The 2D Kramers-Dirac oscillator and a corresponding semi-quantum system

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Abstract

The 2D Kramers-Dirac oscillator and its corresponding semi-quantum Hamiltonian are introduced. The bulk-edge correspondence is shown to hold in terms of spectral flow and Chern number.

1 Introduction

Though the Dirac equation was originally introduced by P.M. Dirac in order to describe electron motions in a relativistic manner, the Dirac equation can be interpreted in a wider sense as describing a quantum system with two energy bands, without reference to relativity, and as giving an element of band rearrangement against a control parameter, where mass is viewed as a control parameter, so that it may take negative values.

In order to explain the band rearrangement against the control parameter, we take up a Hamiltonian which is a modification of the spin-orbital coupling $\boldsymbol{S} \otimes \boldsymbol{L} = \sum S_k \otimes L_k$, where S_k and L_k are the spin and the angular momentum operators, respectively. The Hamiltonian we are interested in is an extended Hamiltonian given by

$$H_\mu = S_3 \otimes (\mu + L_3) + S_- \otimes L_+ + S_+ \otimes L_-, \tag{1}$$

$$S_\pm = S_1 \pm iS_2, \quad L_\pm = L_1 \pm iL_2, \tag{2}$$

where μ is a real-valued control parameter [1].

The H_μ acts on $\mathbb{C}^{2S+1} \otimes \mathcal{H}(S^2)$, where S is the spin parameter and $\mathcal{H}(S^2)$ denotes the space of square integrable functions on the two-sphere S^2. For $S = 1$, the H_μ takes the form

$$H_\mu = \begin{pmatrix} \mu + L_3 & \sqrt{2}L_- & 0 \\ \sqrt{2}L_+ & 0 & \sqrt{2}L_- \\ 0 & \sqrt{2}L_+ & -(\mu + L_3) \end{pmatrix}. \tag{3}$$

We quote a figure from [1] (see Fig. 1), in which eigenvalues of the above Hamiltonian are shown as functions of the control parameter A in place of μ. The eigenvalues are broken up into two classes. The eigenvalues belonging to one class have nothing to do with band rearrangement, but those belonging to the other class are responsible for the band rearrangement. Furthermore, we observe that an elementary band rearrangement or energy level redistribution with/without crossing takes place between two adjacent bands and any band rearrangement may be composed of successive elementary band rearrangements.

We assume that an elementary band rearrangement can be detected through linearization of the Hamiltonian at a "singular" point. As the operator H_μ admit a rotational

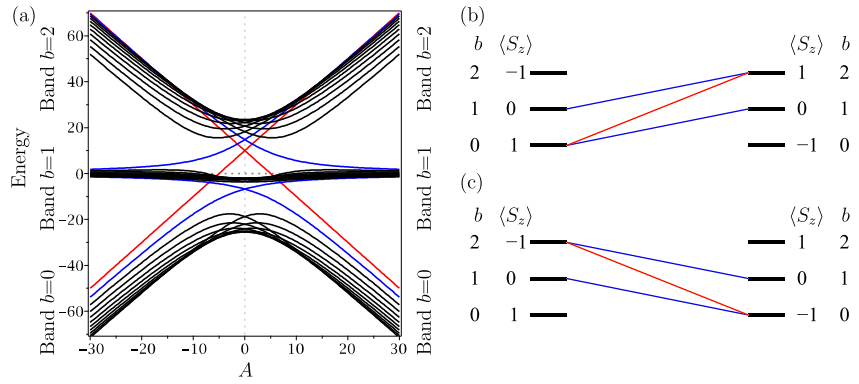

Figure 1: (a) A general view of the quantum energy level pattern for $S = 1$ and $L = 5$. (b) Correlation diagram showing the redistribution of the energy levels between the $A \to -\infty$ and the $A \to \infty$ limits, where A denotes a control parameter. In each limit, the bands can be labeled by increasing energy with an integer $b = 0, 1, 2$, or by the average value of $\langle S_z \rangle$. Only the levels which change bands under the variation in A are shown.

symmetry about the axis $\boldsymbol{e}_3$, the north and the south poles are singular from the viewpoint of group orbits. The linearization of H_μ with $S = \frac{1}{2}$ at the north pole provides a Dirac operator on the tangent plane to S^2 at the north pole;

$$\frac{1}{2}\begin{pmatrix} \mu + L_3 & L_1 - iL_2 \\ L_1 + iL_2 & -\mu - L_3 \end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix} \mu & i\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} \\ i\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1} & -\mu \end{pmatrix}, \tag{4}$$

where use has been made of the standard expression of the angular momentum operators,

$$L_1 = -i\big(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\big),\ L_2 = -i\big(x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\big),\ L_3 = -i\big(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\big). \tag{5}$$

However, to get eigenvalues for the Dirac operator on the plane, we have to restrict the plane $\mathbb{R}^2$ to a bounded domain, say, a disk with radius R and to consider an eigenvalue problem under a boundary condition.

