A SURVEY ON THE GLOBAL GAN-GROSS-PRASAD CONJECTURE FOR FOURIER-JACOBI CASE

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Abstract. The global Gan-Gross-Prasad (GGP) conjecture predicts that the non-vanishing of certain periods is equivalent to the non-vanishing of certain central value of some L-function. There are two types of GGP conjectures : Bessel case, Fourier-Jacobi case. In 2015, Hang Xue proved the Fourier-Jacobi GGP conjecture for skew-hermitian case on the same rank group. But his result is under certain local restriction to apply relative trace formula. We suggest a way to prove one direction of the general Fourier-Jacobi case for skew-hermitian unitary group without such local restricions. This survey article is based on the ongoing joint work with Hiraku Atobe.

1. **Fourier-Jacobi period**

Let E/F be a quadratic extension of number fields with adele rings $\mathbb{A}_E$ and $\mathbb{A}_F$ respectivly. We denote the nontrivial automorphism of E fixing F by $x \to \bar{x}$. Let ω be the non-trivial quadratic character assosiated to $F^\times\backslash\mathbb{A}_F^\times$ by the global class field theory and fix a chracter $\mu : E^\times\backslash\mathbb{A}_E^\times$ such that $\mu|_{\mathbb{A}_F^\times} = \omega$. Sometimes, we view μ as a character of $GL_n(\mathbb{A}_E)$ and in that case, it does mean $\mu \circ \det$. We also fix a nontrivial character ψ of $E\backslash\mathbb{A}_E$. If v is a place of F, we write $E_v = E \otimes F_v$. Let $W_m \subset W_n$ be m and n-dimensional skew-Hermitian spaces over E such that $W_n = X \oplus W_m \oplus X^*$ where $X \oplus X^*$ is the direct sum of r hyperbolic planes and the restriction of hermitian form of W_n to W_m is non-degenerate.

Let G_n, G_m be the isometry group of W_n, W_m respectively and regard G_m as a subgroup of G_n which acts trivially on the orthogonal compliment of W_m in W_n. We fix a complete flag of X and let $P_{n,r}$ the parabolic subgroup of G_n which stabilize this flag, with the unipotent radical $N_{n,r}$. Then the group G_m acts on $N_{n,r}$ through conjugation. Put $H = N_{n,r} \rtimes G_m$. There is an $H(F)$-invariant automorphic Weil representation $\nu_{\psi^{-1},\mu^{-1},W_m}$ of $H(\mathbb{A}_F)$ realized on Schwartz space $\mathcal{S}$. For each $f \in \mathcal{S}$, we can define a certain theta series $\Theta_{\psi^{-1},\mu^{-1}}(h, f)$ defined on $H(\mathbb{A}_F)$.

Let π_1, π_2 be two irreducible cuspidal automorphic representation of $G_n(\mathbb{A}_F)$ and $G_m(\mathbb{A}_F)$ respectively. We regard H as a subgroup of G_n through the map $(n, g) \to ng$. For $\varphi_1 \in \pi_1, \varphi_2 \in \pi_2, f \in \nu_{\psi^{-1},\mu^{-1},W_m}$, we define its Fourier-Jacobi period to be the integral as

$$\mathcal{FJ}_{\psi,\mu}(\varphi_1, \varphi_2, f) := \int_{[N_{n,r}\rtimes G_m]} \varphi_1(ng)\varphi_2(g)\Theta_{\psi^{-1},\mu^{-1}}((n, g), f)dndg,$$

where $[N_{n,r} \rtimes G_m] = N_{n,r}(F) \rtimes G_m(F)\backslash N_{n,r}(\mathbb{A}_F) \rtimes G_m(\mathbb{A}_F)$.

2. **Automorphic forms**

For a connected reductive algebraic group G over F, we fix a minimal F-parabolic subgroup P_0 of G with a Levi decomposition $P_0 = M_0U_0$ and a maximal compact subgroup $K = \prod_v K_v$

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of $G(\mathbb{A}_F)$ which satisfies

$$G(\mathbb{A}_F) = P_0(\mathbb{A}_F)K, \qquad P(\mathbb{A}_F) \cap K = (M(\mathbb{A}_F) \cap K)(U(\mathbb{A}_F) \cap K)$$

and $M(\mathbb{A}_F) \cap K$ is still maximal compact in $M(\mathbb{A}_F)$ for every standard parabolic subgroup $P = UM$ of G where $M_0 \subset M$. (see [10, I.1.4]) Note that the Levi factor M_0 is the centralizer of a maximal split torus T_0. Throughout the rest the paper, P always denote a standard subgroup of G unless mentioned.

