On certain vector valued Siegel modular forms of type $(k, 2)$ over $\mathbb{Z}_{(p)}$ and Sturm bound

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Abstract

We give generators of the module of vector valued Siegel modular forms of type $(k, 2)$ over the ring of scalar valued Siegel modular forms of even weight for $\mathbb{Z}_{(p)}$. This gives an example of the positive solution to more general problem whether the module of vector valued modular forms of arbitrary degree is finitely generated over the ring of modular forms for $\mathbb{Z}_{(p)}$. Moreover we study analogues of Sturm's bounds for vector valued Siegel modular forms of type $(k, 2)$.

1 Introduction

In this report, we state two kinds of results on the vector valued Siegel modular forms of type $(k, 2)$ with degree 2;
- on the finite generations of the module of them having p-integral rational Fourier coefficients,
- on the Strum type bounds for them.

We state our results more precisely. Let p be a prime number and $\mathbb{Z}_{(p)}$ the ring of p-integral rational numbers. Let $M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ be the space consisting of all scalar valued modular forms of weight k of degree 2 whose Fourier coefficients are in $\mathbb{Z}_{(p)}$. We set $M_*^{\text{ev}}(\mathbb{Z}_{(p)}) := \oplus_{k\in 2\mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$. We take Igusa's generators over $\mathbb{Z}$ (given in [3]) φ_6, X_{10}, X_{12} of weight 6, 10, 12 respectively.

A Siegel modular form of type $(k, 2)$ is a holomorphic function f on the Siegel upper-half plane $\mathbb{H}_2$ with values in $\text{Sym}_2(\mathbb{C})$, satisfying

$$f(M\langle Z\rangle) = \det(CZ + D)^k (CZ + D) f(Z)^t(CZ + D)$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the Siegel modular group $\Gamma_2 = Sp_2(\mathbb{Z})$ and for all $Z \in \mathbb{H}_2$. Here $(k, 2)$ comes from the fact that the automorphy factor is the one of representatives in the equivalence class of the representation $\det^k \otimes \text{Sym}(2)$. We denote by $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ the module consisting of all such f whose Fourier coefficients are in $\text{Sym}_2^*(\mathbb{Z}_{(p)}) := \{T = (t_{ij}) \in \text{Sym}_2(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}_{(p)}\}$.

The following two results concern the finite generations.

Theorem 1. *For each even integer k and each prime $p \geq 5$, there exist six generators over $M_*^{ev}(\mathbb{Z}_{(p)})$ of the module $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ whose determinant weights are $10, 14, 16, 16, 18, 22$. If we write them as $\Phi_k \in M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} (k = 10, 14, 16, 18, 22)$ and $\Psi_{16} \in M_{16,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, then we have (as a $\mathbb{Z}_{(p)}$-module)*

$$\begin{aligned} M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} =& M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}} \Phi_{10} \oplus M_{k-14}(\Gamma_2)_{\mathbb{Z}_{(p)}} \Phi_{14} \\ &\oplus M_{k-16}(\Gamma_2)_{\mathbb{Z}_{(p)}} \Phi_{16} \oplus V_{k-16}(\Gamma_2)_{\mathbb{Z}_{(p)}} \Psi_{16} \\ &\oplus V_{k-18}(\Gamma_2)_{\mathbb{Z}_{(p)}} \Phi_{18} \oplus W_{k-22}(\Gamma_2)_{\mathbb{Z}_{(p)}} \Phi_{22}, \end{aligned}$$

where

$$V_k(\Gamma_2)_{\mathbb{Z}_{(p)}} = M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \cap \mathbb{Z}_{(p)}[\varphi_6, X_{10}, X_{12}], \qquad W_k(\Gamma_2)_{\mathbb{Z}_{(p)}} = M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \cap \mathbb{Z}_{(p)}[X_{10}, X_{12}].$$

Theorem 2. *For each odd integer k and each prime $p \geq 5$, there exist four generators over $M_*^{ev}(\mathbb{Z}_{(p)})$ of the module $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ whose determinant weights are $21, 23, 27, 29$. If we write them as $\Phi_k \in M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}(k = 21, 23, 27, 29)$, then we have (as a $\mathbb{Z}_{(p)}$-module)*

$$\begin{aligned} M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} =& M_{k-21}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{21} \oplus M_{k-23}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{23} \\ & \oplus M_{k-27}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{27} \oplus V_{k-29}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{29}, \end{aligned}$$

where

$$V_k(\Gamma_2)_{\mathbb{Z}_{(p)}} = M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \cap \mathbb{Z}_{(p)}[\varphi_6, X_{10}, X_{12}].$$

We will construct explicitly Φ_k and Ψ_k by taking constant multiples of Satoh's generators given in [9] and of Ibukiyama's generators given in [2].

The next two results concern the Sturm type bounds.

Theorem 3. *For each even integer k and each prime $p \geq 5$, suppose that F is a Siegel modular form in $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ having the form*

$$F(\tau, \tau', \omega) = \sum_{\substack{m,n \in \mathbb{Z},\ r \in \frac{1}{2}\mathbb{Z} \\ m,n,mn-r^2 \geq 0}} A(m,n,r) q_\tau^m q_{\tau'}^n q_\omega^{2r}$$

with $q_\tau = e^{2\pi i \tau}, q_{\tau'} = e^{2\pi i \tau'}, q_\omega = e^{2\pi i \omega}$ and $\left(\begin{smallmatrix} \tau & \omega \\ \omega & \tau' \end{smallmatrix}\right)$ in the Siegel upper half space $\mathbb{H}_2$ of degree 2, where $\mathbb{H}_2 := \left\{\left(\begin{smallmatrix} \tau & \omega \\ \omega & \tau' \end{smallmatrix}\right) \in M_2(\mathbb{C}) \mid Im\left(\begin{smallmatrix} \tau & \omega \\ \omega & \tau' \end{smallmatrix}\right) > 0\right\}$. If $A(m,n,r) \equiv 0 \pmod{p}$, i.e. all the elements of $A(m,n,r)$ are congruent to 0 mod p, for every m, n such that

$$0 \leq m \leq \left[\frac{k}{10}\right], \quad 0 \leq n \leq \left[\frac{k}{10}\right],$$

then $F \equiv 0 \pmod{p}$.

Theorem 4. *For each odd integer k and each prime $p \geq 5$, suppose that F is a Siegel modular form in $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ having the form*

$$F(\tau, \tau', \omega) = \sum_{\substack{m,n \in \mathbb{Z},\ r \in \frac{1}{2}\mathbb{Z} \\ m,n,mn-r^2 \geq 0}} A(m,n,r) q_\tau^m q_{\tau'}^n q_\omega^{2r}$$

with $q_\tau = e^{2\pi i \tau}, q_{\tau'} = e^{2\pi i \tau'}, q_\omega = e^{2\pi i \omega}$ and $\left(\begin{smallmatrix} \tau & \omega \\ \omega & \tau' \end{smallmatrix}\right)$ in the Siegel upper half space $\mathbb{H}_2$ of degree 2, where $\mathbb{H}_2 := \left\{\left(\begin{smallmatrix} \tau & \omega \\ \omega & \tau' \end{smallmatrix}\right) \in M_2(\mathbb{C}) \mid Im\left(\begin{smallmatrix} \tau & \omega \\ \omega & \tau' \end{smallmatrix}\right) > 0\right\}$. If $A(n,m,r) \equiv 0 \pmod{p}$ for every m, n such that

$$0 \leq m \leq \left[\frac{k-7}{10}\right], \quad 0 \leq n \leq \left[\frac{k-1}{10}\right],$$

then $F \equiv 0 \pmod{p}$.

