PRIME GEODESIC THEOREM IN $\mathbb{H}^3$

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Abstract. In these notes we give a summary of the main developments in the Prime Geodesic Theorem on hyperbolic three-manifolds since the seminal result of Sarnak (1983). We will focus on arithmetic manifolds and show how strong tools, such as the Kuznetsov formula, allow us to improve on Sarnak's result in this case. In particular, it is currently known that the remainder in the Prime Geodesic Theorem on the Picard manifold is bounded by $O(X^{3/2+\epsilon})$ under Lindelöf hypothesis for quadratic Hecke L-functions over Gaussian integers. We will prove that this is true on average. The main results presented here appeared in a preprint of Chatzakos, Cherubini and Laaksonen (2018).

1. Introduction

These notes are based on the identically titled talk given by the author at RIMS on January 22, 2019. Our aim is to give a brief overview of recent developments in the Prime Geodesic Theorem in three dimensional hyperbolic space and to describe our recent results (with D. Chatzakos and G. Cherubini) in [4].

A prime geodesic theorem (PGT) is a statement about the asymptotic behaviour of the number of simple closed geodesics (tracing their image only once) on a hyperbolic manifold counted according to length. These lengths bear many similarities to the usual prime numbers and we may think of PGT as a geometric analogue of the Prime Number Theorem. In fact, in the case of hyperbolic surfaces the main asymptotic is exactly that of the counting function of prime numbers as originally discovered by Huber [8, 9] and Selberg [11]. In PGT the role of the Riemann zeta function is replaced by the Selberg zeta function. This is curious, because in the arithmetic cases we are focusing on, the Selberg zeta function satisfies the analogue of the Riemann Hypothesis. Yet we are still unable to prove optimal bounds in PGT for any hyperbolic manifold (compact or non-compact) in any dimension. Nevertheless, in these arithmetic cases we can exploit connections to the theory of automorphic L-functions (and Hecke operators) coming from the Maass cusp forms. Even though we will focus on three dimensions, many of the definitions and concepts are completely analogous in two dimensions. There are few crucial differences though, which we will highlight in section 3.

2. Background

2.1. **Three dimensional hyperbolic space.** We will briefly outline our setting, for more details see the comprehensive book [7]. Let

$$\mathbb{H}^3 = \{p = z + jy = x_1 + ix_2 + jy : z \in \mathbb{C}, y > 0\}$$

be the upper half-space. Let Γ be a cofinite discrete subgroup of $G = \mathrm{PSL}_2(\mathbb{C})$. The quotient space $M = \Gamma\backslash\mathbb{H}^3$ is a three dimensional hyperbolic manifold (with possibly some

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singular points) equipped with the metric induced from the following hyperbolic metric on $\mathbb{H}^3$:

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dy^2}{y^2}.$$

We classify elements of G depending on their trace: $\gamma \in G$ is said to be *elliptic*, *parabolic* or *hyperbolic* if $\operatorname{tr}\gamma \in \mathbb{R}$ and $|\operatorname{tr}\gamma| < 2$, $|\operatorname{tr}\gamma| = 2$ or $|\operatorname{tr}\gamma| > 2$, respectively. If $\operatorname{tr}\gamma \in \mathbb{C} \setminus \mathbb{R}$ then γ is called loxodromic. In our case it turns out that there is little difference between loxodromic and hyperbolic elements and therefore we will refer to both of them as "hyperbolic". In particular, both hyperbolic and loxodromic elements have two fixed points on the boundary of $\mathbb{H}^3$ (i.e. $\mathbb{P}^1\mathbb{C}$) and leave the geodesic (in $\mathbb{H}^3$) connecting these fixed points invariant. Therefore this geodesic closes when projected to M. Moreover, we can see that conjugates of γ correspond to the same closed geodesic. This is useful, because each hyperbolic element $\gamma \in G$ is conjugate to a unique diagonal matrix

$$\begin{pmatrix} a(\gamma) & \\ & a(\gamma)^{-1} \end{pmatrix},$$

with $|a(\gamma)| > 1$. The number $N(\gamma) = |a(\gamma)|^2$ is called the *norm* of γ. Since this notion is invariant under conjugation, we define the norm of a hyperbolic conjugacy class to be the norm of any of its representatives. We further define $\{\gamma\}$ to be primitive if the norm of γ is minimal among all elements of the centraliser $C(\gamma)$. The closed geodesic corresponding to such a primitive conjugacy class of Γ is called a *prime geodesic*. From now on, for hyperbolic $\gamma \in \Gamma$, we will abuse notation and use γ to denote both the group element and the corresponding conjugacy class.