2 A review of free Dirac equations with boundary condition

The free Dirac operator on $\mathbb{R}^d$ is given by

$$H_\mu = -i\sum_{k=1}^{d} \gamma_k \partial_k + \mu\gamma_0, \quad \partial_k = \partial/\partial x_k, \tag{6}$$

where μ is a mass, which may serve as a control parameter taking all real values, and where γ_k are the gamma matrices satisfying

$$\begin{array}{ll} \gamma_k\gamma_j + \gamma_j\gamma_k = 2\delta_{jk}I, & j,k = 1,\ldots,d, \\ \gamma_k\gamma_0 + \gamma_0\gamma_k = 0, & (\gamma_0)^2 = I, \\ (\gamma_\nu)^\dagger = \gamma_\nu, & \nu = 0,1,\ldots,d, \end{array} \tag{7}$$

with I denoting the identity matrix of suitable size. The gamma matrices are realized, depending on $d = 1,2,3,4$, as follows:

$$\begin{array}{ll} d=1; & \gamma_1 = \sigma_1, \quad \gamma_0 = \sigma_3, \\ d=2; & \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_2, \quad \gamma_0 = \sigma_3, \\ d=3; & \gamma_k = \sigma_2 \otimes \sigma_k,\ k=1,2,3, \quad \gamma_0 = \sigma_3 \otimes \mathbb{1}, \\ d=4; & \gamma_k = \sigma_2 \otimes \sigma_k,\ k=1,2,3, \quad \gamma_4 = \sigma_1 \otimes \mathbb{1}, \quad \gamma_0 = \sigma_3 \otimes \mathbb{1}, \end{array} \tag{8}$$

where $\mathbb{1}$ denotes the 2×2 identity and where σ_k are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

We now touch on boundary conditions without detail. Let V and B denote a bounded domain in $\mathbb{R}^d$ and its boundary, respectively. A key to finding a boundary condition is Green's formula

$$\langle \Phi, H_\mu \Psi\rangle_V - \langle H_\mu \Phi, \Psi\rangle_V = -i\langle \phi, \vec{\gamma}\cdot\vec{n}\,\psi\rangle_B, \tag{9}$$

where $\phi = \Phi|_B, \psi = \Psi|_B$ and $\vec{\gamma}\cdot\vec{n} = \sum \gamma_j n_j$ and where $\vec{n}$ is the outward unit normal to B. Any boundary conditions for the Dirac equation $H_\mu\Phi = E\Phi$ should require the vanishing of the right-hand side of the above equation. If such a boundary condition is found, the operator H_μ becomes a symmetric operator. Furthermore, with some Sobolev conditions, it becomes self-adjoint. The Atiyah-Patodi-Singer (APS) boundary condition is a well-known one.

In what follows, we give a few examples of eigenvalues as functions of the control parameter, which are picked up from [2, 3]. An eigenvalue is called an edge-state or a bulk-state eigenvalue, according as it is responsible for or has nothing to do with the band rearrangement. The edge-state and the bulk-state eigenvalues of the 2D Dirac equation on a disk under the APS boundary condition are given in Fig.2 and Fig.3, respectively. These examples support our observation that the Dirac equation describes a two-band quantum system and gives an element of band rearrangement against a control parameter.