2 Preliminary

2.1 Siegel modular forms of type $(k, 2)$ and degree 2

The Siegel upper-half space of degree 2 is defined as

$$\mathbb{H}_2 := \{Z = X + iY \in \mathrm{Sym}_2(\mathbb{C}) \mid Y > 0 \text{ (positive definite) }\}.$$

The real symplectic group $\mathrm{Sp}_2(\mathbb{R})$ acts on $\mathbb{H}_2$ in the following way:

$$Z \longrightarrow M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$
$$Z \in \mathbb{H}_2,\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}).$$

A Siegel modular form of type $(k, 2)$ on Γ_2 is a holomorphic function f on $\mathbb{H}_2$ with values in $\mathrm{Sym}_2(\mathbb{C})$, satisfying

$$f(M\langle Z\rangle) = \det(CZ+D)^k (CZ+D) f(Z) {}^t(CZ+D)$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2$ and for all $Z \in \mathbb{H}_2$. Here $(k, 2)$ comes from the fact that the automorphy factor is the one of representatives in the equivalence class of the representation $\det^k \otimes \mathrm{Sym}(2)$.

We denote by $M_{k,2}(\Gamma_2)$ (resp. $S_{k,2}(\Gamma_2)$) the module of Siegel modular forms (resp. cusp forms) of type $(k, 2)$ on Γ_2 .

2.2 Fourier expansions

Any $F(Z) \in M_{k,2}(\Gamma_2)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \Lambda_2} a(T; F) \exp(2\pi i \mathrm{tr}(TZ)), \quad a(T; F) \in \mathrm{Sym}_2(\mathbb{C}),$$

where T runs over all positive semi-definite elements of Λ_2 defined as

$$\Lambda_2 := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}.$$

Taking $q_\tau := \exp(2\pi i\tau)$, $q_\omega := \exp(2\pi i\omega)$ and $q_{\tau'} := \exp(2\pi i\tau')$ for $Z = \left(\begin{smallmatrix} \tau & \omega \\ \omega & \tau' \end{smallmatrix}\right) \in \mathbb{H}_2$, we can write

$$q^T := \exp(2\pi i \mathrm{tr}(TZ)) = q_\omega^{2t_{12}} q_\tau^{t_{11}} q_{\tau'}^{t_{22}}.$$

Using this notation, we have the generalized q-expansion:

$$\begin{aligned} F &= \sum_{0 \leq T \in \Lambda_2} a(T; F) q^T \\ &= \sum_{0 \leq (t_{ij}) \in \Lambda_2} (a(T; F) q_\omega^{2t_{12}}) q_\tau^{t_{11}} q_{\tau'}^{t_{22}} \in \mathrm{Sym}_2(\mathbb{C})[q_\omega^{-1}, q_\omega][\![q_\tau, q_{\tau'}]\!]. \end{aligned}$$

For any subring R of $\mathbb{C}$, we denote by $M_{k,2}(\Gamma_2)_R$ the R-module consisting of those F in $M_{k,2}(\Gamma_2)$ for which $a(T; F)$ is in $\mathrm{Sym}_2^*(R)$ for every $T \in \Lambda_2$ where

$$\mathrm{Sym}_2^*(R) := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{C}) \mid t_{ii}, 2t_{ij} \in R\}.$$

From this, any element F in $M_{k,2}(\Gamma_2)_R$ can be regarded as an element of the ring of formal power series $\mathrm{Sym}_2^*(R)[q_\omega^{-1}, q_\omega][\![q_\tau, q_{\tau'}]\!]$.

2.3 Generators of scalar valued Siegel modular forms

Let φ_4, φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 4, 6, 10, 12, respectively given in [3]. Let $M_k(\Gamma_2)$ (resp. $S_k(\Gamma_2)$) be the $\mathbb{C}$-vector space consisting of the scalar valued Siegel modular forms (resp. cusp forms) of weight k on Γ_2 . We denote by $M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ (resp. $S_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$) the $\mathbb{Z}_{(p)}$-module consisting of the scalar valued Siegel modular forms in $M_k(\Gamma_2)$ (resp. cusp forms in $S_k(\Gamma_2)$) for which $a(T; F)$ is in $\mathbb{Z}_{(p)}$ for every $T \in \Lambda_2$. By the result of Nagaoka [7], we have

$$\begin{aligned} M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)}) &:= \bigoplus_{k \in 2\mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \\ &= \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{10}, X_{12}], \text{ if } p \geq 5. \end{aligned}$$

2.4 p-order of modular forms

We shall define p-order of modular forms. Let p be a prime with $p \geq 5$ and ν_p the additive valuation on $\mathbb{Q}$ normalized as $\nu_p(p) = 1$.

Let F be a formal power series with bounded denominators of the form

$$F = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; F) q^T, \quad a(T; F) \in \mathrm{Sym}_2(\mathbb{Q}).$$

In the scalar valued case, let ν_p be just as in Böcherer-Nagaoka[1] and elsewhere. Define a value ν_p for F with $a(T; F) \in \mathrm{Sym}_2(\mathbb{Q})$ as

$$\nu_p(F) := \inf \left\{ \nu_p(a(T; F)) \mid T \in \frac{1}{N}\Lambda_2 \right\},$$

where $\nu_p\left(\frac{a'}{a}, \frac{b'}{b}, \frac{c'}{c}\right) := \nu_p(\gcd(a', b', c') / \gcd(a, b, c))$ for a, a', b, b', c, $c' \in \mathbb{Z}$. Moreover, we define an order "$\succ$" for two elements of $\frac{1}{N}\Lambda_2$ following [6]. The following statement and its proof are due to Kikuta:

Lemma 1. *(1) For* $f = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; f) q^T$ *and* $g = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; g) q^T$ *with* $a(T; f)$, $a(T; g) \in \mathbb{Q}$, *we have* $\nu_p(fg) = \nu_p(f) + \nu_p(g)$.
(2) Let $F = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; F) q^T$ *with* $a(T; F) \in \mathrm{Sym}_2(\mathbb{Q})$ *and* g *be as in (1). Then we have* $\nu_p(Fg) = \nu_p(F) + \nu_p(g)$.

The proof of Lemma 1 is, for example, in [4].

We remark that, for a formal power series of the form

$$F = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; F) q^T, \quad a(T; F) \in \mathrm{Sym}_2(\mathbb{Q}),$$

we have $a(T; F) \in \mathrm{Sym}_2^*(\mathbb{Z}_{(p)})$ for all $T \in \frac{1}{N}\Lambda_2$ if and only if $\nu_p(F) \geq 0$.

2.5 Generators of vector valued Siegel modular forms

Let R be a subring of $\mathbb{C}$. For a formal power series f of the form

$$f = \sum_{T \in \Lambda_2} a(T; f) q^T \in R[q_\omega^{-1}, q_\omega][\![q_\tau, q_{\tau'}]\!],$$

the theta operator $\Theta^{[1]}$ is defined by

$$\Theta^{[1]}(f) = \sum_{T \in \Lambda_2} T \cdot a(T; f) q^T \in \mathrm{Sym}_2^*(R)[q_\omega^{-1}, q_\omega][\![q_\tau, q_{\tau'}]\!].$$

Let $f \in M_k(\Gamma_2)$ and $g \in M_j(\Gamma_2)$. We put

$$[f, g] := \frac{1}{j} f \Theta^{[1]}(g) - \frac{1}{k} g \Theta^{[1]}(f).$$

Then the results of Satoh [9] states that $[f, g] \in M_{k+j,2}(\Gamma_2)$.