2.2. **Selberg zeta function.** The Selberg zeta function on M is given for $\operatorname{Re} s > 2$ by the Euler product

$$Z(s) = \prod_{\gamma_0} \prod_{(k,l)}^{\infty} \left(1 - a(\gamma_0)^{-2k}\overline{a(\gamma_0)}^{-2l} N(\gamma_0)^{-s}\right),$$

where the first product is over primitive hyperbolic (and, recall, loxodromic) conjugacy classes of Γ and the second product is over pairs (k,l) such that $k, l \geq 0$ and $k \equiv l$ modulo the order of the torsion subgroup of the centraliser $C(\gamma_0)$ (denoted by $m(\gamma)$). While Z does not have a convenient expression in terms of Dirichlet series, we can compute for its logarithmic derivative that (for $\operatorname{Re} s > 2$)

$$\frac{Z'}{Z}(s) = \sum_{\gamma} \frac{N(p)\Lambda_\Gamma(N(\gamma))}{m(\gamma)|a(\gamma) - a(\gamma)^{-1}|^2} N(\gamma)^{-s}, \tag{1}$$

where the sum runs over hyperbolic conjugacy classes of Γ and Λ_Γ is the hyperbolic von Mangoldt function defined as

$$\Lambda_\Gamma(N(\gamma)) = \begin{cases} \log N(\gamma_0), & \text{if } \gamma = \gamma_0^\nu,\ \nu \in \mathbb{N},\ \gamma_0 \text{ primitive hyperbolic,} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that sometimes in the literature [7, 12, 18] the whole Dirichlet coefficient in (1) is taken to be the definition of Λ_Γ. In terms of the remainder in PGT the estimates for these two different von Mangoldt functions differ by a bounded error.

The Selberg zeta function can be analytically continued to an entire function and it has zeros only in the critical strip $0 \leq \operatorname{Re} s \leq 1$. In fact, the zeros of Z are precisely the eigenvalues λ_j of the negative Laplace–Beltrami operator $-\Delta$ on M. More precisely, if we write $\lambda_j = s_j(2-s_j)$ with $s_j = 1 + ir_j$ (here r_j is the spectral parameter), then $Z(s)$ is zero exactly at s_j and $\overline{s}_j$. This is useful because it will allow us to use tools from the spectral theory of automorphic forms to attack the PGT. Notice that with our normalisation, the critical line is at $\operatorname{Re} s = 1$ and small eigenvalues (i.e. purely imaginary $r_j \in (0, i]$) correspond to zeros off the critical line (and on the real line). The number of zeros up to a given height is given by the Weyl law:

$$\#\{0 < r_j \leq T\} - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}(1+ir)dr \sim \frac{\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)}{6\pi^2}T^3, \tag{2}$$

where we have to include the contribution from the continuous spectrum of Δ and φ is the determinant of the scattering matrix. It turns out that for the arithmetic groups that we are interested in, φ is related to a zeta function of a number field. In these cases it is possible to show that the first term on the left in Weyl law dominates. The most crucial observation from (2) is then that Z has many more zeros than ζ (at least in the arithmetic case). This is the reason for the added difficulty in comparison with the prime number theorem as was mentioned in the introduction. We will make this more precise in section 2.4.

2.3. **Length spectrum.** Suppose ℓ is a closed geodesic on M corresponding to the hyperbolic conjugacy class $\{\gamma\}$. It can be shown that $\operatorname{length}(\ell) = \log N(\gamma)$. The collection of all such lengths (counted with multiplicity) is called the *length spectrum* of M and it is connected to the Laplace eigenvalue spectrum of M through the Selberg trace formula (see e.g. [14]). The part of the length spectrum corresponding to prime geodesics is called the *primitive length spectrum*. We define the counting function

$$\pi_\Gamma(X) = \#\{\ell \text{ prime geodesic on } M : \operatorname{length}(\ell) \leq \log X\}.$$

