# Higher uniruledness, Bott towers and "Higher Fano Manifolds"

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**Abstract**

A sufficient criterion for higher uniruledness via Bott towers is given. This sufficient criterion proposes new kinds of "higher Fano manifolds."

## 1 Higher (uni)ruledness and Lower (uni)rationality

Let us recall a coupld of basic concepts of algebraic geometry:

(uni)ruled and (uni)rational

For a projective $n$-dimensional variety $X$,

- $X$ is uniruled (resp. ruled), if there exist a $(n-1)$-dimensional $Z$ and a rational dominant (resp. birational) map
$$\mathbb{P}^1 \times Z - - > X,$$
*May replace a rational dominant (resp. birational) map with an honest domiant (resp. birational) morphism.*

- $X$ is unirational (resp. rational), if there exist a rational dominant (resp. birational) map
$$\mathbb{P}^n - - > X,$$
*May NOT replace with an honest morphism!*

Here, it would be self-evident to propose the following definition:

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**Higher (uni)ruledness**: (uni-)$k$-ruled = (uni-)$\big(-(n-k)\big)$-rational :**Lower (uni)rationality**

For a projective $n$-dimensional variety $X$, and
$1 \leq k \leq n$, let us say:

$X$ is <u>uni-$k$-ruled</u> or <u>$=$ uni-$(k-n)$-rational</u> (resp. <u>$k$-ruled</u> or <u>$= -(n-k)$-rational</u> ),
if there exist a $(n-k)$-dimensional $Z^{n-k}$ and a rational dominant (resp. birational) map

$$\mathbb{P}^k \times Z^{n-k} - - > X,$$

$$\begin{tikzcd}
\text{rational} \arrow[r, equal] \arrow[d, Rightarrow] & \text{0-rational} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \arrow[r, Rightarrow] \arrow[d, Rightarrow] & -(n-k)\text{-rational} = k\text{-ruled} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{1-ruled} \arrow[r, equal] \arrow[d, Rightarrow] & \text{ruled} \arrow[d, Rightarrow] \\
\text{unirational} \arrow[r, equal] & \text{uni-0-rational} \arrow[r, Rightarrow] & \cdots \arrow[r, Rightarrow] & \text{uni-}(-(n-k))\text{-rational} = \text{uni-}k\text{-ruled} \arrow[r, Rightarrow] & \cdots \arrow[r, Rightarrow] & \text{uni-1-ruled} \arrow[r, equal] & \text{uniruled}
\end{tikzcd}$$

- These concepts are birational invariant.
- *However... For $k \geq 2$, may NOT replace a rational dominant (resp. birational) map with an honest domiant (resp. birational) <u>morphism</u>.*

Thus, it makes sense to consider the following
NON birational invariant properties also...

(uni)regular-$\mathcal{R}^k$-ruled

Fix a rational $k$-fold $\mathcal{R}^k$ $(1 \leq l \leq n)$. For a projective $n$-dimensional variety $X$, let us say:

$X$ is <u>uniregular-$\mathcal{R}^k$-ruled</u> (resp. <u>regular-$\mathcal{R}^k$-ruled</u> ), if there exist a $(n-k)$-dimensional $Z^{n-k}$ and a dominant (resp. birational) morphism

$$\mathcal{R}^k \times Z^{n-k} \to X.$$

- Clearly,

$$\begin{tikzcd}
\text{regular-}\mathcal{R}^k\text{-ruled} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & k\text{-ruled} = -(n-k)\text{-rational} \arrow[d, Rightarrow] \\
\text{uniregular-}\mathcal{R}^k\text{-ruled} \arrow[r, Rightarrow] & \text{uni-}k\text{-ruled} = \text{uni-}(-(n-k))\text{-rational}
\end{tikzcd}$$

- (uni)regular-$\mathcal{R}^k$-ruledness' are NOT birational invariant.

Now the purpose of this paper is to report a sufficient criterion for uniregular-$\mathcal{T}^k$-ruledness, with $\mathcal{T}^k$ a $k$-dimensional smooth projective toric variety, and so, for uni-$k$-ruledness (thus for higher uniruledness).

## 2 Past works for sufficient criteria for higher (uni-)ruledness

Let us start with Mori's famous work:

S. Mori, Annals of Math. 79

Any Fano manifold $X$ is <u>covered by $\mathbb{P}^1$</u> , i.e. any general point $x \in X$ is contained in the image of a map from $\mathbb{P}^1$, which is an immersion at $x$.

Then, Kollár pointed out the following:

uniruledness

For a projective $n$-dimensional variety $X$, TFAE:

- $X$ is covered by $\mathbb{P}^1$ .
- $X$ is uniruled , i.e. there exists a $(n-1)$-dimensional $Z$ and a rational, dominant (i.e. the image contains a non-empty open) map

$$\mathbb{P}^1 \times Z - - > X,$$

To study a uniruled manifold, a standard method is to consider the following:

polarized minimal family of rational curves

- For a uniruled manifold $X$ of dimension $n$, with $x \in X$ a general point,

$$\begin{cases} \exists V_x & \underset{\text{irred. open}}{\subset} \ \mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x) \\ G_0 & := Aut\left(\mathbb{P}^1\right)_0 \ \subset \ Aut(\mathbb{P}^1) = PGL(2, \mathbb{C}) \end{cases}$$

$$\begin{tikzcd} \mathbb{P}^1 \times V_x \arrow[r] \arrow[d] \arrow[rr, bend left, "(t,[f])\mapsto f(t)"] & U_x = \left(\mathbf{P}^1 \times V_x\right)//G_0 \arrow[r, "\mathrm{ev}_x"] \arrow[d, "\pi_x"'] & X, \\ V_x \arrow[r] \arrow[u, bend left, "\{o\}\times\mathrm{id}"] & H_x := V_x//G_0 \arrow[u, bend right, "\sigma_x"'] & \end{tikzcd} \tag{1}$$