Let $\mathscr{A}_P(G)$ be the space of automorphic forms on $U(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$. i.e., smooth, K-finite and $\mathfrak{z}$-finite functions on $U(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ of moderate growth, where $\mathfrak{z}$ is the center of the universal enveloping algebra of the complexified Lie algebra of the product of the archimedean localization of $G(\mathbb{A}_F)$. When $P = G$, we simply write $\mathscr{A}(G)$ for $\mathscr{A}_G(G)$. For a cuspidal automorphic representation ρ of $M(\mathbb{A}_F)$, we write $\mathscr{A}_P^{\rho}(G)$ for the subspace of functions $\phi \in \mathscr{A}_P(G)$ such that for all $k \in K$, the function $m \to |\delta_P(m)|^{-1} \cdot \phi(mk)$ belongs to the space of ρ. (Here, ρ_P is the modulus function of $P(\mathbb{A}_F)$.) (see [10, I.2.17])

We extend the definition of automorphic forms from reductive groups to special non-reductive groups. Let N be a unipotent group over F which admits a G-action and denote this action by $\sigma : G \to \mathrm{Aut}(N)$. Using σ, we can consider the semi-direct product $N \rtimes G$ and define automorphic forms on $N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$ as follows.

For a function $\varphi : N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F) \to \mathbb{C}$ and arbitrary $n \in N(\mathbb{A}_F)$, denote $\varphi_n : G(\mathbb{A}_F) \to \mathbb{C}$ by $\varphi_n(g) := \varphi(n, g)$. We say that φ is an automorphic form on $N(F) \rtimes G(F)\backslash N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$ if

- $\varphi\big((\delta, \gamma) \cdot (n, g)\big) = \varphi(n, g) \quad$ for $(\delta, \gamma) \in N(F) \rtimes G(F)$
- φ is smooth
- φ_n is right K-finite for a maximal compact subgroup K of $G(\mathbb{A}_F)$ for any $n \in N(\mathbb{A}_F)$
- φ_n is $\mathfrak{z}$-finite function for any $n \in N(\mathbb{A}_F)$
- φ_n is of moderate growth for any $n \in N(\mathbb{A}_F)$

We denote by $\mathscr{A}(N \rtimes G)$ the space of automorphic forms on $N(F) \rtimes G(F)\backslash N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$. Note that if N is the trivial group $\mathbf{1}$, then $\mathscr{A}(\mathbf{1} \rtimes G)$ equals $\mathscr{A}(G)$. For $\varphi \in \mathscr{A}(N \rtimes G)$, define $\phi_P : N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F) \to \mathbb{C}$ by

$$\varphi_P(n, g) := \int_{U_P(\mathbb{A}_F)} \varphi(n, ug)du \ \text{ for } (n, g) \in N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$$

and define $\varphi^P : G(\mathbb{A}_F) \to \mathbb{C}$ as

$$\varphi^P(g) := \int_{N(\mathbb{A}_F)} \varphi_P(n, g)dn.$$

Proposition 2.1. *For $\varphi \in \mathscr{A}(N \rtimes G)$, $\varphi^P \in \mathscr{A}_P(G)$ for any standard parabolic subgroup P of G.*

Remark 2.2. For $\phi \in \mathscr{A}(G)$, if we regard $\phi \in \mathscr{A}(\mathbf{1} \rtimes G)$, then $\phi_P = \phi^P$. Thus $\phi \to \phi_P$ sends $\mathscr{A}(G)$ to $\mathscr{A}_P(G)$.

3. **Mixed truncation**

To explain mixed truncation, we first recall some notation regarding Arthur truncation. For more explanation on the notation here, see [1, Sec. 1].

For a connected reductive algebraic group G over F, we fix a minimal F-parabolic subgroup P_0 of G with a Levi decomposition $P_0 = U_0 M_0$. Write $X(G)$ for the F-rational characters of G. Let $\mathfrak{a}_0^*$ be the $\mathbb{R}$-vector space spanned by lattice $X(T_0)$ and $\mathfrak{a}_0 = \mathrm{Hom}(X(T_0) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R})$ its dual space. The canonial pairing on $\mathfrak{a}_0^* \times \mathfrak{a}_0$ is denoted by $\langle\ ,\ \rangle$. Let Δ_0 and $\Delta_0^\vee$ be the sets of simple roots and simple coroots in $\mathfrak{a}_0^*$ and $\mathfrak{a}_0$ respectively. Write $\hat{\Delta}_0^\vee$ and $\hat{\Delta}_0$ for the dual basis of Δ_0 and $\Delta_0^\vee$ repectively. (In other words, $\hat{\Delta}_0^\vee$ and $\hat{\Delta}_0$ are set of coweight and weight respectively.) For a standard parabolic subgroup $P = UM$ of G, write T for a maximal split torus in the center of M and $\mathfrak{a}_P^* = X(M) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_P$ for its dual space.

For a pair of standard parabolic subgroups $Q \subset P$ of G, there is a canonial injection $\mathfrak{a}_P \to \mathfrak{a}_Q$ and surjection $\mathfrak{a}_Q \to \mathfrak{a}_P$ induced by two inclusion maps $M_Q \hookrightarrow M_P$ and $T_P \hookrightarrow T_Q$. So we have a canonical decomposition

$$\mathfrak{a}_Q = \mathfrak{a}_Q^P \oplus \mathfrak{a}_P, \qquad \mathfrak{a}_Q^* = (\mathfrak{a}_Q^P)^* \oplus \mathfrak{a}_P^*$$

In particular, if we take $Q = P_0$, we have a decompostion

$$\mathfrak{a}_0 = \mathfrak{a}_0^P \oplus \mathfrak{a}_P, \qquad \mathfrak{a}_0^* = (\mathfrak{a}_0^P)^* \oplus \mathfrak{a}_P^*$$

for all standard subgroup P.