The two dimensional analogue of the counting function of the primitive length spectrum satisfies $\pi_\Gamma(X) \sim X/\log X$. The remainder in 2D has been extensively studied in e.g. [10, 13, 22]. As in the case of rational primes, it is helpful to count the lengths with logarithmic weights. Therefore, we define the hyperbolic Chebyshev function as

$$\psi_\Gamma(X) = \sum_{N(\gamma)\leq X} \Lambda_\Gamma(N(\gamma)),$$

where the sum runs over (primitive) hyperbolic conjugacy classes of Γ of norm at most X. It can be shown that the main term in PGT has a contribution from each of the small eigenvalues of Δ:

$$\psi_\Gamma(X) = \sum_{1<s_j\leq 2} \frac{X^{s_j}}{s_j} + E_\Gamma(X),$$

where $E_\Gamma(X)$ is the remainder. Clearly the dominant term corresponds to the zero eigenvalue ($s_j = 2$).

The first non-trivial bound on the remainder of PGT in three dimensions is due to Sarnak.

Theorem 1 ([21])**.** *Let Γ be a cofinite subgroup of* $\mathrm{PSL}_2(\mathbb{C})$*. Then*

$$E_\Gamma(X) \ll X^{5/3+\epsilon}. \tag{3}$$

His proof uses the Selberg trace formula. In the rest of this note we will summarise various results which improve upon this bound.

Before proceeding we briefly remark that the primitive length spectrum is also closely connected to class numbers of primitive binary quadratic forms, see [20, 21]. Here we will merely point out that consequently the elements of the length spectrum come with high multiplicity (equal to the class number), which is very different from the situation with primes.

2.4. **The Explicit formula.** Recall that the Prime Number Theorem is connected to the zeros of the Riemann zeta function through Riemann's explicit formula, which says for $X > 1$ that

$$\sum_{n\le X}\Lambda(n) = X - 2\,\mathrm{Re}\left(\sum_{0<|\mathrm{Im}\,\rho|<T}\frac{X^\rho}{\rho}\right) + O\left(\frac{X}{T}\log X + 1,\right) \tag{4}$$

where the sum is over non-trivial zeros of ζ up to height T. In particular, if we assume the Riemann Hypothesis then we can bound the exponential sum in (4) by $X^{1/2}\log X$ to obtain the optimal error term in the Prime Number Theorem.

There is also an analogous formula for the Prime Geodesic Theorem. In three dimensions it was proven by Nakasuji.

Theorem 2 ([17, 18])**.** *Suppose Γ is a cocompact group or a Bianchi group. Let $1 \le T < X^{1/2}$. Then*

$$E_\Gamma(X) = 2\,\mathrm{Re}\left(\sum_{0<r_j\le T}\frac{X^{1+ir_j}}{1+ir_j}\right) + O\left(\frac{X^2}{T}\log X\right), \tag{5}$$

Note that by the Weyl law (2), we immediately recover the bound of Sarnak (3) for cocompact and Bianchi groups. Therefore in order to improve on Sarnak's result, we have to detect cancellation in the spectral exponential sum

$$S(T,X) = \sum_{0<r_j\le T} X^{ir_j}.$$

3. Pointwise bounds and higher moments

For general cofinite groups, it is very difficult to say anything meaningful about $S(T,X)$. In the case of arithmetic groups we can use input from the theory of automorphic L-functions, in particular of Hecke operators, to say more. For the remainder of this note we will fix $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$. In this case there are no small eigenvalues (i.e. Z satisfies the Riemann Hypothesis) and so the asymptotic in the PGT becomes just

$$\psi_\Gamma(X) = \frac{X^2}{2} + E_\Gamma(X).$$

The first improvement over Sarnak's 5/3 was given by Koyama [12] who proved (conditionally) that

$$E_\Gamma(X) \ll X^{11/7+\epsilon}.$$

His result depends on assuming the following mean Lindelöf hypothesis for Rankin–Selberg L-functions attached to Hecke–Maass cusp forms u_j:

$$\sum_{r_j\le T}\frac{r_j}{\sinh \pi r_j}L(\tfrac12+it, u_j\otimes u_j) \ll |t|^A T^{3+\epsilon},$$

where $A > 0$ is a constant and $L(s, u_j \otimes u_j)$ is defined by the Dirichlet series (for $\mathrm{Re}\, s > 1$)

$$L(s, u_j \otimes u_j) = \sum_{n \in \mathbb{Z}[i] \setminus \{0\}} \frac{|\rho_j(n)|^2}{N(n)^s}.$$