- By results of Miyaoka [Kol96, V,3.7.5.Prop] and Kebekus [Keb02, Th.3.3], every curve parametrized by $H_x$ is immersed at $x$, and the subvarietry $H_x^{\mathrm{Sing},x}$, parametrizing curves singular at $x$, is at most finite.
- There is a normalization onto its image (Kebekus [Keb02, Th.3.3,3.4] Hwang-Mok [HM04] ) of the finite morphism

$$\tau_x : H_x \to \mathbb{P}\left(T_{X,x}\right) \cong \mathbb{P}^{n-1},$$

giving a polarization $(H_x, \tau_x^* \mathcal{O}(1)) =: (H_x, L_x)$, called a PMFRC ( polarized minimal family of rational curves ) through $x$.
- Denote this situation by $\boxed{X \mapsto H_x}$.

It turns out that PMFRC $(H_x, L_x)$ possesses a very rich information about $X$ :

Cho-Miyaoka-Shepherd-Barron, Kebekus

For PMFRC $(H_x, L_x)$ of a uniruled manifold $X$, we know

$$l := \dim H_x + 2 \leq (n-1) + 2 = n+1 \tag{2}$$

becuase

$$\tau_x : H_x \to \mathbb{P}(T_{X,x}) \cong \mathbb{P}^{n-1},$$

is a finite morphism (Miyaoka, Kebekus).
If (2) becomes an equality, i.e. if

$$l := \dim H_x + 2 = n+1,$$

then $X \cong \mathbb{P}^n$.

Here, let us compare Mori's theorem and the theorem of Cho-Miyaoka-Shepherd-Barron and Kebekus:

| Theorem | Condition | Conclusion | in particular... |
|---|---|---|---|
| Mori | $c_1(X) > 0$ | uniruledness | lowest uniruledness |
| Cho-Miyaoka-Shepherd-Barron, Kebekus | $l := \dim H_x + 2 = \dim X + 1$ | $X \cong \mathbb{P}^{\dim X}$ | highest ruledness |

So, these two theorems suggest sufficent criteria, which give a hierachy of uniruledness, might be expressed as possitivity of certain polynomial of $c_i(X)$ $(1 \leq i \leq n)$ and some restrictions on $l := \dim H_x$, or the pseudo-index of $X$ :

$$i_X := \min \{-K_X \cdot C \mid C \subset X \text{ rational curve}\},$$

which enjoys

$$l := \dim H_x + 2 \geq i_X \quad (\forall x \in X)$$

In fact, most past work which considered the hierachy of uniruledness were stated under such conditions, i.e. in terms of $i_X$ or $l := \dim H_x + 2$, and the so-called "higher Fano conditions":

Various definitions of "$k$-Fano" (de Jong-Starr (Harris))

For $k \in \mathbb{N}$, let us call $X$

$$\begin{cases} \underline{\text{strong } k\text{-Fano}} \\ \underline{k\text{-Fano}} \\ \underline{\text{weak } k\text{-Fano}} \end{cases} \text{if } \begin{cases} \mathrm{ch}_i(X) > 0 \ (1 \leq i \leq k), \\ \mathrm{ch}_i(X) > 0 \ (1 \leq i \leq k-1), \ \mathrm{ch}_k(X) \geq 0, \\ \mathrm{ch}_1(X) > 0, \mathrm{ch}_i(X) \geq 0 \ (2 \leq i \leq k), \end{cases} , \quad \text{respectively.}$$

Note:

$$\left\{\text{degree } d \text{ hypersurface } X_n^d \subset \mathbb{P}^{n+1} \mid d^k \leq n+1\right\}$$
$$\subseteq \{\text{strong } k\text{-Fano's}\} \subseteq \{k\text{-Fano's}\} \subseteq \{\text{weak } k\text{-Fano's}\}$$

but confined to statements like

"Any general point of $X$ is contained in the image of a generically injective morphism $\mathcal{R}^k \to X$, where $\mathcal{R}^k$ is a some rational $k$-fold (which is in many cases just $\mathbb{P}^k$). "

And, the first attempt was given by de Jong-Starr:

de Jong-Starr, DukeJM 07

For any 2-Fano manifold, its general point is contained in the image of a generically injective morphism $\mathcal{R}^2 \to X$ for some rational 2-fold $\mathcal{R}^2$.
(However, it is not clear what kind of rational 2-fold $\mathcal{R}^2$ show up for each general point.)

To go furhter, Araujo-Castravet and Taku Suzuki chose to consider iterated PMFRC:

Is $H_x$ Fano again? - Druel, Math. Ann. 2006

$$\begin{tikzcd}
\mathbb{P}^1 \times V_x \arrow[r] \arrow[d, shift left] \arrow[rrr, bend left, "{(t,[f])\mapsto f(t)}"] & U_x = \left(\mathbf{P}^1 \times V_x\right)//G_0 \arrow[rr, "\mathrm{ev}_x"] \arrow[d, shift right, "\pi_x"'] & & X, \\
V_x \arrow[r] \arrow[u, bend left, "\{o\}\times\mathrm{id}"] & H_x := V_x//G_0 \arrow[u, bend right, "\sigma_x"'] & &
\end{tikzcd}$$

- The unique section $\sigma_x$, characterized by

$$\mathrm{ev}_x\left(\sigma_x(H_x)\right) = x.$$

determines a divisor (line bundle)

$$\mathcal{O}_{U_x}(\sigma_x), \text{ or simply, } (\sigma_x),$$

on $U_x$, giving:

  - $U_x \cong \mathbb{P}\big((\pi_x)_*\mathcal{O}_{U_x}(\sigma_x)\big)$.
  - $T_{H_x} \cong (\pi_x)_*\Big(\big((ev_x^* T_X)/T_{\pi_x}\big)(-\sigma_x)\Big)$.