For every standard subgroup P, let $\Delta_P \subset \Delta_0$ be the set of non-trivial restriction of simple roots to $\mathfrak{a}_P$. For any pair of standard sugroups $Q \subset P$, denote by Δ_Q^P the subset of Δ_Q appearing in the root decomposition of the Lie algebra of unipotent radical $U_Q \cap M_P$. Then for $H \in \mathfrak{a}_P$, $\langle \alpha, H \rangle = 0$ for all $\alpha \in \Delta_Q^P$ and so $\Delta_Q^P \subset (\mathfrak{a}_Q^P)^*$. Note that $\Delta_P^G = \Delta_P$. For any $\alpha \in \Delta_Q^P$, there is a $\tilde{\alpha} \in \Delta_0$ whose restriction to $\mathfrak{a}_Q^P$ is α. Write $\alpha^\vee$ for the projection of $\tilde{\alpha}^\vee$ to $\mathfrak{a}_Q^P$. Define

$$(\Delta_Q^P)^\vee = \{\alpha^\vee |\ \alpha \in \Delta_Q^P\}.$$

Define $(\hat{\Delta}^\vee)_Q^P \subset (\mathfrak{a}_Q^P)^*$ and $\hat{\Delta}_Q^P \subset \mathfrak{a}_Q^P$ to be the dual basis of Δ_Q^P and $(\Delta_Q^P)^\vee$ respectively. We simply write $\hat{\Delta}_P^\vee$ for $(\hat{\Delta}^\vee)_P^G$ and $\hat{\Delta}_P$ for $\hat{\Delta}_P^G$, respectively.

Let τ_Q^P be the characteristic function of the subset

$$\{H \in \mathfrak{a}_0 : \langle \alpha, H \rangle > 0 \text{ for all } \alpha \in \Delta_Q^P\} \subset \mathfrak{a}_0$$

and let $\hat{\tau}_Q^P$ be the characteristic function of the subset

$$\{H \in \mathfrak{a}_0 : \langle \varpi, H \rangle > 0 \text{ for all } \varpi \in \hat{\Delta}_Q^P\} \subset \mathfrak{a}_0.$$

Note that these two functions depends only on the projection of $\mathfrak{a}_0$ to $\mathfrak{a}_Q^P$. We write τ_P and $\hat{\tau}_P$ for τ_P^G and $\hat{\tau}_P^G$, respectively.

For each parabolic subgroup $P = UM$, we have height map

$$H_P : G(\mathbb{A}_F) \to \mathfrak{a}_P$$

characterized by the following properties : (see [1, page 917])

- $|\chi|(m) = e^{\langle \chi, H_P(m) \rangle}$ for all $\chi \in X(M)$ and $m \in M(\mathbb{A}_F)$
- $H_P(nmk) = H_P(m)$ for all $n \in U(\mathbb{A}_F), m \in M(\mathbb{A}_F), k \in K$.

The restriction of H_P on $M(\mathbb{A}_F)$ is surjective homomorphism. Denote the kernel of $H_P|_{M(\mathbb{A}_F)}$ by $M(\mathbb{A}_F)^1$ and the connected component of 1 in $T(\mathbb{R})$ by $T(\mathbb{R})^0$. Then $M(\mathbb{A}_F)$ is the direct product of normal subgroup $M(\mathbb{A}_F)^1$ with $T(\mathbb{R})^0$ and H_P gives an isomorphism between $T(\mathbb{R})^0$ and $\mathfrak{a}_P$. Denote the inverse of this map by $X \to e^X$. We simply write $H(g)$ for $H_{P_0}(g)$. Note that $H_P(g)$ is the projection of $H(g)$ onto $\mathfrak{a}_P$.

Let $T \in \mathfrak{a}_0$. For $\phi, \phi' \in \mathscr{A}(N \rtimes G)$, we define a mixed truncation by

$$\Lambda_m^T(\phi \otimes \phi')(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{\gamma \in P(F)\backslash G(F)} \Big(\int_{N(F)\backslash N(A_F)} \phi_P(n, \gamma g)\phi'_P(n, \gamma g)dn \Big) \hat{\tau}_P(H(\gamma g) - T)$$

for $g \in G$. More generally, we define a partial mixed truncation by

$$\Lambda_m^{T,P}(\phi \otimes \phi')(g) = \sum_{Q \subset P} (-1)^{\dim \mathfrak{a}_Q^P} \sum_{\delta \in Q(F)\backslash P(F)} \Big(\int_{N(F)\backslash N(A_F)} \phi_Q(n, \delta g)\phi'_Q(n, \delta g)dn \Big) \hat{\tau}_Q^P(H(\delta g) - T)$$

for $\phi, \phi' \in \mathscr{A}(N \rtimes G)$.

Lemma 3.1. *Let $\phi, \phi' \in \mathscr{A}(N \rtimes G)$. Then $\Lambda_m^T(\phi \otimes \phi')$ is rapidly decreasing on $G(F)\backslash G(\mathbb{A}_F)^1$.*

For $(\phi, \phi') \in \mathscr{A}(N \rtimes G)^2$ and $\phi'' \in \mathscr{A}(G)$, we consider the following integral

$$\int_{G(F)\backslash G(\mathbb{A}_F)^1} \Lambda_m^T(\phi \otimes \phi')(g)\phi''(g)dg. \tag{3.1}$$

Thanks to Lemma 3.1, this integral converges.