The ρ_j are the Fourier coefficients of u_j and are related to the Hecke eigenvalues $\lambda_j(n)$ by $\rho_j(n) = v_j(n)\sqrt{\sinh(\pi r_j)/r_j}$ and $v_j(n) = v_j(1)\lambda_j(n)$. In fact, in Koyama's paper the assumption is for the symmetric square L-function of u_j, but it is easy to relate this to the Rankin–Selberg L-function.

The first unconditional improvement of (3) was given by the authors in [1]. Namely, we showed (for the Picard group) that

$$E_\Gamma(X) \ll X^{13/8+\epsilon}. \tag{6}$$

This was done by proving a mean subconvexity estimate for Rankin–Selberg L-functions. We managed to go half-way between the trivial $O(T^4)$ and the optimal $O(T^3)$ bound, that is, we proved that

$$\sum_{r_j \le T} \frac{r_j}{\sinh \pi r_j} |L(\tfrac{1}{2} + t, u_j \otimes u_j)| \ll |t|^A T^{7/2+\epsilon}, \tag{7}$$

for some constant $A > 0$. The above result (6) for the PGT was then improved by Balkanova and Frolenkov [2] through a very different method to

$$E_\Gamma(X) \ll X^{3/2+\theta/2+\epsilon},$$

where θ is the subconvexity exponent for quadratic Hecke L-functions over $\mathbb{Z}[i]$. The best known exponent is $\theta = 103/512$ by the work of Wu [23]. We remark that our earlier result (6) corresponds here to the convexity bound $\theta = 1/4$. Clearly under Lindelöf they obtain the exponent $3/2+\epsilon$, which coincidentally is also the barrier in the explicit formula of Nakasuji. It is unclear whether this is the "real truth". On the one hand, for random exponential sums of the form of $S(T, X)$, we expect to save one power of T, so the bound on $S(T, X)$ (that is $O(T^2(TX)^\epsilon)$) corresponding to $E_\Gamma(X) \ll X^{3/2+\epsilon}$ is reasonable. On the other hand, even in two dimensions the analogous bound (corresponding to Lindelöf for quadratic Dirichlet L-functions, see [10]) is different from the conjectured square root cancellation for the error in PGT. There are also omega results [19] of the form $\Omega_\pm(X^{1+\delta})$ where $\delta > 0$ for arithmetic groups and $\delta < 0$ for other cofinite groups. This bound, however, seems too optimistic to be the actual order of growth since it would require too strong cancellation in $S(T, X)$.

In our recent work we obtain an unconditional square mean estimate of the remainder $E_\Gamma(X)$, which shows that the exponent $3/2 + \epsilon$ is true on average.

Theorem 3 ([4]). *Let $V \ge Y \gg 1$ and $\epsilon > 0$. Then*

$$\frac{1}{Y} \int_V^{V+Y} |E_\Gamma(X)|^2 dX \ll V^{3+\epsilon} \left(\frac{V}{Y}\right)^{2/3}. \tag{8}$$

This follows from a second moment estimate on $S(T, X)$.

Theorem 4 ([4]). *Let $V \ge Y \gg 1$ and $T \ll V^{1/2-\epsilon}$. Then*

$$\frac{1}{Y} \int_V^{V+Y} |S(T, X)|^2 dX \ll T^{3+\epsilon} V^{3/2+\epsilon} Y^{-1}.$$

We note that the short interval estimate in Theorem 4 follows trivially from the one for the full dyadic interval $[V, 2V]$. This is, however, a crucial technicality for deducing Theorem 3. We also want to point out that analogous results in two dimensions were obtained in [3, 5]. Curiously in 2D these second moment estimates beat (on average) the bound that is obtained under Lindelöf for Dirichlet L-functions, whereas in 3D the two exponents are the same. In the final section we will briefly outline the proof of our Theorem 4 since Theorem 3 follows then from the explicit formula.