Is $H_x$ Fano again? - ARAUJO-CASTRAVET, Prop.1.3 AJM 2012

- Let $X$ be smooth complex projective uniruled,
- Let $(H_x, L_x)$: PMFRC through a general point $x \in X$.

Then, for any $k \geq 1$, ($B_j$ : $j$-th Bernouilli number with $B_1 = -\frac{1}{2}$ )

Fundamental Formula

$$ch_k(H_x) \;=\; \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L_x)^j \pi_{x*}\,\mathrm{ev}_x^*\big(ch_{k+1-j}(X)\big) - \frac{1}{k!} c_1(L_x)^k.$$

suggested basic strategy

- Given a strong $k$-Fano $X$ ( $\mathrm{ch}_i(X) > 0,\ 1 \le i \le k$), construct inductively a sequence of PMFRC's:

$$X = H_0 \mapsto H_1 \mapsto H_2 \mapsto \cdots H_{k-1}$$

  so that $X_j$ is strong $(k-j)$-Fano ( $\mathrm{ch}_i(X) > 0,\ 1 \le i \le k-j$).

- Araujo-Castravet did this for (strong) 2-Fano and (strong) 3-Fano, $\underline{\underline{\text{under some extra condition.}}}$

- Taku Suzuki did this for weak $k$-Fano ( $\mathrm{ch}_1(X) > 0, \mathrm{ch}_i(X) \ge 0$) for general $k \ge 2$, $\underline{\underline{\text{under some extra condition.}}}$

Araujo-Castravet, AmerJM 12

- For any strong 2-Fano with $i_X \ge 3$, its general point $x \in X$ is contained in the image of a generaically injective morphism $\mathbb{P}^2 \to X$, $\underline{\underline{\text{if } (H_x, L_x) \not\cong (\mathbb{P}^d, \mathcal{O}(2)), (\mathbb{P}^1, \mathcal{O}(3)).}}$

- For any 3-Fano with $i_X \ge 4$, its general point $x \in X$ posseses a sequence of PMFRC's: $X \mapsto H_x \mapsto W_h$, and is contained in the image of a generaically injective morphism $\mathbb{P}^3 \to X$, $\underline{\underline{\text{if } (H_x, L_x) \not\cong (\mathbb{P}^d, \mathcal{O}(2)) \text{ and } (W_h, M_h) \not\cong (\mathbb{P}^k, \mathcal{O}(2)), (\mathbb{P}^1, \mathcal{O}(3)).}}$

Taku Suzuki (Nagaoka, M.)

For any weak $k$-Fano $X$ with $\underline{\underline{i_X \ge k^2 - k + 1}}$, its general point $x \in X$ its general point $x \in X$ posseses a sequence of PMFRC's:

$$X \mapsto H_x =: H_1 \mapsto \cdots \mapsto H_{k-1},$$

and is contained in the image of a generaically injective morphism $\mathbb{P}^k \to X$, $\underline{\underline{\text{if } H_1 \not\cong Q^{\dim H_1}, H_i \not\cong \mathbb{P}^{\dim H_i} (1 \le i < k).}}$

(*There is also a similar $k$-Fano version, but still with* $\underline{\underline{i_X \ge k^2 - 2k + 1}}$ *and* $\underline{\underline{\textit{if } (H_i, L_i) \not\cong (\mathbb{P}^{\dim H_i}, \mathcal{O}(2))\ (1 \le i < k),\ (H_{k-1}, L_{k-1}) \not\cong (\mathbb{P}^1, \mathcal{O}(3)).}}$ *(M.)*)

Thus, there are $\boxed{\text{two major drawbacks}}$:

- In general, we do not know appriori whether the assumptions on $(H_i, L_i)$ are satisfied or not. So, we can only claim $X$ s covered by $\mathbb{P}^k$ under these conditions, $\underline{\underline{\text{when we are fortunate!}}}$

- $\underline{\underline{\text{The condition } i_X \ge k^2 - k + 1 \text{ (or } i_X \ge k^2 - 2k + 1) \text{ is too strong.}}}$
  For instance, when $X = \mathbb{P}^n$, as $i_X = n + 1$, we can not apply these results to derive the trivial uni-$n$-ruledness of $\mathbb{P}^n$. $\underline{\underline{\text{even if we are fortunate!}}}$

Actually, our sufficient criterion for uniregular $\mathcal{T}^l$-ruledness, with $\mathcal{T}^k$ a $k$-dimensional smooth projective toric variety, takes care of these two drawbacks appropriately.

# 3 How to avoid the extra conditions on $(H_i, L_i)$

Let us first axiomatize our situation:

Height $k$ toric tower through $x \in X$

For $1 \le l \le k$, a <u>$l$-story highet $k$ toric tower through $x \in X$</u>, schemetically denoted as:

$$\mathcal{T}^{k-l+1} \to M_1 \downarrow^{\to} M_2 \downarrow^{\to} \cdots \downarrow^{\to} M_{l-1} \downarrow^{\to} M_l = X \ni x,$$

<u>(For our applications, these occur as PMFRC's: $X = M_l \mapsto M_{l-1} \mapsto \cdots \mapsto M_2 \mapsto M_1$.)</u>

consists of the following data: It starts at the first floor with a generically finite onto its image morphism

$$f^{k-l+1} : \mathcal{T}^{k-l+1} \to M_1$$

from a $k-l+1$-dimensional smooth projective toric variety $\mathcal{T}^{k-l+1}$ to a variety $M_1$, such that:

**<u>if $l = 1$:</u>** $f^k : \mathcal{T}^k \to M_1 = X$ is passing through $x \in X$.

**<u>if $l \ge 2$:</u>** for each $1 \le m \le l-1$, there exists a diagram of varieties:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_m) & \xrightarrow{e_m} & M_{m+1} \\ {\scriptstyle \pi_m}\downarrow & & \\ M_m & & \end{array}$$

such that

- $\mathbb{P}(\mathcal{E}_m) \xrightarrow{\pi_m} M_m$ is a projectivized bundle associated with a rank 2 vector bundle $\mathcal{E}_m$, admitting a short exact sequence
$$0 \to \ell'_m \to \mathcal{E}_m \to \ell''_m \to 0$$
with <u>$\ell'_m \otimes (\ell''_m)^{-1}$ global generated.</u>
- $e_m$ is generically finite onto its image.
- For each $1 \le m \le l-1$, there exists $p_m \in \mathbb{P}(\mathcal{E}_m)$ such that:
  - $f^{k-l+1} : \mathcal{T}^{k-l+1} \to M_1$ is passing through $\pi_1(p_1) \in M_1$.
  - For each $2 \le m \le l-1$, $e_{m-1}(p_{m-1}) = \pi_m(p_m)$:

$$\begin{array}{ccc} & & \mathbb{P}(\mathcal{E}_m) \\ & & \downarrow{\scriptstyle \pi_m} \\ \mathbb{P}(\mathcal{E}_{m-1}) & \xrightarrow[e_{m-1}]{} & M_m \end{array} \qquad \begin{array}{ccc} & & p_m \\ & & \downarrow \\ p_{m-1} & \longmapsto & e_m \end{array}$$

  - $e_{l-1}(p_{l-1}) = x \in M_l = X$.

In the definition of "Height $k$ toric tower through $x \in X$", existence of such $p_m \in \mathbb{P}(\mathcal{E}_m)$ $(1 \le m \le l-1)$ is always guaranteed if the $e_{l-1}$ image of any fiber of $\pi_{l-1}$ contains $x \in X = M_l$ :

$$x \in e_{l-1}\left(\pi_{l-1}^{-1}(m_{l-1})\right) \subseteq M_l = X \text{ for any } m_{l-1} \in M_{l-1}.$$

This follows immediately from the surjectivity of $\pi_m : \mathbb{P}(\mathcal{E}_m)$ $(1 \le m \le l-1) \to M_m$ $(1 \le m \le l-1)$.

A very important observation is that any $l$-story height $k$ toric tower produces a simplest 1-story (i.e. no ceiling $\downarrow^{\rightarrow}$) height $k$ $\boxed{\text{toric}}$ tower, not merely a $(l-1)$-fold iterated $\mathbb{P}^1$-bundles over a toric $k-l+1$-fold $\mathcal{T}^{k-l+1}$:

"Toric Tower Ceilings Removable"Theorem

If there is a $l$-story height $k$ toric tower passing through $x \in X$, then there is a generically finite onto its image morphism

$$f : \mathcal{T}^k \to X$$

from a $k$-dimensional smooth projective toric variety $\mathcal{T}^k$ to $X$, passing through $x \in X$.

*Proof.* **if $l = 1$:** Nothing to prove.

**if $l \geq 2$:** Starting at the first story with the generically finite onto its image morphism

$$f^{k-l+1} : \mathcal{T}^{k-l+1} \to M_1,$$

passing through $\pi_1(p_1)$, provided by the definition, we shall ascend the stories so that at the $s$-th story ($2 \leq s \leq l$) we have the generically finite onto its image morphism

$$f^{k-l+s} : \mathcal{T}^{k-l+s} \to M_s,$$

inductively construcing by the following pullback diagram:

$$\begin{tikzcd}
\mathcal{T}^{k-l+s} = \mathbb{P}\left((f^{k-l+s})^* \mathcal{E}_{s-1}\right) \arrow[r] \arrow[d] \arrow[rr, bend left, "f^{k-l+s}"] & \mathbb{P}(\mathcal{E}_{s-1}) \arrow[r, "e_{s-1}"'] \arrow[d, "\pi_{s-1}"'] & M_s \\
\mathcal{T}^{k-l+s-1} \arrow[r, "f^{k-l+s-1}"'] & M_{s-1} &
\end{tikzcd}$$

From this, if $f^{k-l+s-1}$ is generically finite onto its image morphism, passing through $\pi_{s-1}(p_{s-1})$ $(= e_{s-2}(p_{s-2})$ if $s \geq 3$ ), we see immediately $f^{k-l+s}$ is also generically finite onto its image morphism, passing through $e_{s-1}(p_{s-1})$. Of course, we are not done yet. The problem is to show, if $\mathcal{T}^{k-l+s-1}$ is smooh toric, then $\mathcal{T}^{k-l+s} = \mathbb{P}\left((f^{k-l+s})^* \mathcal{E}_{s-1}\right)$ is still smooth toric. Then this can be shown in the following order:

- Since the pullback preserves a short exact sequence of vector bundles (see e.g. Fulton-Lang p.104), we have an exact sequence:

$$0 \to (f^{k-l+s})^*(l'_{s-1}) \to f^{k-l+s})^* \mathcal{E}_{s-1} \to (f^{k-l+s})^*(l''_{s-1}) \to 0 \tag{3}$$

- This extension (3) is classified by

$$\begin{aligned}
&\mathrm{Ext}^1\left((f^{k-l+s})^*(l''_{s-1}), (f^{k-l+s})^*(l'_{s-1})\right) \cong \mathrm{Ext}^1\left(\mathcal{O}_{\mathcal{T}^{k-l+s-1}}, (f^{k-l+s})^*(l'_{s-1} \otimes (l''_{s-1})^{-1})\right) \\
\cong &H^1\left(\mathcal{T}^{k-l+s-1}, (f^{k-l+s})^*(l'_{s-1} \otimes (l''_{s-1})^{-1})\right),
\end{aligned}$$

  which is 0 by the Demazurre vanishing [Dem] [Ful93, §3.5] [CLS], for:

– $\mathcal{T}^{k-l+s-1}$ is a smooth toric variety by the inductive assumption.