Write ρ_0 for half the sum of positive roots in $\mathfrak{a}_0^*$ and denote by ρ_P the projection of ρ_0 to $\mathfrak{a}_P^*$. Recall that $e^{2\langle \rho_P, H_P(p)\rangle} = \delta_P(p)$ for $p \in P(\mathbb{A}_F)$. It is known that an automorphic form $\phi \in \mathscr{A}_P(G)$ admits a finite decomposition

$$\phi(ue^X mk) = \sum_i Q_i(X)\phi_i(mk)e^{\langle \lambda_i + \rho_P, X\rangle}$$

for $u \in U(\mathbb{A}_F), X \in \mathfrak{a}_P, m \in M(\mathbb{A}_F)^1$ and $k \in K$, where $\lambda_i \in \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$, $Q_i \in \mathbb{C}[\mathfrak{a}_P]$ and $\phi_i \in \mathscr{A}_P(G)$ satisfies $\phi_i(e^X g) = \phi_i(g)$ for $X \in \mathfrak{a}_P$ and $g \in G$. (see [10, I.3.2]) We denote the finite set of exponents λ_i appearing in this decomposition by $\mathcal{E}_P(\phi)$.

Proposition 3.2. *Integral in* (3.1) *is a function of the form $\sum_\lambda p_\lambda(T)e^{\langle \lambda, T\rangle}$, where p_λ is a polynomial in T and λ can be taken from the set*

$$\bigcup_P \{\lambda + \lambda' + \lambda'' + \rho_P \mid \lambda \in \mathcal{E}_P(\phi^P),\ \lambda' \in \mathcal{E}_P(\phi'^P),\ \lambda'' \in \mathcal{E}_P(\phi''_P)\ (i = 1, 2, 3)\}$$

Definition 3.3. Let $\mathscr{A}_0(N \rtimes G)$ be the subspace of triplets $(\phi, \phi', \phi'') \in \mathscr{A}(N \rtimes G)^2 \times \mathscr{A}(G)$ such that the polynomial corresponding to the zero exponent of (3.1) is constant. For $(\phi, \phi', \phi'') \in \mathscr{A}_0(N \rtimes G)$, we define its regularized period $\mathcal{P}(\phi, \phi', \phi'')$ as its value $p_0(T)$. We also write

$$\mathcal{P}(\phi, \phi', \phi'') = \int^*_{G(F)\backslash G(\mathbb{A}_F)^1} \int^*_{N(F)\backslash N(\mathbb{A}_F)} \phi(n, g)\phi'(n, g)\phi''(g)dg.$$

Let $\mathscr{A}(N \rtimes G)^*$ be the space of all triplets $(\phi, \phi', \phi'') \in \mathscr{A}(N \rtimes G)^2 \times \mathscr{A}(G)$ such that

$$\langle \lambda + \lambda' + \lambda'' + \rho_P, \omega^\vee \rangle \neq 0 \quad (\omega^\vee \in (\hat{\Delta}^\vee)_P,\ \lambda \in \mathcal{E}_P(\phi^P),\ \lambda' \in \mathcal{E}_P(\phi'^P),\ \lambda' \in \mathcal{E}_P(\phi''_P))$$

for all parabolic subgroups P of G. If $(\phi, \phi', \phi'') \in \mathscr{A}(N \rtimes G)^*$, then the $\#$-integral

$$\mathcal{P}_P^T(\phi, \phi', \phi'') = \int^{\#}_{P(F)\backslash G(\mathbb{A}_F)^1} \Lambda_m^{T,P}(\phi \otimes \phi')(g)\phi''_P(g)\tau_P(H(g) - T)dg$$

is defined as the triple integral

$$\int_K \int_{M(F)\backslash M(\mathbb{A}_F)^1} \int_{\mathfrak{a}_P} \Lambda_m^{T,P}(\phi \otimes \phi')(e^X mk)\phi''_P(e^X mk)e^{-2\langle \rho_P, X\rangle}\tau_P(X - T)dXdmdk.$$

Proposition 3.4. *The following statements hold.*

(i) $\mathscr{A}(N \rtimes G)^* \subset \mathscr{A}_0(N \rtimes G)$
(ii) *If* $(\phi, \phi', \phi'') \in \mathscr{A}(N \rtimes G)^*$, *then*

$$\mathcal{P}(\phi, \phi', \phi'') = \sum_P \mathcal{P}_P^T(\phi, \phi', \phi'')$$

It says that $\sum_P \mathcal{P}_P^T$ *is independent of* T.
(iii) *The regularized period is a* $G(\mathbb{A})^1$*-invariant linear functional on* $\mathscr{A}(N \rtimes G)^*$.