Let φ_4, φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 4, 6, 10, 12, respectively given in [3]. It is known that the $M_*^{\mathrm{ev}}(\Gamma_2)$-module of Siegel modular forms of type $(k, 2)$ has six generators:

Theorem 5 (Satoh [9]). *For each even integer k, we have (as a $\mathbb{C}$-vector space)*

$$\begin{aligned} M_{k,2}(\Gamma_2) = & M_{k-10}(\Gamma_2)[\varphi_4,\varphi_6] \oplus M_{k-14}(\Gamma_2)[\varphi_4, X_{10}] \\ & \oplus M_{k-16}(\Gamma_2)[\varphi_4, X_{12}] \oplus V_{k-16}(\Gamma_2)[\varphi_6, X_{10}] \\ & \oplus V_{k-18}(\Gamma_2)[\varphi_6, X_{12}] \oplus W_{k-22}(\Gamma_2)[X_{10}, X_{12}], \end{aligned}$$

where

$$V_k(\Gamma_2) = M_k(\Gamma_2) \cap \mathbb{C}[\varphi_6, X_{10}, X_{12}], \qquad W_k(\Gamma_2) = M_k(\Gamma_2) \cap \mathbb{C}[X_{10}, X_{12}].$$

We construct Φ_k ($k = 10$, 14, 16, 18, 22) and Ψ_{16} by taking constant multiples of these generators:

$$\Phi_{10} = -\frac{1}{144}[\varphi_4,\varphi_6], \qquad \Phi_{14} = 10[\varphi_4, X_{10}], \qquad \Phi_{16} = 12[\varphi_4, X_{12}],$$
$$\Psi_{16} = 10[\varphi_6, X_{10}], \qquad \Phi_{18} = 12[\varphi_6, X_{12}], \qquad \Phi_{22} = -60[X_{10}, X_{12}].$$

Then we have

$$a\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \Phi_{10}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \Phi_{14}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \qquad a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \Phi_{16}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$
$$a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \Psi_{16}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \qquad a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}; \Phi_{18}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \qquad a\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \Phi_{22}\right) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

For modular forms $F_1 \in M_{k_1}(\Gamma_2)$, $F_2 \in M_{k_2}(\Gamma_2)$ and $F_3 \in M_{k_3}(\Gamma_2)$, we define

$$\begin{aligned} [F_1,F_2,F_3] := -\frac{1}{4\pi^2}\Biggl(& k_1F_1 \begin{pmatrix} \frac{\partial F_2}{\partial \tau}\frac{\partial F_3}{\partial \omega} - \frac{\partial F_2}{\partial \omega}\frac{\partial F_3}{\partial \tau} & \frac{\partial F_2}{\partial \tau}\frac{\partial F_3}{\partial \tau'} - \frac{\partial F_2}{\partial \tau'}\frac{\partial F_3}{\partial \tau} \\ \frac{\partial F_2}{\partial \tau}\frac{\partial F_3}{\partial \tau'} - \frac{\partial F_2}{\partial \tau'}\frac{\partial F_3}{\partial \tau} & \frac{\partial F_2}{\partial \omega}\frac{\partial F_3}{\partial \tau'} - \frac{\partial F_2}{\partial \tau'}\frac{\partial F_3}{\partial \omega} \end{pmatrix} \\ -k_2F_2 & \begin{pmatrix} \frac{\partial F_1}{\partial \tau}\frac{\partial F_3}{\partial \omega} - \frac{\partial F_1}{\partial \omega}\frac{\partial F_3}{\partial \tau} & \frac{\partial F_1}{\partial \tau}\frac{\partial F_3}{\partial \tau'} - \frac{\partial F_1}{\partial \tau'}\frac{\partial F_3}{\partial \tau} \\ \frac{\partial F_1}{\partial \tau}\frac{\partial F_3}{\partial \tau'} - \frac{\partial F_1}{\partial \tau'}\frac{\partial F_3}{\partial \tau} & \frac{\partial F_1}{\partial \omega}\frac{\partial F_3}{\partial \tau'} - \frac{\partial F_1}{\partial \tau'}\frac{\partial F_3}{\partial \omega} \end{pmatrix} + k_3F_3 \begin{pmatrix} \frac{\partial F_1}{\partial \tau}\frac{\partial F_2}{\partial \omega} - \frac{\partial F_1}{\partial \omega}\frac{\partial F_2}{\partial \tau} & \frac{\partial F_1}{\partial \tau}\frac{\partial F_2}{\partial \tau'} - \frac{\partial F_1}{\partial \tau'}\frac{\partial F_2}{\partial \tau} \\ \frac{\partial F_1}{\partial \tau}\frac{\partial F_2}{\partial \tau'} - \frac{\partial F_1}{\partial \tau'}\frac{\partial F_2}{\partial \tau} & \frac{\partial F_1}{\partial \omega}\frac{\partial F_2}{\partial \tau'} - \frac{\partial F_1}{\partial \tau'}\frac{\partial F_2}{\partial \omega} \end{pmatrix}\Biggr). \end{aligned}$$

Then the results of Ibukiyama [2] states that $[F_1,F_2,F_3] \in M_{k_1+k_2+k_3+1,2}(\Gamma_2)$.

Let φ_4, φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 4, 6, 10, 12, respectively given in [3]. It is known that the $M_*^{\mathrm{ev}}(\Gamma_2)$-module of Siegel modular forms of type $(k,2)$ with an odd integer k has four generators:

Theorem 6 (Ibukiyama [2]). *For each odd integer k, we have (as a $\mathbb{C}$-vector space)*

$$\begin{aligned} M_{k,2}(\Gamma_2) = & M_{k-21}(\Gamma_2)[\varphi_4,\varphi_6,X_{10}] \oplus M_{k-23}(\Gamma_2)[\varphi_4,\varphi_6, X_{12}] \\ & \oplus M_{k-27}(\Gamma_2)[\varphi_4, X_{10}, X_{12}] \oplus V_{k-29}(\Gamma_2)[\varphi_6, X_{10}, X_{12}], \end{aligned}$$

where

$$V_k(\Gamma_2) = M_k(\Gamma_2) \cap \mathbb{C}[\varphi_6, X_{10}, X_{12}].$$

We construct Φ_k ($k = 21$, 23, 27, 29) by taking constant multiples of these generators:

$$\Phi_{21} = \frac{1}{3456}[\varphi_4,\varphi_6,X_{10}], \qquad \Phi_{23} = \frac{1}{3456}[\varphi_4,\varphi_6,X_{12}],$$
$$\Phi_{27} = -\frac{1}{48}[\varphi_4,X_{10},X_{12}], \qquad \Phi_{29} = -\frac{1}{72}[\varphi_6,X_{10},X_{12}].$$

Then we have

$$a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \mid \varphi_{21}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad a\left(\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \mid \varphi_{23}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$a\left(\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \mid \varphi_{27}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad a\left(\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \mid \varphi_{29}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2.6 The Witt operator

Let F be a holomorphic function on $\mathbb{H}_2$. Then the Witt operator is defined by

$$W(F)(\tau,\tau') := F\begin{pmatrix}\tau & 0\\ 0 & \tau'\end{pmatrix}, \quad (\tau,\tau')\in\mathbb{H}_1\times\mathbb{H}_1.$$

This operator was first introduced in Witt[8]. We extend the Witt operator to the case of vector valued forms. Let $G = \left(\begin{smallmatrix}G_{11} & G_{12}\\ G_{12} & G_{22}\end{smallmatrix}\right) \in M_{k,2}(\Gamma_2,\nu)$ be a vector valued Siegel modular form of type $(k,2)$ on Γ_2 with character ν, then we define

$$W(G)(\tau,\tau') := \begin{pmatrix}W(G_{11}) & W(G_{12})\\ W(G_{12}) & W(G_{22})\end{pmatrix}, \quad (\tau,\tau')\in\mathbb{H}_1\times\mathbb{H}_1.$$