– $(f^{k-l+s})^*(l'_{s-1} \otimes (l''_{s-1})^{-1})$, a pullback of a globally generated line bundle $l'_{s-1} \otimes (l''_{s-1})^{-1}$, remains globally generated.

Consequently, the extension (3) splits.

- Then we see:

$$\mathcal{T}^{k-l+s} := \mathbb{P}\left((f^{k-l+s})^* \mathcal{E}_{s-1}\right) \overset{(3) \text{ splits}}{\cong} \mathbb{P}\left((f^{k-l+s})^*(l'_{s-1}) \oplus (f^{k-l+s})^*(l''_{s-1})\right),$$

which becomes a smooth toric variety [Oda78, §7] [CLS].

Finally, at the last $l^{th}$ story ($s = l$), we find a generically finite onto its image morphism $f^k : \mathcal{T}^k \to M_l$, passing through $\pi_{l-1}(p_{l-1}) = x \in X$, from a smooh toric $k$-fold $\mathcal{T}^k$, as desired.

□

The importance of having a toric manifold $\mathcal{T}^k$ in the above theorem is that, there are at most countably many distinct isomorphism classes of toric manifolds. Actually, "Toric Tower Ceilings Removabele"Theorem is used to deduce our sufficient criterion for uniregular-$\mathcal{T}^k$-ruledness:

A sufficient criterion for uniregular-$\mathcal{T}^k$-ruledness

For a a smooth projective variety $X$, suppose there is a subset $S \subset X$, which contains a non-emppty open subset of $X$, whose arbitrary point $x \in S$ posseses a $l$-story height $k$ toric tower through $x$:

$$TT_x = \left(\mathcal{T}^{k-l+1} \to M_1 \downarrow^{\to} M_2 \downarrow^{\to} \cdots \downarrow^{\to} M_{l-1} \downarrow^{\to} M_l = X \ni x\right)$$

Let $\mathcal{T}^k_x$ be the ceilings removed toric $k$-fold from $TT_x$ by "Toric Tower Ceiling Removable"Theorem. Then, $X$ is uniregular-$\mathcal{T}^k$-ruled for some toric $k$-fold:

$$\mathcal{T}^k \in \left\{\mathcal{T}^k_x \mid x \in S\right\}.$$

Consequently, $X$ is uni-$k$-ruled.

*Proof.*

- Let $\Lambda$ be the set of isomorphism classes of the toric $k$-folds which show up in $\left\{\mathcal{T}^k_x \mid x \in S\right\}$. Observe that: $\Lambda$ is a countable set.

- Then, the evaluation morphism

$$\coprod_{\mathcal{T}^k_i \in \Lambda} \coprod_{h_i} \left(\mathcal{T}^k_i \times \mathrm{Hom}^{h_i}(\mathcal{T}^k_i, X)\right) = \coprod_{\mathcal{T}^k_i \in \Lambda} \left(\mathcal{T}^k_i \times \left(\coprod_{h_i} \mathrm{Hom}^{h_i}(\mathcal{T}^k_i, X)\right)\right) \to X,$$

where $h_i$ runs over those generically finite onto its image components of $\mathrm{Hom}(\mathcal{T}^k_i, X)$, is dominant, for its image contains $S$ which contains a non-empty open. Since the coproduct runs over a countable quasi-projective schemes , there is some $\mathcal{T}^k_i \in \Lambda$ and $h_i$ such that the corresponding evaluation morphism at a generically finite onto its image component

$$\mathcal{T}^k_i \times \mathrm{Hom}^{h_i}(\mathcal{T}^k_i, X) \to X$$

is dominant.

- Now we may take appropriate hypersections succesively to get a desired dominant morphism

$$\mathcal{T}_i^k \times Z^{n-k} \to X$$

for some $n-k$-dimensional $Z^{n-k}$.

$\square$

# 4 How to replace the too strong condition $i_X \geq k^2 - k + 1$ ($i_X \geq k^2 - 2k; 1$)

$$\begin{cases} A^*(X) & := \oplus_{0\leq k\leq n:=\dim X} A^k(X),\ \text{Chow ring of } X,\ A_k(X) := A^{n-k}(X) \\ N^k(X) & := A^k(X)/\sim,\ N_k(X) := A_k(X)/\sim,\ \text{intersection quotients, i.e. modulo numerical equivalence} \\ & \implies N^k(X) \otimes N_k(X) \to \mathbb{Z} \text{ is a perfect pairing.} \\ N^k(X)_{\mathbb{R}} & := N^k(X) \otimes_{\mathbb{Z}} \mathbb{R} \end{cases}$$

Let us assume there exists a sequence of PMFRC's $X =: H_0 \mapsto H_1 \mapsto \cdots \mapsto H_{i-1} \mapsto H_i$ as follows:

$$\begin{tikzcd}
 & & & & U_1 \arrow[r, "e_1"] \arrow[d, "\pi_1"'] & H_0 := X \\
 & & & U_2 \arrow[r, "e_2"] \arrow[d, "\pi_2"'] & H_1 & \\
 & & & H_2 \arrow[dl, dotted, no head] & & \\
 & U_k \arrow[r, "e_k"] \arrow[d, "\pi_k"'] & H_{k-1} & & & \\
 & H_k \arrow[dl, dotted, no head] & & & & \\
U_i \arrow[r, "e_i"] \arrow[d, "\pi_i"'] & H_{i-1} & & & & \\
H_i & & & & &
\end{tikzcd} \tag{4}$$