Let $\mathscr{A}(N \rtimes G)^{**}$ be the subspace of all triplets $(\phi, \phi', \phi'') \in \mathscr{A}(N \rtimes G)^2 \times \mathscr{A}(G)$ such that

$$\langle \lambda + \lambda' + \lambda'' + \rho_P, \omega^\vee \rangle \neq 0 \quad (\omega^\vee \in (\hat{\Delta}^\vee)_Q^P,\ \lambda \in \mathcal{E}_Q(\phi^Q),\ \lambda' \in \mathcal{E}_Q(\phi'^Q),\ \lambda'' \in \mathcal{E}_Q(\phi''_Q))$$

for all pairs of parabolic subgroups $Q \subset P$ of G. Clearly $\mathscr{A}(N \rtimes G)^{**} \subset \mathscr{A}(N \rtimes G)^*$.

If $(\phi, \phi', \phi'') \in \mathscr{A}(N \rtimes G)^{**}$, then the regularized integral

$$\int^*_{P(F)\backslash G(\mathbb{A})^1} \Big(\int_{N(F)\backslash N(\mathbb{A}_F)} \phi_P(n,g)\phi'_P(n,g)dn \Big) \phi''_P(g)dg$$

$$= \int_K \int^*_{M(F)\backslash M(\mathbb{A}_F)} \int^\#_{\mathfrak{a}_P} \Big(\int_{N(F)\backslash N(\mathbb{A}_F)} \phi_P(n, e^X mk)\phi'_P(n, e^X mk)dn \Big) \phi''_P(e^X mk)\hat{\tau}_P(X-T)e^{-2\langle \rho_P, X\rangle} dX dm dk$$

is well defined for every P.

Proposition 3.5. *If* $(\phi, \phi', \phi'') \in \mathscr{A}(N \rtimes G)^{**}$, *then*

$$\int_{G(F)\backslash G(\mathbb{A}_F)^1} \Lambda_m^T(\phi \otimes \phi')(g)\phi''(g)dg$$

$$= \sum_P (-1)^{\dim \mathfrak{a}_P} \int^*_{P(F)\backslash G(\mathbb{A}_F)^1} \Big(\int_{N(F)\backslash N(\mathbb{A}_F)} \phi_P(n,g)\phi'_P(n,g)dn \Big) \phi''_P(g)\hat{\tau}_P(H(g)-T)dg.$$

4. Jacquet module corresponding to Fourier-Jacobi character

In this section, E/F can be either quadratic extension of number fields or a non-archimedean quadratic extension of local fields whose characteristics are zero. In the local field case, ψ and μ denote a nontrivial character of F and $E^\times$ respecively. Write $|\cdot|$ and $|\cdot|_E$ for the normailzed absolute value on F and E respectively, viewed as a character of general linear group composed with det.

Let $(W_n, (\cdot,\cdot))$ be a skew-hermition space over E of dimension n and let G_n its unitary group. Let a be the dimension of a maximal totally isotropic subspace of W_n and we assume $a > 0$. We fix maximal totally isotropic subspaces X and Y of W_n, in duality, with respect to $(\cdot,\cdot)$. Fix a complete flag in X

$$0 = X_0 \subset X_1 \subset \cdots \subset X_a = X,$$

and choose a basis $\{e_1, e_2, \cdots, e_a\}$ of X_a such that $\{e_1, \cdots, e_k\}$ is a basis of X_k for $1 \leq k \leq a$. Let $\{f_1, f_2, \cdots, f_a\}$ be the basis of X^* which is dual to the fixed basis of X, i.e., $(e_i, f_j) = \delta_{ij}$ for $1 \leq i, j \leq r$, where $\delta_{i,j}$ denotes the Kronecker delta. We write X_k^* for the subspace of X^* spanned by $\{f_1, f_2, \cdots, f_k\}$ and W_{n-2k} for the orthogonal complement of $X_k + X_k^*$ in W_n.

Denote by $P_{n,k}$ the parabolic subgroup of G_n stabilizing X_k, by $U_{n,k}$ its unipotent radical and $M_{n,k}$ the Levi subgroup of $P_{n,k}$ stabilizing the above decomposition. Then $M_{n,k} \simeq GL(X_k) \times G_{n-2k}$. (Here, we regard $GL(X_k) \simeq GL_k$ as the subgroup of $M_{n,k}$ which acts as the identity map on W_{n-2k}.)

For a smooth representation σ of $GL(X_k)$ and a smooth representation π of G_{n-2k}, we denote by $\mathrm{Ind}_{P_{n,k}}^{G_n}(\sigma \boxtimes \pi)$ the normailized induced representation of G_n and by $\mathrm{ind}_{P_{n,k}}^{G_n}(\sigma \boxtimes \pi)$ the unnormalized induction. For $1 \le i \le a-k$, we write $\sigma^{(i)}$ for the Bernstein-Zelevinski (i)-th derivative of σ. (For the definition of Bernstein-Zelevinski derivative, refer to [2, Section 4.3].)

For $0 \le k \le [\frac{n}{2}]$, we write $N_{n,k}$ (resp. $\mathcal{N}_k$) for the unipotent radical of the parabolic subgroup of G_n (resp. $GL(X_k)$) stabilizing the flag $\{0\} \subset X_1 \subset \cdots \subset X_k$. If we regard $\mathcal{N}_a$ as a subgroup of $M_{n,a} \simeq GL(X) \times G_{n-2a}$, it acts on $U_{n,a}$ and so $N_{n,a} = U_{n,a} \rtimes \mathcal{N}_a$.