For later use, we introduce some examples:

$$\begin{aligned}
&W(\varphi_4)(\tau,\tau') = E_4(\tau)E_4(\tau'), \qquad W(\varphi_6)(\tau,\tau') = E_6(\tau)E_6(\tau'),\\
&W(X_{10})(\tau,\tau') \equiv 0, \qquad W(X_{12})(\tau,\tau') = 12\Delta(\tau)\Delta(\tau'),\\
&W(\Delta_5)(\tau,\tau') \equiv 0,\\
&W(\Phi_{10}) = -\begin{pmatrix}\Delta(\tau)E_4(\tau')E_6(\tau') & 0\\ 0 & E_4(\tau)E_6(\tau)\Delta(\tau')\end{pmatrix},\\
&W(\Phi_{16}) = -12\begin{pmatrix}E_6(\tau)\Delta(\tau)E_4(\tau')\Delta(\tau') & 0\\ 0 & E_4(\tau)\Delta(\tau)E_6(\tau')\Delta(\tau')\end{pmatrix},\\
&W(\Phi_{18}) = -12\begin{pmatrix}E_4(\tau)^2\Delta(\tau)E_6(\tau')\Delta(\tau') & 0\\ 0 & E_6(\tau)\Delta(\tau)E_4(\tau')^2\Delta(\tau')\end{pmatrix},\\
&W(\Phi_{14}) = W(\Psi_{16}) = W(\Phi_{22}) \equiv 0,\\
&W(\Phi_9) = -2E_4(\tau)E_4(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix},\\
&W(\Phi_{11}) = -2E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix},\\
&W(\Phi_{17}) = -144\Delta(\tau)\Delta(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix},\\
&W(\Phi_{23}) = 12\Delta(\tau)\Delta(\tau')(E_4(\tau)^3\Delta(\tau') - \Delta(\tau)E_4(\tau')^3)\begin{pmatrix}0\,1\\1\,0\end{pmatrix}\\
&W(\Phi_{21}) = W(\Phi_{27}) = W(\Phi_{29}) \equiv 0,
\end{aligned}$$

where

$$\Phi_9 := 10[\varphi_4,\Delta_5], \qquad \Phi_{11} := 10[\varphi_6,\Delta_5], \qquad \Phi_{17} := -60[\Delta_5, X_{12}],$$

and η is the usual Dedekind eta function defined as $\eta(\tau) = q_\tau^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q_\tau^m)$. Then we have $\Phi_{14} = \Delta_5\cdot\Phi_9'$, $\Psi_{16} = \Delta_5\cdot\Phi_{11}$, $\Phi_{22} = \Delta_5\cdot\Phi_{17}$.

In [7], Nagaoka proved the following property.

Lemma 2 (Nagaoka[7]). *Let $F\in\mathbb{Q}[\![q_\tau, q_{\tau'}]\!]$ be a formal power series of the form*

$$F = \sum_{a,b,c\geq 0}\gamma_{abc}W(\varphi_4)^aW(\varphi_6)^bW(X_{12})^c, \quad \gamma_{abc}\in\mathbb{Q}.$$

If $\nu_p(F)\geq 0$, then we have γ_{abc} as $\nu_p(\gamma_{abc})\geq 0$ for all $a,b,c\geq 0$.

From this Lemma, we get the following corollary.

Corollary 1. *Let* $F \in \mathbb{Q}[\![q_\tau, q_{\tau'}]\!]$ *be a formal power series of the form*

$$F = \sum_{a,b,c \geq 0} \gamma_{abc} W(\varphi_4)^a W(\varphi_6)^b W(X_{12})^c, \quad \gamma_{abc} \in \mathbb{Q}.$$

If $\nu_p(F) \geq 1$, *then we have* γ_{abc} *as* $\nu_p(\gamma_{abc}) \geq 1$ *for all* $a, b, c \geq 0$.

Proof. Since $\nu_p(F) \geq 1$, we get $\nu_p(\frac{1}{p}F) \geq 0$. Hence from Lemma 2, we can take

$$\frac{1}{p}F = \sum_{a,b,c \geq 0} \frac{1}{p}\gamma_{abc} W(\varphi_4)^a W(\varphi_6)^b W(X_{12})^c, \quad \nu_p(\frac{1}{p}\gamma_{abc}) \geq 0.$$

Hence we can take $\nu_p(\gamma_{abc}) \geq 1$ for all $a, b, c \geq 0$. □

3 Proofs

3.1 Proof of Theorem 1

Since Φ_i and Ψ_{16} have Fourier coefficients in $\mathbb{Z}$, the inclusion "$\supset$" is clear. We can prove also the converse inclusion "$\subset$" by an inductive argument on the determinant weight with application of the Witt operator. For more details, see [4].

3.2 Proof of Theorem 3

We prove it by an inductive argument on the determinant weight. By Theorem 1 , for any $F \in M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, we can write F in the form

$$\begin{aligned} F =& (P_1 + X_{10}Q_1)\Phi_{10} + (P_2 + X_{10}Q_2)\Phi_{14} + (P_3 + X_{10}Q_3)\Phi_{16} \\ &+ (P_4 + X_{10}Q_4)\Psi_{16} + (P_5 + X_{10}Q_5)\Phi_{18} + (P_6 + X_{10}Q_6)\Phi_{22}, \end{aligned}$$

where $P_1 \in M_{k-10}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, $Q_1 \in M_{k-20}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_2 \in M_{k-14}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, $Q_2 \in M_{k-24}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_3 \in M_{k-16}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, $Q_3 \in M_{k-26}(\Gamma_2)_{\mathbb{Z}_{(p)}}$,
$P_4 \in V_{k-16}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_6, X_{12}]$, $Q_4 \in V_{k-26}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_5 \in V_{k-18}(\Gamma_2) \cap \mathbb{Z}_{(p)}[\varphi_6, X_{12}]$, $Q_5 \in V_{k-28}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, $P_6 \in W_{k-22}(\Gamma_2) \cap \mathbb{Z}_{(p)}[X_{12}]$, $Q_6 \in W_{k-32}(\Gamma_2)_{\mathbb{Z}_{(p)}}$.

Here we regard P_i as polynomials (with coefficient $\mathbb{Z}_{(p)}$) $P_1 = P_1(\varphi_4, \varphi_6, X_{12})$, $P_2 = P_2(\varphi_4, \varphi_6, X_{12})$, $P_3 = P_3(\varphi_4, \varphi_6, X_{12})$, $P_4 = P_4(\varphi_6, X_{12})$, $P_5 = P_5(\varphi_6, X_{12})$, $P_6 = P_6(X_{12})$.

We apply Witt operator to F. Since $W(X_{10}) = W(\Phi_{14}) = W(\Psi_{16}) = W(\Phi_{22}) = 0$, we get

$$\begin{aligned} W(F) &= W(P_1)W(\Phi_{10}) + W(P_3)W(\Phi_{16}) + W(P_5)W(\Phi_{18}) \\ &= \begin{pmatrix} \sum_{f \in M_{k+2}(\Gamma_1), g \in M_k(\Gamma_1)} f(\tau)g(\tau') & 0 \\ 0 & \sum_{f \in M_{k+2}(\Gamma_1), g \in M_k(\Gamma_1)} g(\tau)f(\tau') \end{pmatrix} \\ &= \begin{pmatrix} ME_1 & 0 \\ 0 & ME_2 \end{pmatrix} \\ &:= \begin{pmatrix} \sum_{m,n \geq 0} B_{11}(m,n) q_\tau^m q_{\tau'}^n & 0 \\ 0 & \sum_{m,n \geq 0} B_{22}(m,n) q_\tau^m q_{\tau'}^n \end{pmatrix}, \end{aligned}$$

where

$$ME_1 = \sum_{\substack{12i+4j+6t=k+2\\12i'+4j'+6t'=k\\t,t'=0,1}} C_1(i,i')\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t \Delta(\tau')^{i'} E_4(\tau')^{j'} E_6(\tau')^{t'}$$

$$ME_2 = \sum_{\substack{12i+4j+6t=k\\12i'+4j'+6t'=k+2\\t,t'=0,1}} C_2(i,i')\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t \Delta(\tau')^{i'} E_4(\tau')^{j'} E_6(\tau')^{t'}.$$

The q_τ-expansion of $\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t$ has the form

$$\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t = q_\tau^i + \cdots .$$

The numbers j and t are uniquely determined by choosing a value of i.

For each m, n such that $0 \le m, n \le \left[\frac{k}{10}\right]$, $A(m,n,r) \equiv 0 \pmod p$. We have that if $m \le \left[\frac{k}{10}\right]$ and $n \le \left[\frac{k}{10}\right]$, then $B_{11}(m,n) \equiv B_{22}(m,n) \equiv 0 \pmod p$. This implies that $C_1(i,i') \equiv C_2(i,i') \equiv 0 \pmod p$ for $i, i' \le \left[\frac{k}{10}\right]$. Note that $i, i' \le \left[\frac{k}{10}\right]$ since $12i+4j+6t = k$ or $k+2$ and $12i'+4j'+6t' = k$ or $k+2$ and $\left[\frac{k}{12}\right] \le \left[\frac{k+2}{12}\right] \le \left[\frac{k}{10}\right]$. Thus we have $W(F) \equiv 0 \pmod p$.