This induces:

$$\begin{tikzcd}
N^r(X)_{\mathbb{R}} \arrow[r, "\pi_{1*}e_1^*"'] \arrow[rrrrrrr, bend left, "\overline{T}^i"] & N^{r-1}(H_1)_{\mathbb{R}} \arrow[r, "\pi_{2*}e_2^*"'] & N^{r-2}(H_2)_{\mathbb{R}} \arrow[r, dotted, no head] & N^{r-k+1}(H_{k-1})_{\mathbb{R}} \arrow[r, "\pi_{k*}e_k^*"'] & N^{r-k}(H_k)_{\mathbb{R}} \arrow[r, dotted, no head] & N^{r-i+1}(H_{i-1})_{\mathbb{R}} \arrow[r, "\pi_{i*}e_i^*"'] & N^{r-i}(H_i)_{\mathbb{R}}
\end{tikzcd}$$

Here, to simplify notations, we have set:

$$\overline{T} = \pi_{k*}e_k^*, \quad 1 \leq \forall k \leq i.$$

Thus, we have

$$\overline{T}^m : N^r(H_j)_{\mathbb{R}} \to N^{r-m}(H_{j+m}) \quad (0 \le m \le r,\ 0 \le j \le j+m \le i)$$

Then, motivated by Suzuki's work [Suz16], we made the following explicit calculation [Min17][Min18]:

M.

Assume there exists a sequence of PMFRC's

$$X \mapsto H_1 \mapsto \cdots \mapsto H_{i-1} \mapsto H_i$$

such that $\overline{T}\,(c_1(L_{k-1})) = 1$ $(2 \le k \le i)$. Then, for $j \ge 1, i \ge 1$ we have

$$\begin{aligned}
&\mathrm{ch}_j(H_i)\\
&= \left( g(i,0)_j + \sum_{k=1}^{i} \underbrace{T^k\left(\mathrm{ch}_k(X)\right)}_{scalar} g(i,k)_j + \sum_{k=i+1}^{\min\{\dim X, i+j\}} \underbrace{T^i\left(\mathrm{ch}_k(X)\right) c_1(L_i)^{i-k}}_{\text{degree } 0} g(i,k)_j \right) c_1(L_i)^j \\
&= \left( g(i,0)_j + \sum_{k=1}^{i} \underbrace{T^k\left(\mathrm{ch}_k(X)\right)}_{scalar} g(i,k)_j \right) c_1(L_i)^j + \sum_{k=i+1}^{\min\{\dim X, i+j\}} T^i\left(\mathrm{ch}_k(X)\right) g(i,k)_j c_1(L_i)^{i+j-k},
\end{aligned} \tag{5}$$

where, using the Stirling number of the first kind $\begin{bmatrix} i+q \\ k \end{bmatrix}$ (see e.g. [AIK14]),

$$g(i,k)_j = \begin{cases} -\frac{i}{j!} & k = 0 \\ \frac{(-1)^j k!}{j!} \sum_{q=\max\{k-i,1\}}^{j} \begin{bmatrix} i+q \\ k \end{bmatrix} \frac{1}{(i+q)!} \underbrace{\left( \sum_{l=1}^{q} (-1)^l \binom{q}{l} l^j \right)}_{=\ c(j,q)} & k \ge 1, j \ge \max\{k-i,1\} \\ 0 & k \ge 1, j < \max\{k-i,1\} \end{cases} \tag{6}$$

In the above formula (5), we assumed $j \ge 1$, but the case $j = 0$, corresponding to $\mathrm{ch}_0(H_i) = \dim H_i$ can be taken care of by the following classically known result:

[AC12, p.92,15th line], quoted from [Kol96, IV.2.9]

For the sequence of PMFRC's $X =: H_0 \mapsto H_1 \mapsto \cdots \mapsto H_{i-1} \mapsto H_i$,

$$\dim H_k = \mathrm{ch}_0(H_k) = \overline{T}\,(\mathrm{ch}_1(H_{k-1})) - 2, \quad (2 \le k \le i) \tag{7}$$

We should also guarantee the condition $\overline{T}\,(c_1(L_{k-1})) = 1$ $(2 \le k \le i)$ is satisfied. For this, the following result of Araujo-Castravet is crucial:

[AC12, Lem.4.5(1)]

For $2 \le k \le i$, suppose

$$\mathrm{ch}_2(H_{k-2}) > 0,\ \dim H_{k-1} \ge 1, \tag{8}$$

and $H_{k-1} \ncong \mathbb{P}^{\dim H_{k-1}}$, then $\overline{T}\,(c_1(L_{k-1})) = 1$.

All the above are already known to the author when the author announced [Min17][Min18].

However, as was already emphasized, the resulting conclusions were too restrictive. Now, the idea fo overcome this difficulty turned out to be very simple:

BASIC STRATEGY

Rewrite (5) (7)(8) in the form

$$\overline{T}^i\left(\text{degree } d \text{ polynomial of } \mathrm{ch}_k(X)\ (1 \le k \le n) \text{ and } \underbrace{\text{the scalar } l := \dim H_1 + 2 = \overline{T}(c_1(X)),}_{\text{May be regarded as a variable}}\right) > 0. \tag{9}$$

This is because, Araujo-Castravet [AC12, Lem.2.7(3)] preserves "positiveness." Thus, the condition (9) is satisfied if the following holds:

$$\begin{cases} \left(\text{degree } d \text{ polynomial of } \mathrm{ch}_k(X)\ (1 \le k \le n) \text{ and } \underbrace{\text{the scalar } l := \dim H_1 + 2 = \overline{T}(c_1(X))}_{\text{May be regarded as a variable}}\right) > 0 \\ \underbrace{0 \le d - i < \dim H_i + 1}_{\text{essentially } d-i \le \dim H_i \iff \dim H_i - d + 1 > 0, \text{ which is also taken care of along the same line.}} \end{cases} \tag{10}$$

Then we shall simply define $\boxed{\text{“}N\text{-Fano manifold”}}$ to be one which satisfy (10)s correspond to those (5)(7)(8) needed to inductively construct a sequence of PMFRC's

$$X = H_0 \mapsto H_1 \mapsto \cdots \mapsto H_{N-1} \mapsto H_N$$

with $H_i$ $(0 \le i \le N-1)$ Fano (also $\dim H_N \ge 0$, and $\overline{T}(c_1(L_i)) = 1$ $(1 \le i \le N-2)$ ). Then, by what we had shown in the previous section, we see immediately that any "$N$-Fano" is automarically uni-$N$-ruled.