4. Outline of proof

4.1. **Kuznetsov trace formula and Kloosterman sums.** A key ingredient in our proof is the Kuznetsov trace formula. It relates the Fourier coefficients of cusp forms to Kloosterman sums. For Gaussian integers, Kloosterman sums are defined as

$$S(n,m,c) = \sum_{a \in (\mathbb{Z}[i]/(c))^*} e\big(\langle m, a/c\rangle\big) e\big(\langle n, a^*/c\rangle\big),$$

where $m, n, c \in \mathbb{Z}[i]$, $c \neq 0$; a^* denotes the inverse of a modulo the ideal (c); and $\langle x, y\rangle$ denotes the standard inner product on $\mathbb{R}^2 \cong \mathbb{C}$. The Kloosterman sums obey Weil's bound [16, (3.5)]

$$|S(n,m,c)| \leq |(n,m,c)| d(c) N(c)^{1/2}. \tag{9}$$

Here $d(c)$ is the number of divisors of c and $N(c)$ denotes the norm of c.

Theorem 5 (Kuznetsov formula [15, 16])**.** *Let h be an even function, holomorphic in $|\operatorname{Im} r| < 1/2 + \epsilon$, for some $\epsilon > 0$, and assume that $h(r) = O((1+|r|)^{-3-\epsilon})$ in the strip. Then, for any non-zero $m, n \in \mathbb{Z}[i]$:*

$$D + C = U + S,$$

with

$$\begin{aligned}
D &= \sum_{j=1}^{\infty} \frac{r_j \rho_j(n)\overline{\rho_j(m)}}{\sinh \pi r_j} h(r_j),\\
C &= 2\pi \int_{-\infty}^{\infty} \frac{\sigma_{ir}(n)\sigma_{ir}(m)}{|mn|^{ir}|\zeta_K(1+ir)|^2} h(r)dr,\\
U &= \frac{\delta_{m,n} + \delta_{m,-n}}{\pi^2} \int_{-\infty}^{\infty} r^2 h(r) dr,\\
S &= \sum_{c \in \mathbb{Z}[i]\setminus\{0\}} \frac{S(n,m,c)}{|c|^2} \int_{-\infty}^{\infty} \frac{ir^2}{\sinh \pi r} h(r) H_{ir}\left(\frac{2\pi\sqrt{mn}}{c}\right) dr,
\end{aligned}$$

where $\sigma_s(n) = \sum_{d|n} N(d)^s$ is the divisor function,

$$H_\nu(z) = 2^{-2\nu}|z|^{2\nu} J^*_\nu(z) J^*_\nu(\overline{z}),$$

J_ν is the J-Bessel function of order ν, and $J^*_\nu(z) = J_\nu(z)(z/2)^{-\nu}$.

In general, the Kuznetsov formula allows one to prove finer results compared to what is obtainable from the Selberg trace formula. This comes at the expense of having to analyse more complicated oscillatory integrals and exponential sums.

4.2. **Proof of Theorem 4.** Since the proof is fairly technical, we will only describe the general strategy and the most important estimates. For the full proof see [4]. We study the smoothed sum

$$\sum_{r_j} X^{ir_j} e^{-r_j/T}.$$

This can be related back to $S(T, X)$ by a standard argument (see e.g. [13]). We then approximate the summands by a more regular function in order to apply the Kuznetsov formula. We pick

$$h(r) = \frac{\sinh((\pi + 2i\alpha)r)}{\sinh \pi r}, \qquad 2\alpha = \log X + i/T,$$

which satisfies $h(r) = X^{ir}e^{-r/T} + O(e^{-\pi r})$. This choice is not arbitrary and appears already in [6]. In order to apply Kuznetsov formula, we have to insert the Fourier coefficients of cusp forms. This is a standard argument ([13, 12]), but we will summarise it below.