Lemma 3. $P_1, P_3, P_5 \equiv 0 \pmod p$.

Proof of Lemma 3. Using fact that $W(\Delta_5) = 0$, we get

$$\begin{aligned}
W(F) =& W(P_1)W(\Phi_{10}) + W(P_3)W(\Phi_{16}) + W(P_5)W(\Phi_{18})\\
=& -W(P_1)\begin{pmatrix}\Delta(\tau)E_4(\tau')E_6(\tau') & 0\\ 0 & E_4(\tau)E_6(\tau)\Delta(\tau')\end{pmatrix}\\
&-12W(P_3)\begin{pmatrix}E_6(\tau)\Delta(\tau)E_4(\tau')\Delta(\tau') & 0\\ 0 & E_4(\tau)\Delta(\tau)E_6(\tau')\Delta(\tau')\end{pmatrix}\\
&-12W(P_5)\begin{pmatrix}E_4(\tau)^2\Delta(\tau)E_6(\tau')\Delta(\tau') & 0\\ 0 & E_6(\tau)\Delta(\tau)E_4(\tau')^2\Delta(\tau')\end{pmatrix}\\
=& \left\{-W(P_1)\begin{pmatrix}E_4(\tau')E_6(\tau') & 0\\ 0 & E_4(\tau)E_6(\tau)\end{pmatrix}\right.\\
&-12W(P_3)\begin{pmatrix}E_6(\tau)E_4(\tau')\Delta(\tau') & 0\\ 0 & E_4(\tau)\Delta(\tau)E_6(\tau')\end{pmatrix}\\
&\left.-12W(P_5)\begin{pmatrix}E_4(\tau)^2E_6(\tau')\Delta(\tau') & 0\\ 0 & E_6(\tau)\Delta(\tau)E_4(\tau')^2\end{pmatrix}\right\}\begin{pmatrix}\Delta(\tau) & 0\\ 0 & \Delta(\tau')\end{pmatrix}\\
=& \begin{pmatrix}f_{11} & 0\\ 0 & f_{22}\end{pmatrix}\begin{pmatrix}\Delta(\tau) & 0\\ 0 & \Delta(\tau')\end{pmatrix}.
\end{aligned}$$

where the $(1,1)$-component and $(2,2)$-component of $W(F)$ are

$$\begin{aligned}
f_{11}\Delta(\tau) &= (-W(P_1)E_4(\tau')E_6(\tau') - 12W(P_3)E_6(\tau)E_4(\tau')\Delta(\tau') - 12W(P_5)E_4(\tau)^2E_6(\tau')\Delta(\tau'))\Delta(\tau),\\
f_{22}\Delta(\tau') &= (-W(P_1)E_4(\tau)E_6(\tau) - 12W(P_3)E_4(\tau)\Delta(\tau)E_6(\tau') - 12W(P_5)E_6(\tau)\Delta(\tau)E_4(\tau')^2)\Delta(\tau').
\end{aligned}$$

Since $\nu_p(W(F)) \ge 1$, and $\nu_p(\Delta(\tau)) = \nu_p(\Delta(\tau')) = 0$, we have $\nu_p(f_{11}) = \nu_p(f_{22}) \ge 1$. Then we get

$$\begin{aligned}
&\nu_p(E_4(\tau)\Delta(\tau)E_6(\tau')f_{11} - E_6(\tau)E_4(\tau')\Delta(\tau')f_{22})\\
&= \nu_p((E_4(\tau)^3\Delta(\tau') - \Delta(\tau)E_4(\tau')^3)(E_4(\tau)E_4(\tau')W(P_1) + 2^8\cdot 3^4\Delta(\tau)\Delta(\tau')W(P_5)))\\
&\ge 1
\end{aligned}$$

and

$$\begin{aligned}
&\nu_p(-E_4(\tau)E_6(\tau)f_{11} + E_4(\tau')E_6(\tau')f_{22}) \\
&= \nu_p(2^2 \cdot 3(E_4(\tau)^3\Delta(\tau') - \Delta(\tau)E_4((\tau')^3))(E_4(\tau)E_4(\tau')W(P_3) + E_6(\tau)E_6(\tau')W(P_5))) \\
&\geq 1.
\end{aligned}$$

Since $\nu_p(E_4(\tau)^3\Delta(\tau') - \Delta(\tau)E_4((\tau')^3)) = 0$, we get

$$\nu_p(E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4\Delta(\tau)\Delta(\tau')W(P_5)) \geq 1, \tag{1}$$

$$\nu_p(E_4(\tau)E_4(\tau')W(P_3) + E_6(\tau)E_6(\tau')W(P_5)) \geq 1. \tag{2}$$

(Case $k \not\equiv 0 \pmod 6$**)** we have $W(P_5) = 0$. Hence we have $\nu_p(W(P_1)) \geq 1$ and $\nu_p(W(P_3)) \geq 1$. From Corollary1, we get $\nu_p(P_1(\varphi_4, \varphi_6, X_{12})) \geq 1$ and $\nu_p(P_3(\varphi_4, \varphi_6, X_{12})) \geq 1$.

(Case $k \equiv 0 \pmod{12}$**)** We can write

$$\begin{aligned}
W(P_1) &= E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 2 \pmod 3 \\ 4a+12b+12c=k-16}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c, \\
W(P_3) &= \sum_{\substack{a\equiv 2 \pmod 3 \\ 4a+12b+12c=k-16}} \gamma'_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c, \\
W(P_5) &= E_6(\tau)E_6(\tau') \sum_{12b+12c=k-24} \gamma''_{bc} W(\varphi_6^2)^b W(X_{12})^c.
\end{aligned}$$

Using these formulas, we can write

$$\begin{aligned}
&E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4\Delta(\tau)\Delta(\tau')W(P_5) \\
=&E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3 \\ 4a+12b+12c=k-12}} \gamma_{a-1bc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c \\
&+ 2^6 \cdot 3^3 E_6(\tau)E_6(\tau') \sum_{\substack{c\geq 1 \\ 12b+12c=k-12}} \gamma''_{bc-1} W(\varphi_6^2)^b W(X_{12})^c \\
=&E_6(\tau)E_6(\tau') \left\{ \sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3 \\ 4a+12b+12c=k-12}} \gamma_{a-1bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c \right. \\
&\left. +2^6 \cdot 3^3 \sum_{\substack{c\geq 1 \\ 12b+12c=k-12}} \gamma''_{bc-1} W(\varphi_6)^{2b} W(X_{12})^c \right\}.
\end{aligned}$$

Since $\nu_p(\text{LHS}) \geq 1$, we have $\nu_p(\text{RHS}) \geq 1$ for both of two formulas above. From Corollary1, we get $\nu_p(P_1(\varphi_4, \varphi_6, X_{12})) \geq 1$ and $\nu_p(P_5(\varphi_4, \varphi_6, X_{12})) \geq 1$. From the formula (2), $\nu_p(P_3(\varphi_4, \varphi_6, X_{12})) \geq 1$.

Case $k \equiv 6 \pmod{12}$($k \equiv 2 \pmod 4$ and $k \equiv 0 \pmod 6$)**:** Similarly to the case of $k \equiv 0 \pmod{12}$, we can prove the assertion of Lemma 3.