For any $0 < k < \frac{n}{2}$, let $\mathcal{H}_{n-2k}$ be the Heisenberg group of skew hermitian space W_{n-2k} over E and $\Omega_{\psi^{-1},\mu^{-1}W_{n-2k}}$ be the Weil representation of $\mathcal{H}_{n-2k} \rtimes G_{n-2k}$ with respect to ψ^{-1}, μ^{-1}. Then since $U_{n,k-1}\backslash U_{n,k} \simeq \mathcal{H}_{n-2k}$, we can pull back $\Omega_{\psi^{-1},\mu^{-1},W_{n-2k}}$ to $U_{n,k} \rtimes G_{n-2k}$ and denote it by the same symbol $\Omega_{\psi^{-1},\mu^{-1},W_{n-2k}}$. We define a character $\lambda_k : \mathcal{N}_k \to \mathbb{C}^\times$ by

$$\lambda_k(n) = \psi((Tr^*_{E/F}(n_{1,2} + n_{2,3} + \cdots + n_{k-1,k})), \quad u \in \mathcal{N}_k.$$

Here, $n_{i,i+1}$ is the $(i, i+1)$-component of n when we regard n as an element in GL_k and

$$Tr^*_{E/F} = \begin{cases} Tr_{E/F} & , \text{ local fields case} \\ Tr_{\mathbb{A}_E/\mathbb{A}_F} & , \text{ number fields case.} \end{cases}$$

Put $\nu_{\psi^{-1},\mu^{-1},W_{n-2k}} = \Omega_{\psi^{-1},\mu^{-1},W_{n-2k}} \otimes \lambda_k$ and denote $H_{n,k} = N_{n,k} \rtimes G_{n-2k}$. We can embed $H_{n,k}$ into $G_n \times G_{n-2k}$ by inclusion on the first factor and projection on the second factor. Then $\nu_{\psi^{-1},\mu^{-1},W_{n-2k}}$ is a smooth representation of $H_{n,k} = N_{n,k} \rtimes G_{n-2k}$ and upto conjugation of the normalizer of $H_{n,k}$ in $G_n \times G_{n-2k}$, it is uniquely determined by ψ modulo $\mathrm{Nm}_{E/F}E^\times$ and μ. We shall denote by $\omega_{\psi^{-1},\mu^{-1},W_{n-2k}}$ the restriction of $\nu_{\psi^{-1},\mu^{-1},W_{n-2k}}$ to G_{n-2k}.

For $0 \le l \le \frac{n-2}{2}$, we define a character ψ_l of $N_{n,l+1}$, which factors through the quotient $n : N_{n,l+1} \to U_{n,l+1}\backslash N_{n,l+1} \simeq \mathcal{N}_{l+1}$, by setting

$$\psi_l(u) = \lambda_{l+1}(n(u)).$$

In the local fields case, for a smooth representation π' of G_n, we write $J_{\psi_l}(\pi' \otimes \Omega_{\psi^{-1},\mu^{-1},W_{n-2l-2}})$ for the Jacquet module of $\pi' \otimes \Omega_{\psi^{-1},\mu^{-1},W_{n-2l-2}}$ with respect to the group $N_{n,l+1}$ and its character ψ_l, regarded as a representation of the unitary group G_{n-2l-2}.

Lemma 4.1. *Let n, m, a be positive integers such that $n - m \ge 0$ and even. Write q for the residual characteristic of E. Let $\mathcal{E}, \sigma$ and π be smooth representations of finite lengths of $G_{n+2a}, GL(X_a)$ and G_m, respectively. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_{m+2a}} \left(\mathcal{E} \otimes \nu_{\psi^{-1},\mu^{-1},W_{m+2a}} \otimes \mathrm{ind}_{P_{m,a}}^{G_{m+2a}}(\sigma|\cdot|_E^s \boxtimes \pi) \right)$$

is equal or less than

$$\dim_{\mathbb{C}} \sigma^{(a)} \cdot \dim_{\mathbb{C}} \mathrm{Hom}_{G_m}(J_{\psi_{\frac{n-m}{2}+a-1}}(\mathcal{E} \otimes \Omega_{\psi^{-1},\mu^{-1},W_m}), \pi^\vee))$$

except for finitely many q^{-s}.

Proof. The proof is similar with [15, Lemma 4.1] except for the symplectic group is replaced by unitary group. ☐

5. **Residual representation**

For an irreducible cuspidal automorphic representations π of $G_n(\mathbb{A}_F)$ and σ of $GL_a(\mathbb{A}_E)$, we write $L(s,\sigma\times\pi)$ for the Rankin-Selberg L-function $L(s,\sigma\times BC(\pi))$. We also wrtie $L(s,\sigma,As^+)$ for the Asai L-function of σ and $L(s,\sigma,As^-)$ for the μ-twisted Asai L-function $L(s,\sigma\otimes\mu,As^+)$. (cf. [5, Section 7])

