quad \textbf{for} \quad i = (i_1, i_2, \cdots, i_n) \quad \textbf{in} \quad \mathbb{Z}^n. \ .$$

(T$_p$)

For

$^{\forall}c \in C = S^{\mathbb{Z}^n}$,

$^{\forall}J \subseteq \mathbb{Z}^n, \quad |J| < \infty$

define

$$V_J(c) = (\textstyle\prod_{j \in J} c(j)) \ \times S^{\mathbb{Z}^n \setminus J}$$

: the J-neighbourhood of c

Then, since S is discrete, we have $c_j \in \mathcal{O}_j$ for any $j \in J$. This enable us to take

$$\{V_J(c) \mid c \in C, J \subseteq \mathbb{Z}^n, \quad |J| < \infty\}$$

as O_0 an open basis.

Then, we have :

Theorem B.

(T$_p$) $\simeq$ (T$_m$) : homeomorphic

Proof. (a) We show (T$_p$) $\leq$ (T$_m$), i.e., $^{\exists}U_\epsilon(c) \subseteq {}^{\forall}V_J(c)$

For

$^{\forall}V_J(c) = \prod_{j \in J} c(j) \ \times S^{\mathbb{Z}^n \setminus J}$

with

$^{\forall}c \in C = S^{\mathbb{Z}^n}$,

$^{\forall}J \subseteq S^{\mathbb{Z}^n}$ and $|\mathrm{J}| < \infty$

choose $\lambda \in \mathbb{N}$

such that (A) $^{\forall}j \in J, \quad \delta(0,j) < \lambda$

and set

$^{\forall}\epsilon = 2^{-\lambda}$

Then,

$c' \in U_\epsilon(c) \;\Rightarrow\; d(c,c') < \epsilon = 2^{-\lambda}$, **where**

$d(c,c') = 2^{-min\{\delta(0,i) \;|\; c(i) \neq c'(i)\}}$ for c, c' in C,

$\Rightarrow\; \lambda < \min\{\delta(0,\mathrm{i}) \mid \mathrm{c(i)} \neq \mathrm{c'(i)}\}$

$\Rightarrow$ **By (A)**

if $j \in J$**, we have** $\delta(0,j) < \lambda$

and so $c(j) = c'(j)$

$\Rightarrow\; c' \in V_J(c) = \prod_{j \in J} c(j) \;\times S^{\mathbb{Z}^n \setminus J}$

Thus,

$U_\epsilon(c) \subseteq V_J(c)$ **and so** $(\mathbf{T}_p) \leq (\mathbf{T}_m)$.

(b) Next we show $(\mathbf{T}_m) \leq (\mathbf{T}_p)$, **i.e. ,** ${}^{\exists}V_J(c) \subseteq {}^{\forall}U_\epsilon(c)$.

For

${}^{\forall}U_\epsilon(c)$

with

$c \in C = S^{\mathbb{Z}^n}$,

$\epsilon = 2^{-\lambda}, \quad \lambda \in \mathbb{N}$

let

(B) $J = \{j \in I \mid \delta(0,j) \leq \lambda\}$

Then, since

$V_J(c) = \prod_{j \in J} c(j) \;\times S^{\mathbb{Z}^n \setminus J}$,

we have

$c' \in V_J(c) \;\Rightarrow\; {}^{\forall}j \in J, \quad c(j) = c'(j)$

$\Rightarrow$ **by (B) if** $\delta(0,j) \leq \lambda$ **, we have** $j \in J$

and so $c(j) = c'(j)$

$\Rightarrow\; \lambda < \min\{\delta(0,\mathrm{j}) \mid \mathrm{c(j)} \neq \mathrm{c'(j)}\}$

$\Rightarrow\; 2^{-min\{\delta(0,j) \;|\; c(j) \neq c'(j)\}} < 2^{-\lambda} = \epsilon$

$\Rightarrow\; d(c,c') < \epsilon$

$\Rightarrow\; c' \in U_\epsilon(c)$

Thus

$V_J(c) \subseteq U_\epsilon(c)$ **and so** $(\mathbf{T}_m) \leq (\mathbf{T}_p)$.

Q.E.D.

References

[1] B. Durand, Global Properties of Cellular automata, (E. Goles, S. Martinez, Eds. , Cellular automata Complex Systems, Kluwer Academic Pblishers), (1999) 1-22.

[2] H. Ishibashi, Injectivity of Cellular Automata, Jp Journal of Algebra, Number Theory and Appl. 40(2) (2018) , 199-205.

[3] H. Ishibashi, Product of Discrete Topolgicl Spaces, Jp Journal of Algebra, Number Theory and Appl. 40(5) (2018) , 685-690.

[4] H. Nishio, T. Saito, Information of Cellular Automata I - An Algebraic Sutady -, Fundamenta Informaticae 58 (2003) 399-420.

Chuou 2-3-8, Moroyama, Iruma, Saitama, Japan
***E-mail address*:** hishi@yuzu-tv.ne.jp