□

Since the lemma above, we get

$$F \equiv \Delta_5 \cdot \{\Delta_5(Q_1\Phi_{10} + Q_3\Phi_{16} + Q_5\Phi_{18}) + (P_2 + X_{10}Q_2)\Phi_9 + (P_4 + X_{10}Q_4)\Phi_{11} + (P_6 + X_{10}Q_6)\Phi_{17}\}.$$

We denote RHS by $\Delta_5 \cdot G$. It is known that $\Delta_5 \not\equiv 0 \pmod p$ and $q_\tau^{\frac{1}{2}} q_{\tau'}^{\frac{1}{2}} \mid \Delta_5$ but $q_\tau q_{\tau'} \nmid \Delta_5$. Next we apply Witt operator to G.

$$\begin{aligned} W(G) &= W(P_2)W(\Phi_9) + W(P_4)W(\Phi_{11}) + W(P_6)W(\Phi_{17}) \\ &= \sum_{f\in M_{k-10}(\Gamma_1), g\in M_{k-10}(\Gamma_1)} f(\tau)g(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix} \\ &= \sum_{\substack{12i+4j+6t=k-10\\12i'+4j'+6t'=k-10\\t,t'=0,1}} C_3(i,i')\Delta(\tau)^i E_4(\tau)^j E_6(\tau)^t \Delta(\tau')^{i'} E_4(\tau')^{j'} E_6(\tau')^{t'} \eta(\tau)^{12}\eta_(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix} \\ &= \sum_{m,n\geq 0} B_{12}(m,n) q_\tau^m q_{\tau'}^n \eta(\tau)^{12}\eta_(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix}. \end{aligned}$$

It is known that $\eta(\tau)^{12} \not\equiv 0 \pmod p$ and $q_\tau^{\frac{1}{2}} \mid \eta(\tau)^{12}$ but $q_\tau \nmid \eta(\tau)^{12}$. Hence we have that if $m \leq \left[\frac{k}{10}\right] - 1$ and $n \leq \left[\frac{k}{10}\right] - 1$, then $B_{12}(m,n) \equiv 0 \pmod p$. This implies that $C_3(i,i') \equiv 0 \pmod p$ for $i, i' \leq \left[\frac{k}{10}\right] - 1$. Note that $i, i' \leq \left[\frac{k}{10}\right] - 1$ since $12i + 4j + 6t = k - 10$ and $12i' + 4j' + 6t' = k - 10$ and $\left[\frac{k-10}{12}\right] \leq \left[\frac{k}{10}\right] - 1$. Thus we have $W(G) \equiv 0 \pmod p$.

Lemma 4. $P_2, P_4, P_6 \equiv 0 \pmod p$.

Proof of Lemma 4. Using fact that $W(\Delta_5) = 0$, we get

$$\begin{aligned} W(G) =& W(P_2)W(\Phi_9) + W(P_4)W(\Phi_{11}) + W(P_6)W(\Phi_{17}) \\ =& W(P_2)\left(-2E_4(\tau)E_4(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix}\right) \\ &+ W(P_4)\left(-2E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix}\right) \\ &+ W(P_6)\left(-2^4\cdot 3^2\Delta(\tau)\Delta(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix}\right). \end{aligned}$$

Case $k \not\equiv 4 \pmod 6$: In this case we have $P_4 = P_6 = 0$ as polynomials. Therefore we get $\nu_p(W(P_2)) \geq 1$. From Corollary1, we get $\nu_p(P_2(\varphi_4, \varphi_6, X_{12})) \geq 1$.

Case $k \equiv 4 \pmod{12}$: We can write

$$\begin{aligned} W(P_2) &= E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 2 \pmod 3\\4a+12b+12c=k-20}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c, \\ W(P_4) &= \sum_{12b+12c=k-16} \gamma'_{bc} W(\varphi_6^2)^b W(X_{12})^c, \\ W(P_6) &= 0. \end{aligned}$$

Using these formulas, we can write

$$W(G) = -2\Big(\sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3\\ 4a+12b+12c=k-16}} \gamma_{a-1bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c + \sum_{12b+12c=k-16} \gamma'_{bc} W(\varphi_6)^{2b} W(X_{12})^c\Big) E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix}.$$

Again from Corollary1, we have $\nu_p(\gamma_{a-1bc}) \geq 1$ and $\nu_p(\gamma'_{bc}) \geq 1$. These mean that $\nu_p(P_2(\varphi_4,\varphi_6,X_{12})) \geq 1$ and $\nu_p(P_4(\varphi_6,X_{12})) \geq 1$.

Case $k \equiv 10 \pmod{12}$: We can write

$$W(P_2) = \sum_{\substack{a\equiv 2 \pmod 3\\ 4a+12b+12c=k-14}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_4) = E_6(\tau)E_6(\tau') \sum_{12b+12c=k-22} \gamma'_{bc} W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_6) = \gamma''_{\frac{k-22}{12}} W(X_{12})^{\frac{k-22}{12}}.$$

Using these formulas, we can write

$$W(G) = \Big(-2\sum_{\substack{a\equiv 0 \pmod 3,\ a\geq 3\\ 4a+12b+12c=k-10}} \gamma_{a-3bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c - 2\sum_{\substack{b\geq 1\\ 12b+12c=k-10}} \gamma'_{b-1c} W(\varphi_6)^{2b} W(X_{12})^c - 2^2\cdot 3\gamma''_{\frac{k-22}{12}} W(X_{12})^{\frac{k-10}{12}}\Big)\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}01\\10\end{pmatrix}.$$

Again from Corollary1, we have $\nu_p(-2\gamma_{a-3bc}) \geq 1$, $\nu_p(-2\gamma'_{b-1c}) \geq 1$ and $\nu_p(-2^2\cdot 3\gamma''_{\frac{k-22}{12}}) \geq 1$. These mean that $\nu_p(P_2(\varphi_4,\varphi_6,X_{12})) \geq 1$, $\nu_p(P_4(\varphi_6,X_{12})) \geq 1$ and $\nu_p(P_6(X_{12})) \geq 1$.

This completes the proof of Lemma 4.

□

Since Lemma 4, we get

$$F \equiv X_{10}\cdot(Q_1\Phi_{10} + Q_2\Phi_{14} + Q_3\Phi_{16} + Q_4\Psi_{16} + Q_5\Phi_{18} + Q_6\Phi_{22}) \pmod p).$$

Then $H_1 := Q_1\Phi_{10} + Q_2\Phi_{14} + Q_3\Phi_{16} + Q_4\Psi_{16} + Q_5\Phi_{18} + Q_6\Phi_{22} \in M_{k-10,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ and $A((m,n,r);H_1) \equiv 0 \pmod p$ for every m, n such that $0 \leq m \leq \left[\frac{k-10}{10}\right]$, $0 \leq n \leq \left[\frac{k-10}{10}\right]$. Moreover $\nu_p(F) = \nu_p(H_1)$ since $\nu_p(X_{10}) = 0$.

By repeating this argument, there exists the modular form H_t of weight $k-10t$ and t_0 such that

$$F \equiv H_t \cdot X_{10}^t \pmod p$$

where $1 \leq t \leq t_0$ and

$$A((m,n,r);H_t) \equiv 0 \pmod p$$

for every m, n such that $0 \leq m \leq \left[\frac{k-10t}{10}\right]$, $0 \leq n \leq \left[\frac{k-10t}{10}\right]$. Thus we have

$$\nu_p(F) = \nu_p(H_{t_0}).$$

Since the weight of $H_{t_0} \leq 22$, we should check the case $k \leq 22$ directly.

(Case $k \equiv 0 \pmod{10}$**)** $H_{t_0} \in M_{10,2}(\Gamma_2)$ and $t_0 = \frac{k-10}{10}$. Since

$$X_{10} = (-2 + q_\omega + q_\omega^{-1}) + \cdots ,$$

we have that if $n \leq \left[\frac{k}{10}\right] - t_0 = 1$ and $m \leq \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r); H_{t_0}) \equiv 0 \pmod{p}$. With $a_1 \in \mathbb{Z}_{(p)}$ we can write

$$H_{t_0} = a_1 \cdot \Phi_{10} = \begin{pmatrix} a_1 0 \\ 0\, 0 \end{pmatrix} q_\tau + \cdots .$$

Hence we have $a_1 \equiv 0 \pmod{p}$. Hence we have $F \equiv 0 \pmod{p}$.