Proposition 5.1 ([9], Proposition 5.3)**.** *Let π be an irreducible gloally generic cuspidal automorphic representation of $G_n(\mathbb{A}_F)$ and σ an irreducible cuspidal automorphic representation of $GL_a(\mathbb{A}_E)$. For $\phi\in\mathscr{A}^{\sigma\boxtimes\pi}_{P_{n,a}}(G_{n+2a})$, the Eisenstein sereis $E(\phi,z)$ has at most a simple pole at $z=\frac{1}{2}$ and $z=1$. Moreover, it has a pole at $z=\frac{1}{2}$ as ϕ varies if and only if $L(s,\sigma\times\pi^\vee)$ is non-zero at $s=\frac{1}{2}$ and $L(s,\sigma,As^{(-1)^n})$ has a pole at $s=1$. Furthermore, it has a pole at $z=1$ as ϕ varies if and only if $L(s,\sigma\times\pi^\vee)$ has a pole at $s=1$.*

For $\phi\in\mathscr{A}^{\sigma\boxtimes\pi}_{P_a}(G_{n+2a})$, we define the residue of the Eisenstein series to be the limit

$$\mathcal{E}^0(\phi)=\lim_{z\to\frac{1}{2}}(z-\frac{1}{2})E(\phi,z),\quad \mathcal{E}^1(\phi)=\lim_{z\to1}(z-1)E(\phi,z).$$

For $i=0,1$, let $\mathcal{E}^i(\sigma,\pi)$ be the residual representations of $G_{n+2a}(\mathbb{A}_F)$ generated by $E^i(\phi)$.

The assumption that π is globally generic ensures the existence of the weak base change $BC(\pi)$ and we can write it as an isobaric sum of the form $\sigma_1\boxplus\cdots\boxplus\sigma_t$, where $\sigma_1,\cdots,\sigma_t$ are distinct irreducible cuspidal automorphic representations of the general linear groups such that the (twisted) Asai L-function $L(s,\sigma_i,As^{(-1)^{n-1}})$ has a pole at $s=1$.

Remark 5.2. Since $L(s,\sigma\times\pi^\vee)=\Pi_{i=1}^t L(s,\sigma\times\sigma_i^\vee)$, Proposition 5.1 implies that $\mathcal{E}^1(\sigma,\pi)$ is non-zero if and only if $\sigma\simeq\sigma_i$ for some $1\le i\le t$.

Remark 5.3. Let c be the automorphism of $GL_n(E)$ induced by $^-:E\to E$ and for a representation σ of $GL_n(\mathbb{A}_E)$, we define $\sigma^c:=\sigma\circ c$. Note that $L(s,\sigma,As^\pm)$ are nonzero at $s=1$ by [12, Theorem 5.1]. Thus if $L(s,\sigma,As^{(-1)^{n-1}})$ has a pole at $s=1$, the Rankin-Selberg L-function

$$L(s,\sigma\times\sigma^c)=L(s,\sigma,As^+)\cdot L(s,\sigma,As^-)$$

has a simple pole at $s=1$ and so $\sigma^c\simeq\sigma^\vee$.

6. **Lemmas**

In this section, E/F denotes a quadration extension of number fields.

Let $W_m\subset W_n$ be two skew-hermitian spaces over E of dimension m,n such that $W_n=X\oplus W_m\oplus X^*$. Let V be the $\mathrm{Res}_{E/F}(W_m)$, which is the restriction of scarlar of W_m to F. Write $n-m=2a$. Let $V=Y+Y^*$ be the complete polarization of V. Then the global Weil representation $\Omega_{\psi^{-1},\mu^{-1},W_m}$ of $\mathcal{N}(X)\rtimes G_m$ has a realization on the Schrodinger model $\mathcal{S}(Y(\mathbb{A}_F))$. For $f\in\mathcal{S}(Y(\mathbb{A}_F))$, we define theta funtion $\Theta_{\psi^{-1},\mu^{-1}}(\cdot,f)$ on $H(\mathbb{A}_F)=N_{n,a}(\mathbb{A}_F)\rtimes G_m(\mathbb{A}_F)$ by

$$\Theta_{\psi^{-1},\mu^{-1}}(h,f)=\sum_{x\in Y(F)}\big(\nu_{\psi^{-1},\mu^{-1},W_m}(h)f\big)(x)=\sum_{x\in Y(F)}\psi_{a-1}(n)\cdot\big(\Omega_{\psi^{-1},\mu^{-1},W_m}(u,g)f\big)(x)$$

where $h=((u,n),g)\in\big(U_{n,a}\rtimes\mathcal{N}(X)\big)\rtimes G_m$. Then $\Theta_{\psi^{-1},\mu^{-1}}(f)\in\mathscr{A}(H)$ and the space of these theta functions $\{\Theta_{\psi^{-1},\mu^{-1}}(\cdot,f)\mid f\in\mathcal{S}(Y(\mathbb{A}_F))\}$ is another realization of Weil representation $\nu_{\psi^{-1},\mu^{-1},W_m}$ of $H(\mathbb{A}_F)$.

Since we haved fixed μ, ψ, we simply wrtie ν_{W_m} for $\nu_{\psi^{-1},\mu^{-1},W_m}$ and its associated theta function $\Theta_{\psi^{-1},\mu^{-1}}(\cdot, f)$ as $\Theta(\cdot, f)$.