Actually, it is easy to rewrite (5) in the form of (9); we only have to apply the following easy lemmas:

Compare with [AC12, Lem.2.7(1)] [Suz16, Lem.2.10(2)]

- For any $f(x) \in \mathbb{Q}[[y]]$,

$$f(c_1(L_x)) = T\left(f\left(\frac{c_1(X)}{l}\right)\frac{c_1(X)}{l}\right). \tag{11}$$

- For any $\overline{A} \in \oplus_{k>0} N^k(X), g(y) \in \mathbb{Q}[[y]]$,

$$T\left(\overline{A}\right) g\left(c_1(L_x)\right) = T\left(\overline{A} \cdot g\left(\frac{c_1(X)}{l}\right)\right). \tag{12}$$

Applying (11) (12) repeatedly, we obtain:

$$\begin{cases} c_1(L_i)^j & = T^i\left(\left(\frac{c_1(X)}{l}\right)^{i+j}\right) \quad (j \ge 0) \\ T^i(\mathrm{ch}_k(X))c_1(L_i)^p & = T^i\left(\mathrm{ch}_k(X)\left(\frac{c_1(X)}{l}\right)^p\right) \quad (i \le k, p \ge 0) \\ T^k(\mathrm{ch}_k(X))c_1(L_i)^q & = T^i\left(\mathrm{ch}_k(X)\left(\frac{c_1(X)}{l}\right)^{i-k+q}\right) \quad (i \ge k, q \ge 0) \end{cases} \tag{13}$$

Applying (13), we can immediately rewrite (5) as follows:

**Polished version of $\mathrm{ch}_J(H_i)$ and a sufficient criterion for its positivity (M.)**

Assume there exists a sequence of PMFRC's

$$X \mapsto H_1 \mapsto \cdots \mapsto H_{i-1} \mapsto H_i$$

<u>such that $\overline{T}\left(c_1(L_{k-1})\right) = 1\ (2 \le k \le i)$.</u> Then, <u>for $j \ge 1, i \ge 1$,</u> we have

$$\begin{aligned}
&\mathrm{ch}_j(H_i) \\
&= \left( g(i,0)_j + \sum_{k=1}^{i} \underbrace{T^k\left(\mathrm{ch}_k(X)\right)}_{scalar} g(i,k)_j \right) c_1(L_i)^j + \sum_{k=i+1}^{\min\{\dim X, i+j\}} T^i\left(\mathrm{ch}_k(X)\right) g(i,k)_j c_1(L_i)^{i+j-k} \\
&= T^i \left( g(i,0)_j \left(\frac{c_1(X)}{l}\right)^{i+j} + \sum_{k=1}^{i} g(i,k)_j \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+j} + \sum_{k=i+1}^{\min\{\dim X, i+j\}} g(i,k)_j \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+j} \right) \\
&= T^i \left( g(i,0)_j \left(\frac{c_1(X)}{l}\right)^{i+j} + \sum_{k=1}^{\min\{\dim X, i+j\}} g(i,k)_j \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+j} \right) \\
&= T^i \Bigg( -\frac{i}{j!} \left(\frac{c_1(X)}{l}\right)^{i+j} + \sum_{k=1}^{\min\{\dim X, i+j\}} \frac{(-1)^j k!}{j!} \sum_{q=\max\{k-i,1\}}^{j} \begin{bmatrix} i+q \\ k \end{bmatrix} \frac{1}{(i+q)!} \underbrace{\left( \sum_{l=1}^{q} (-1)^l \binom{q}{l} l^j \right)}_{=\ c(j,q)} \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+j} \Bigg) \\
&= \frac{1}{j!} T^i \left( -i \left(\frac{c_1(X)}{l}\right)^{i+j} + (-1)^j \sum_{k=1}^{\min\{\dim X, i+j\}} k! \sum_{q=\max\{k-i,1\}}^{j} \frac{\sum_{l=1}^{q} (-1)^l \binom{q}{l} l^j}{(i+q)!} \begin{bmatrix} i+q \\ k \end{bmatrix} \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+j} \right).
\end{aligned} \tag{14}$$

In particular, if $\dim H_i \ge j$, then we may conclude:

$$\begin{aligned}
&-i \left(\frac{c_1(X)}{l}\right)^{i+j} + (-1)^j \sum_{k=1}^{\min\{\dim X, i+j\}} k! \sum_{q=\max\{k-i,1\}}^{j} \frac{\sum_{l=1}^{q} (-1)^l \binom{q}{l} l^j}{(i+q)!} \begin{bmatrix} i+q \\ k \end{bmatrix} \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+j} \ > \ 0 \\
&\Longrightarrow\ ch_j(H_i) > 0.
\end{aligned} \tag{15}$$