(Case $k \equiv 4 \pmod{10}$**)** $H_{t_0} \in M_{14,2}(\Gamma_2)$ and $t_0 = \frac{k-14}{10}$. Then we have that if $n \leq \left[\frac{k}{10}\right] - t_0 = 1$ and $m \leq \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r); H_{t_0}) \equiv 0 \pmod{p}$. With $a_1, a_2 \in \mathbb{Z}_{(p)}$ we can write

$$\begin{aligned} H_{t_0} =& a_1 \cdot \varphi_4 \Phi_{10} + a_2 \Phi_{14} \\ =& \begin{pmatrix} a_1 0 \\ 0\, 0 \end{pmatrix} q_\tau + \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix} q_{\tau'} \\ &+ \left\{ \begin{pmatrix} 30a_1 + 2a_2 & 0 \\ 0 & 30a_1 + 2a_2 \end{pmatrix} + \begin{pmatrix} -28a_1 - a_2 & -14a_1 - \frac{1}{2}a_2 \\ -14a_1 - \frac{1}{2}a_2 & -28a_1 - a_2 \end{pmatrix} q_\omega \right. \\ &+ \left. \begin{pmatrix} -28a_1 - a_2 & 14a_1 + \frac{1}{2}a_2 \\ 14a_1 + \frac{1}{2}a_2 & -28a_1 - a_2 \end{pmatrix} q_\omega^{-1} + \begin{pmatrix} a_1 a_1 \\ a_1 a_1 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1 & -a_1 \\ -a_1 & a_1 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'} + \cdots . \end{aligned}$$

Hence we have $a_1 \equiv 0 \pmod{p}$ and $-28a_1 - a_2 \equiv 0 \pmod{p}$. Hence we get $a_2 \equiv 0 \pmod{p}$. Hence we have $F \equiv 0 \pmod{p}$.

(Case $k \equiv 6 \pmod{10}$**)** $H_{t_0} \in M_{16,2}(\Gamma_2)$ and $t_0 = \frac{k-16}{10}$. Then we have that if $n \leq \left[\frac{k}{10}\right] - t_0 = 1$ and $m \leq \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r); H_{t_0}) \equiv 0 \pmod{p}$. With $a_1, a_2, a_3 \in \mathbb{Z}_{(p)}$ we can write

$$\begin{aligned} H_{t_0} =& a_1 \cdot \varphi_6 \Phi_{10} + a_2 \Phi_{16} + a_3 \Psi_{16} \\ =& \begin{pmatrix} a_1 0 \\ 0\, 0 \end{pmatrix} q_\tau + \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix} q_{\tau'} \\ &+ \left\{ \begin{pmatrix} -714a_1 + 10a_2 - 2a_3 & 0 \\ 0 & -714a_1 + 10a_2 - 2a_3 \end{pmatrix} + \begin{pmatrix} -28a_1 + a_2 + a_3 & -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 \\ -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega \right. \\ &+ \left. \begin{pmatrix} -28a_1 + a_2 + a_3 & 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 \\ 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega^{-1} + \begin{pmatrix} a_1 a_1 \\ a_1 a_1 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1 & -a_1 \\ -a_1 & a_1 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'} + \cdots . \end{aligned}$$

Hence we have $a_1 \equiv -714a_1 + 10a_2 - 2a_3 \equiv -28a_1 + a_2 + a_3 \equiv 0 \pmod{p}$. Hence we get $a_2 = \frac{1}{2^2 \cdot 3}(10a_2 - 2a_3 + 2(a_2 + a_3)) \equiv 0 \pmod{p}$ and $a_3 = \frac{1}{2^2 \cdot 3}(-(10a_2 - 2a_3) + 10(a_2 + a_3)) \equiv 0 \pmod{p}$. Hence we have $F \equiv 0 \pmod{p}$.

(Case $k \equiv 8 \pmod{10}$**)** $H_{t_0} \in M_{18,2}(\Gamma_2)$ and $t_0 = \frac{k-18}{10}$. Then we have that if $n \leq \left[\frac{k}{10}\right] - t_0 = 1$ and $m \leq \left[\frac{k}{10}\right] - t_0 = 1$, then $a((m,n,r); H_{t_0}) \equiv 0 \pmod{p}$. With $a_1, a_2, a_3 \in \mathbb{Z}_{(p)}$ we can write

$$\begin{aligned} H_{t_0} =& a_1 \cdot \varphi_4^2 \Phi_{10} + a_2 \varphi_4 \Phi_{14} + a_3 \Phi_{18} \\ =& \begin{pmatrix} a_1 0 \\ 0\, 0 \end{pmatrix} q_\tau + \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix} q_{\tau'} \\ &+ \left\{ \begin{pmatrix} 270a_1 - 2a_2 + 10a_3 & 0 \\ 0 & 270a_1 - 2a_2 + 10a_3 \end{pmatrix} + \begin{pmatrix} -28a_1 + a_2 + a_3 & -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 \\ -14a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega \right. \\ &+ \left. \begin{pmatrix} -28a_1 + a_2 + a_3 & 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 \\ 14a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3 & -28a_1 + a_2 + a_3 \end{pmatrix} q_\omega^{-1} + \begin{pmatrix} a_1 a_1 \\ a_1 a_1 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1 & -a_1 \\ -a_1 & a_1 \end{pmatrix} q_\omega^{-2} \right\} q_\tau q_{\tau'} + \cdots . \end{aligned}$$

Hence we have $a_1 \equiv 270a_1 - 2a_2 + 10a_3 \equiv -28a_1 + a_2 + a_3 \equiv 0 \pmod p$. Hence we get $a_2 = \frac{1}{2^2 \cdot 3}(-(-2a_2 + 10a_3) + 10(a_2 + a_3)) \equiv 0$ and $a_3 = \frac{1}{2^2 \cdot 3}(-2a_2 + 10a_3 + 2(a_2 + a_3)) \equiv 0 \pmod p$. Hence we have $F \equiv 0 \pmod p$.

(Case $k \equiv 2 \pmod{10}$) $H_{t_0} \in M_{22,2}(\Gamma_2)$ and $t_0 = \frac{k-22}{10}$. Then we have that if $n \le \left[\frac{k}{10}\right] - t_0 = 2$ and $m \le \left[\frac{k}{10}\right] - t_0 = 2$, then $a((m,n,r); H_{t_0}) \equiv 0 \pmod p$. With $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{Z}_{(p)}$ we can write