Lemma 6.1. *Let σ be an irreducible cuspidal automorphic representation of $GL_a(\mathbb{A}_E)$, π_1, π_2 an irreducible globally generic cuspidal automorphic representation of $G_n(\mathbb{A}_F)$ and $G_m(\mathbb{A}_F)$ respectively. We write $BC(\pi_1)$ as an isobaric sum $\sigma_1 \boxplus \cdots \boxplus \sigma_t$, where $\sigma_1, \cdots, \sigma_t$ are distinct irreducible cuspidal automorphic representations of the general linear groups such that the (twisted) Asai L-function $L(s, \sigma_i, As^{(-1)^{n-1}})$ has a pole at $s = 1$. If $\sigma \simeq \sigma_i$ for some $1 \le i \le t$, then $\mathcal{P}(\varphi, \Theta(f), E(\phi, z)) = 0$ for all , $\varphi \in \mathcal{E}^1(\sigma, \pi_1), \phi \in \mathscr{A}_{P_{m,a}}^{\mu\cdot\sigma^c\boxtimes\pi_2}(G_{m+2a})$ and $f \in \nu_{W_{m+2a}}$.*

Lemma 6.2. *With the same notation as in Lemma 6.1, we assume $\sigma \simeq \sigma_i$ for some $1 \le i \le t$. If $\varphi \in \mathcal{E}^1(\sigma, \pi_1), \phi \in \mathscr{A}_{P_{m,a}}^{\mu\cdot\sigma^\vee\boxtimes\pi_2}(G_{m+2a})$ and $f \in \nu_{W_{m+2a}}$, then*

$$\mathcal{P}(\varphi, \Theta(f), \mathcal{E}(\phi)) =$$

$$\int_{K_{m+2a}} \int_{M_{m+2a}(F)\backslash M_{m+2a}(\mathbb{A})^1} \phi(mk)\Big(\int_{N_{n+2a,r}(F)\backslash N_{n+2a,r}(\mathbb{A}_F)} \varphi_{P_a}(nmk)\Theta_{P_a}((n, mk), f)dn\Big)\, dmdk.$$

Proof. The proof is almost same with [15, Proposition 6.3]. □

Lemma 6.3. *With the same notation as in Lemma 6.1, we assume $\sigma \simeq \sigma_i$ for some $1 \le i \le t$. If there are $\xi_1 \in \pi_1$, $\xi_2 \in \pi_2$ and $\xi \in \nu_{W_m}$ such that $\mathcal{FJ}(\xi_1, \xi_2, \xi) \neq 0$, then there are $\varphi \in \mathcal{E}^1(\sigma, \pi_1), \phi \in \mathscr{A}_{P_a}^{\mu\cdot\sigma^\vee\boxtimes\pi_2}(G_{m+2a})$ and $f \in \nu_{W_{m+2a}}$ such that*

$$\int_{K_{m+2a}} \int_{M_{m+2a}(F)\backslash M_{m+2a}(\mathbb{A})^1} \phi(mk)\Big(\int_{N_{n+2a,r}(F)\backslash N_{n+2a,r}(\mathbb{A}_F)} \varphi_{P_a}(nmk)\Theta^{\psi}_{P_a}((n, mk), f)dn\Big)\, dmdk \neq 0.$$

7. Main theroem

Theorem 7.1. *Let π_1, π_2 be an irreducile globally generic cuspidal automorphic representations of $G_n(\mathbb{A}_F)$ and $G_m(\mathbb{A}_F)$ respectively. If there are $\varphi_1 \in \pi_1, \varphi_2 \in \pi_2$ and $f \in \nu_{W_m}$ such that $\mathcal{FJ}_{\psi,\mu}(\varphi_1, \varphi_2, f) \neq 0$, then $L(\frac{1}{2}, BC(\pi_1) \times BC(\pi_2) \otimes \mu^{-1}) \neq 0$.*

Proof. Since π_1 is globally generic, $BC(\pi_1)$ is an isobaric sum of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_t$, where $\sigma_1, \cdots, \sigma_t$ are distinct irreducible cuspidal automorphic representations of the general linear groups such that the (twisted) Asai L-function $L(s, \sigma_i, As^{(-1)^{n-1}})$ has a pole at $s = 1$. Then for each $1 \le i \le t$, $L(s, \mu^{-1} \cdot \sigma_i, As^{(-1)^n})$ has a pole at $s = 1$. On the other hand, $\mathcal{E}^0(\mu \cdot \sigma_i^\vee, \pi_2)$ is nonzero by Lemma 6.2 and Lemma 6.3. Thus by Proposition 5.1, we have $L(\frac{1}{2}, BC(\pi_2^\vee) \times \mu \cdot \sigma_i^\vee) \neq 0$ and so $L(\frac{1}{2}, BC(\pi_2) \times \mu^{-1}\sigma_i) \neq 0$. Thus

$$L(\frac{1}{2}, BC(\pi_1) \times BC(\pi_2) \otimes \mu^{-1}) = \prod_{i=1}^{t} L(\frac{1}{2}, BC(\pi_2) \times \mu^{-1}\sigma_i) \neq 0.$$

□

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