Let f be a function of Schwartz class on $[\sqrt{N}, \sqrt{2N}]$ with mass $\int f(x)\, dx = N$. After a change of variables, we may write f in terms of its Mellin transform $\tilde{f}$ through Mellin inversion:

$$f(x) = \frac{1}{\pi i} \int_{(3)} \tilde{f}(2s)|x|^{-2s} ds.$$

We set $x = |n|$, insert our test function and the Fourier coefficients, and sum over non-zero $n \in \mathbb{Z}[i]$ and r_j:

$$\frac{1}{N} \sum_{n \in \mathbb{Z}[i]\setminus\{0\}} \sum_{r_j} f(|n|)h(r_j)|v_j(n)|^2 = \frac{1}{\pi i N} \sum_{r_j} \frac{r_j h(r_j)}{\sinh \pi r_j} \int_{(3)} \tilde{f}(2s) L(s, u_j \otimes u_j) ds.$$

The Rankin–Selberg L-function $L(s, u_j \otimes u_j)$ has a simple pole at $s = 1$ with residue independent of j. Thus we move the contour to $\operatorname{Re} s = 1/2$, pick up the pole, and obtain for some absolute constants c_1, c_2 that

$$\sum_{r_j} X^{ir_j} e^{-r_j/T} = \frac{c_1}{N} \sum_{n, r_j} f(|n|)h(r_j)|v_j(n)|^2 + \frac{c_2}{N} \int_{(\frac{1}{2})} \tilde{f}(2s) \sum_{r_j} h(r_j) L(s, u_j \otimes u_j) ds.$$

For the integral on the right, we can almost immediately apply our bound (7).

It remains to treat the sum with the Fourier coefficients. In particular, we need to estimate

$$\int_V^{V+Y} \left| \frac{1}{N} \sum_{r_j, n} f(|n|)h(r_j)|v_j(n)|^2 \right|^2 dX.$$

Notice that h satisfies all the conditions of the Kuznetsov formula. It is not difficult to see that the contribution of the continuous spectrum as well as the delta term (i.e. C and U in Theorem 5) are negligible. Thus we are left with

$$\int_V^{V+Y} \sum_{N \le N(n) \le 2N} |\mathcal{S}_n(\omega)|^2 \, dX,$$

where $\mathcal{S}_n$ is the following weighted sum of Kloosterman sums:

$$\mathcal{S}_n(\omega) = \sum_{c \in \mathbb{Z}[i]\setminus\{0\}} \frac{S(n, n, c)}{N(c)} \omega\left(\frac{2\pi \overline{n}}{c}\right),$$

and ω is the highly oscillatory function

$$\omega(z) = \int_{-\infty}^{\infty} \frac{ir^2}{\sinh \pi r} J_{ir}(z) J_{ir}(\overline{z}) h(r) dr.$$

The main idea is to cut the sum over c into small enough pieces which allow us to evaluate the weight function ω explicitly. For the Kloosterman sums it suffices to use the Weil bound.

Let $z = 2\pi\overline{n}/c$. In the initial part, where $|z| \geq 1$ (i.e. $N(c) \leq 4\pi^2 N(n)$), we appeal to a formula of Motohashi [16, (2.10)] and transform the J-Bessel functions to an integral of a K-Bessel function (which are often easier to deal with). In the remaining range, $|z| < 1$, we can use the series definition of the J-Bessel functions. It is not hard to see that only the first term from the series for each J contributes and thus we are left with

$$\mathcal{S}_n(\omega) = \sum_{N(c) > 4\pi^2 N(n)} \frac{S(n,n,c)}{N(c)} \omega_0(z) + O(N^{1/2+\epsilon} T^{1+\epsilon}),$$

where

$$\omega_0(z) = \int_{-\infty}^{\infty} \frac{1}{\Gamma(1+ir)^2} \left| \frac{z}{2} \right|^{2ir} \frac{ir^2 h(r)}{\sinh \pi r} dr.$$

We then observe that this integral defining ω_0 is actually (almost) a K-Bessel function of order 0. In particular, this means that as $|z| \to 0$ (i.e. in the tail of the sum over c) we have sufficient decay, $\omega_0(z) \ll |z|^{1/2+\epsilon}$. The final piece of the proof is therefore to understand expressions of the form

$$\int_V^{V+Y} \sum_n \left| \sum_{C_1 < N(c) \leq C_2} \frac{S(n,n,c)}{N(c)^2} K_0(|z| X^{1/2}) \right|^2 dX,$$

for some C_1, C_2 which are functions of T, V and $N(n)$. Here we use again the Weyl bound for the Kloosterman sums and a standard estimate $|K_0(w)| \leq 2|w|^{-1/2} \exp(-\operatorname{Re} w)$. □

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