When $j = 1, 2$, and $\boxed{i \ge 1}$, (15) is simplified to the following forms:

$$\begin{aligned}
&\left(-i + \frac{l}{i+1}\right) \left(\frac{c_1(X)}{l}\right)^{i+1} + \sum_{k=2}^{i} \frac{k!}{(i+1)!} \begin{bmatrix} i+1 \\ k \end{bmatrix} \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+1} + \mathrm{ch}_{i+1}(X)\ > \ 0 \\
&\Longrightarrow\ \mathrm{ch}_1(H_i)\ > \ 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
&i \left(-1 + \frac{l}{(i+2)(i+1)}\right) \left(\frac{c_1(X)}{l}\right)^{i+2} + \sum_{k=2}^{i} \frac{k!}{(i+2)!} \left( 2 \begin{bmatrix} i+1 \\ k-1 \end{bmatrix} + i \begin{bmatrix} i+1 \\ k \end{bmatrix} \right) \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+2} \\
&\qquad + \mathrm{ch}_{i+1}(X) \left(\frac{c_1(X)}{l}\right) + 2\mathrm{ch}_{i+2}(X)\ > \ 0 \\
&\Longrightarrow\ \mathrm{ch}_2(H_i)\ > \ 0.
\end{aligned} \tag{17}$$

For $\dim H_i$, we have the following sufficient criterion:

$\dim H_i \geq d_i$ sufficient criterion

$$\begin{aligned}
&\dim H_i \geq d_i \iff \dim H_i - d_i + 1 > 0 \\
&\iff T^i \left( \left( -(i+d_i) + \frac{l}{i} \right) \left( \frac{c_1(X)}{l} \right)^i + \sum_{k=2}^{i-1} \frac{k!}{i!} \begin{bmatrix} i \\ k \end{bmatrix} \mathrm{ch}_k(X) \left( \frac{c_1(X)}{l} \right)^{i-k} + \mathrm{ch}_i(X) \right) > 0 \\
&\Longleftarrow \left( -(i+d_i) + \frac{l}{i} \right) \left( \frac{c_1(X)}{l} \right)^i + \sum_{k=2}^{i-1} \frac{k!}{i!} \begin{bmatrix} i \\ k \end{bmatrix} \mathrm{ch}_k(X) \left( \frac{c_1(X)}{l} \right)^{i-k} + \mathrm{ch}_i(X) \quad > \quad 0.
\end{aligned} \tag{18}$$

For the definition of "$N$-Fano,"given a uniruled mfd $X$, for $1 \leq i \leq N-1$, we shall inductively construct a sequence of PMFRC's as follows:

$$X \mapsto H_1 \cdots H_{i-1} \mapsto H_i \qquad \Big( H_k : \text{Fano}, \ \dim H_k \geq N-k \ (1 \leq k \leq i-1); \ \overline{T}(L_{k-1}) = 1 \quad (2 \leq k \leq i-1) \Big)$$

$\xRightarrow{+\alpha}$ either one of the following holds:

$$\begin{cases} X \mapsto H_1 \cdots H_{i-1} \mapsto H_i = \mathbb{P}^{\dim H_i} & (\dim H_i \geq N-i : \ \underline{\text{WE ARE DONE IN THIS CASE!}}) \\ X \mapsto H_1 \cdots H_{i-1} \mapsto H_i \mapsto H_{i+1} & \big( H_k : \text{Fano}, \dim H_k \geq N-k \ (1 \leq k \leq i); \ \overline{T}(L_{k-1}) = 1 \quad (2 \leq k \leq i) \big) \end{cases} \tag{19}$$

Here, noticing that $\dim H_i \geq 0 \implies \dim H_{i-1} \geq 1 \implies \dim H_{i-2} \geq 2$, the extra conditions $+\alpha$ needed are reduced to the following:

$$\dim H_i \geq N-i, \ \underbrace{\mathrm{ch}_2(H_{i-2}) > 0}_{\text{only when } i \geq 2}, \ \mathrm{ch}_1(H_i) > 0,$$

$$\overset{(18)(16)(17)}{\Longleftarrow} \begin{cases} \left(-N + \frac{l}{i}\right) \left(\frac{c_1(X)}{l}\right)^i + \sum_{k=2}^{i-1} \frac{k!}{i!} \begin{bmatrix} i \\ k \end{bmatrix} \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k} + \mathrm{ch}_i(X) \quad > \quad 0. \\ \qquad (\underline{\text{For } i = 1, \text{ replace this with } l \geq N+1.}) \\ (i-2)\left(-1 + \frac{l}{i(i-1)}\right) \left(\frac{c_1(X)}{l}\right)^i + \sum_{k=2}^{i-2} \frac{k!}{i!} \left( 2 \begin{bmatrix} i-1 \\ k-1 \end{bmatrix} + (i-2) \begin{bmatrix} i-1 \\ k \end{bmatrix} \right) \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k} \\ \qquad + \mathrm{ch}_{i-1}(X) \left(\frac{c_1(X)}{l}\right) + 2\mathrm{ch}_i(X) \ > \ 0 \\ \qquad (\underline{\text{For } i = 1, \text{ this condition should be omitted}}) \\ \qquad (\underline{\text{For } i = 2, \text{ replace this with } \mathrm{ch}_2(X) > 0.}) \\ \qquad (\underline{\text{This condition apperas exactly like this only for } i \geq 3.}) \\ \left(-i + \frac{l}{i+1}\right) \left(\frac{c_1(X)}{l}\right)^{i+1} + \sum_{k=2}^{i} \frac{k!}{(i+1)!} \begin{bmatrix} i+1 \\ k \end{bmatrix} \mathrm{ch}_k(X) \left(\frac{c_1(X)}{l}\right)^{i-k+1} + \mathrm{ch}_{i+1}(X) \ > \ 0 \end{cases} \tag{20}$$

Now, we are ready to define our "Higher Fano" manifolds:

“$N$-Fano” manifolds

We call $X$ a “$N$-Fano”, if $X$ is a Fano manifold, i.e. $c_1(X) > 0$ and the positivity conditions (20) hold for $1 \leq i \leq N-1$.
Of course, any “$N$-Fano” is uniregular-$\mathcal{T}^N$-ruld for some toric $N$-fold $\mathcal{T}^N$ (actually, $\mathcal{T}^N$ can be taken by a generalized Bott $N$-fold), and so, uni-$N$-ruled.

A detailed version will be put on the arxiv very soon.

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