$$
\begin{aligned}
H_{t_0} =& (a_1 \cdot \varphi_4^3 + a_2 \cdot \varphi_6^2 + a_3 \cdot X_{12})\Phi_{10} + a_4 \varphi_4^2 \Phi_{14} + a_5 \varphi_6 \Phi_{16} + a_6 \cdot \varphi_6 \Psi_{16} + a_7 \cdot \Phi_{22} \\
=& \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & 0 \end{pmatrix} q_\tau + \begin{pmatrix} 0 & 0 \\ 0 & a_1 + a_2 \end{pmatrix} q_{\tau'} \\
&+ \Bigg\{ \begin{pmatrix} 510a_1 - 1218a_2 - 2a_4 - 2a_5 - 2a_6 & 0 \\ 0 & 510a_1 - 1218a_2 - 2a_4 - 2a_5 - 2a_6 \end{pmatrix} \\
&+ \begin{pmatrix} -28a_1 - 28a_2 + a_4 + a_5 + a_6 & -14a_1 - 14a_2 + \frac{1}{2}a_4 + \frac{1}{2}a_5 + \frac{1}{2}a_6 \\ -14a_1 - 14a_2 + \frac{1}{2}a_4 + \frac{1}{2}a_5 + \frac{1}{2}a_6 & -28a_1 - 28a_2 + a_4 + a_5 + a_6 \end{pmatrix} q_\omega \\
&+ \begin{pmatrix} -28a_1 - 28a_2 + a_4 + a_5 + a_6 & 14a_1 + 14a_2 - \frac{1}{2}a_4 - \frac{1}{2}a_5 - \frac{1}{2}a_6 \\ 14a_1 + 14a_2 - \frac{1}{2}a_4 - \frac{1}{2}a_5 - \frac{1}{2}a_6 & -28a_1 - 28a_2 + a_4 + a_5 + a_6 \end{pmatrix} q_\omega^{-1} \\
&+ \begin{pmatrix} a_1 + a_2 & a_1 + a_2 \\ a_1 + a_2 & a_1 + a_2 \end{pmatrix} q_\omega^2 + \begin{pmatrix} a_1 + a_2 & -a_1 - a_2 \\ -a_1 - a_2 & a_1 + a_2 \end{pmatrix} q_\omega^{-2} \Bigg\} q_\tau q_{\tau'} \\
&+ \begin{pmatrix} 696a_1 - 1032a_2 & 0 \\ 0 & 0 \end{pmatrix} q_\tau^2 + \begin{pmatrix} 0 & 0 \\ 0 & 696a_1 - 1032a_2 \end{pmatrix} q_{\tau'}^2 \\
&+ \Bigg\{ \begin{pmatrix} -54324a_1 + 350028a_2 - 1404a_4 + 564a_5 + 2052a_6 & 0 \\ 0 & 279432a_1 + 1088136a_2 + 10a_3 - 168a_4 + 1800a_5 + 408a_6 \end{pmatrix} \\
&+ \begin{pmatrix} -46464a_1 + 1920a_2 + 704a_4 - 280a_5 - 1024a_6 & -23232a_1 + 960a_2 + 352a_4 - 140a_5 - 512a_6 \\ -23232a_1 + 960a_2 + 352a_4 - 140a_5 - 512a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 896a_5 - 200a_6 \end{pmatrix} q_\omega \\
&+ \begin{pmatrix} -46464a_1 + 1920a_2 + 704a_4 - 280a_5 - 1024a_6 & 23232a_1 - 960a_2 - 352a_4 + 140a_5 + 512a_6 \\ 23232a_1 - 960a_2 - 352a_4 + 140a_5 + 512a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 896a_5 - 200a_6 \end{pmatrix} q_\omega^{-1} \\
&+ \begin{pmatrix} 510a_1 - 1218a_2 - 2a_4 - 2a_5 - 2a_6 & 510a_1 - 1218a_2 - 2a_4 - 2a_5 - 2a_6 \\ 510a_1 - 1218a_2 - 2a_4 - 2a_5 - 2a_6 & 1020a_1 - 2436a_2 - 4a_4 - 4a_5 - 4a_6 \end{pmatrix} q_\omega^2 \\
&+ \begin{pmatrix} 510a_1 - 1218a_2 - 2a_4 - 2a_5 - 2a_6 & -510a_1 + 1218a_2 + 2a_4 + 2a_5 + 2a_6 \\ -510a_1 + 1218a_2 + 2a_4 + 2a_5 + 2a_6 & 1020a_1 - 2436a_2 - 4a_4 - 4a_5 - 4a_6 \end{pmatrix} q_\omega^{-2} \Bigg\} q_\tau q_{\tau'}^2 \\
&+ \Bigg\{ \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} + \begin{pmatrix} A_2 & C_2 \\ C_2 & B_2 \end{pmatrix} q_\omega + \begin{pmatrix} A_2 & -C_2 \\ -C_2 & B_2 \end{pmatrix} q_\omega^{-1} \\
&+ \begin{pmatrix} A_3 & C_3 \\ C_3 & B_3 \end{pmatrix} q_\omega^2 + \begin{pmatrix} A_3 & -C_3 \\ -C_3 & B_3 \end{pmatrix} q_\omega^{-2} \\
&+ \begin{pmatrix} 17952a_1 + 114720a_2 + a_3 + 88a_4 - 896a_5 - 200a_6 & 41184a_1 + 113760a_2 + a_3 - 264a_4 - 756a_5 + 312a_6 \\ 41184a_1 + 113760a_2 + a_3 - 264a_4 - 756a_5 + +312a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 896a_5 - 200a_6 \end{pmatrix} q_\omega^3 \\
&+ \begin{pmatrix} 17952a_1 + 114720a_2 + a_3 + 88a_4 - 896a_5 - 200a_6 & -41184a_1 - 113760a_2 - a_3 + 264a_4 + 756a_{-}312a_6 \\ -41184a_1 - 113760a_2 - a_3 + 264a_4 + 756a_5 + -312a_6 & 17952a_1 + 114720a_2 + a_3 + 88a_4 - 896a_5 - 200a_6 \end{pmatrix} q_\omega^{-3} \\
&+ \begin{pmatrix} 696a_1 - 1032a_2 & 696a_1 - 1032a_2 \\ 696a_1 - 1032a_2 & 696a_1 - 1032a_2 \end{pmatrix} q_\omega^4 + \begin{pmatrix} 696a_1 - 1032a_2 & -696a_1 + 1032a_2 \\ -696a_1 + 1032a_2 & 696a_1 - 1032a_2 \end{pmatrix} q_\omega^{-4} \Bigg\} q_\tau^2 q_{\tau'}^2 + \cdots,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 = B_1 &= -59095920a_1 - 257328624a_2 - 2288a_3 - 75776a_4 - 364544a_5 - 329216a_6 - 18a_7 \\
A_2 = B_2 &= -20671008a_1 - 53005344a_2 - 577a_3 + 33960a_4 + 153984a_5 + 144840a_6 + 8a_7 \\
C_2 &= -24544032a_1 + 15272544a_2 - 139a_3 + 239160a_4 + 14556a_5 + 511608a_6 + 35a_7 \\
A_3 = B_3 &= -688416a_1 + 217056a_2 - 8a_3 + 3840a_4 + 29184a_5 + 19968a_6 + a_7 \\
C_3 &= -344208a_1 + 108528a_2 - 4a_3 + 1920a_4 + 14592a_5 + 9984a_6 + \frac{1}{2}a_7
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
a_1 &= \frac{1}{2^6 \cdot 3^3}(1032(a_1 + a_2) + (696a_1 - 1032a_2)) \equiv 0 \pmod p, \\
a_2 &= \frac{1}{2^6 \cdot 3^3}(696(a_1 + a_2) - (696a_1 - 1032a_2)) \equiv 0 \pmod p.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&-\frac{89}{2^2\cdot 3}(a_4+a_5+a_6)-\frac{1}{2^4\cdot 3}(-1404a_4+564a_5+2052a_6)+\frac{1}{2^4\cdot 3}(10a_3-168a_4+1800a_5+408a_6)\\
&-\frac{5}{2^3\cdot 3}(a_3+88a_4-896a_5-200a_6)=205a_5=5\cdot 41a_5\equiv 0 \pmod p
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{41}{3}(a_4+a_5+a_6)+\frac{1}{2^7}(4a_3-1920a_4-14592a_5-9984a_6-a_7)\\
&-\frac{1}{2^9\cdot 3^2}(-2288a_3-75776a_4-364544a_5-329216a_6-36a_7)\\
&-\frac{19}{2^2\cdot 3^2}(a_3+88a_4-896a_5-200a_6)-\frac{1}{2\cdot 3}(-352a_4+140a_5+512a_6)=401a_5\equiv 0 \pmod p.
\end{aligned}
$$

Hence we get $a_5 \equiv 0 \pmod p$. Then we get $a_4+a_6 \equiv 0 \pmod p$ and $-352a_4+512a_6 \equiv 0 \pmod p$. Hence we get $a_4 = \frac{1}{2^5\cdot 3^3}(512(a_4+a_6)-(-352a_4+512a_6)) \equiv 0 \pmod p$ and $a_6 = \frac{1}{2^5\cdot 3^3}(352(a_4+a_6)+(-352a_4+512a_6)) \equiv 0 \pmod p$. Hence we get $a_3 \equiv a_7 \equiv 0 \pmod p$. Hence we have $F \equiv 0 \pmod p$. □

3.3 Proof of Theorem 2 and 4

We can prove Theorem 2 and 4 similarly as the proofs of Theorem 1 and 3, respectively. Namely, the proofs are inductions on the determinant weight with applications of the Witt operator. We omit the details here (see [5]).

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