3 A review of Dirac oscillators

The band rearrangements so far observed is for free Dirac equations with boundary condition. A question arises as to whether a band rearrangement can be observed also for a Dirac equation in an electro-magnetic field. From the linearization point of view, the Dirac equation in question is required to be linear in momentum operators and position

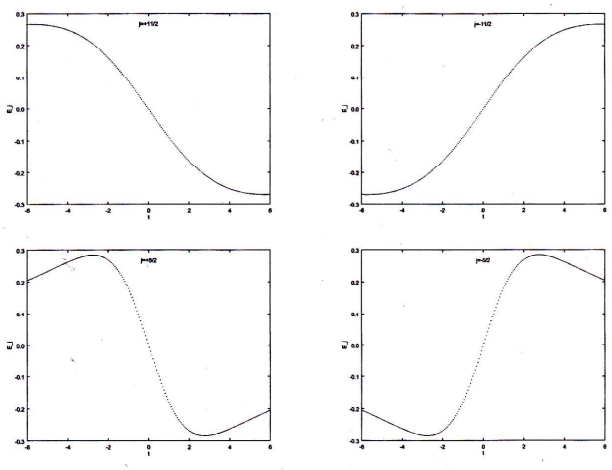

Figure 2: Edge eigenvalues as functions of a control parameter t for the 2D Dirac equation with the APS boundary condition. Left panels are for $j = 11/2$, $j = 5/2$, and right graphs for $j = -11/2$ and $j = -5/2$, where j denotes the eigenvalue of the spin-angular momentum operator.

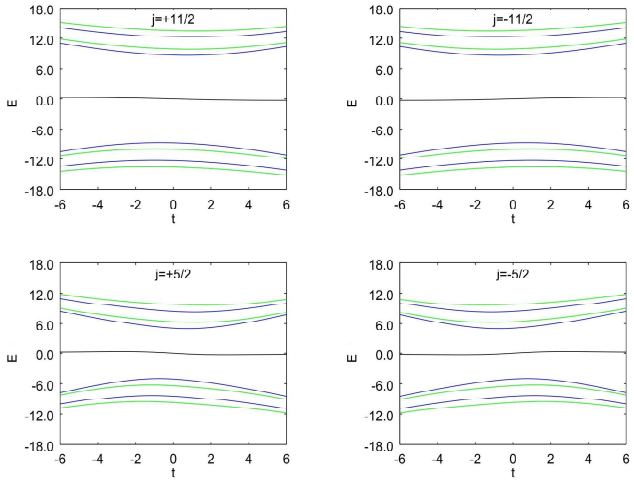

Figure 3: Bulk-state eigenvalues as functions of the control parameter t under the APS boundary condition.

variables. The Dirac oscillator was initially introduced as a Dirac operator linear in momentum and position variables in [4] and later studied in [5, 6], for example, as solvable

problems. Let a_k and $a_k^\dagger$ be the annihilation and the creation operators given by

$$a_k = \frac{1}{\sqrt{2}}\big(\frac{\partial}{\partial x_k} + x_k\big), \quad a_k^\dagger = \frac{1}{\sqrt{2}}\big(-\frac{\partial}{\partial x_k} + x_k\big), \tag{10}$$

respectively, which satisfy the commutation relations

$$[a_k, a_j^\dagger] = \delta_{kj}, \quad [a_k, a_j] = [a_k^\dagger, a_j^\dagger] = 0. \tag{11}$$

A normalized basis of $L^2(\mathbb{R}^d)$ is given by

$$|n_1, \dots, n_d\rangle = \frac{1}{\sqrt{n_1! \cdots n_d!}} (a_1^\dagger)^{n_1} \cdots (a_d^\dagger)^{n_d} |0\rangle, \tag{12}$$

where n_k are non-negative integers, and $|0\rangle = |0, \dots, 0\rangle$ is the ground state satisfying

$$a_k|0\rangle = 0, \quad k = 1, \dots, d.$$

The n-dimensional Dirac oscillators, $n = 1, 2, 3$, are defined to be

$$H_\mu^{(1)} = \begin{pmatrix} \mu & a \\ a^\dagger & -\mu \end{pmatrix}, \tag{13}$$

$$H_\mu^{(2)} = \begin{pmatrix} \mu & a_1 - ia_2 \\ a_1^\dagger + ia_2^\dagger & -\mu \end{pmatrix}, \tag{14}$$

$$H_\mu^{(3)} = \begin{pmatrix} \mu \mathbb{1} & \boldsymbol{\sigma} \cdot \boldsymbol{a} \\ \boldsymbol{\sigma} \cdot \boldsymbol{a}^\dagger & -\mu \mathbb{1} \end{pmatrix}, \tag{15}$$

respectively, where $\mathbb{1}$ denotes the 2×2 identity and where $\boldsymbol{\sigma} \cdot \boldsymbol{a} = \sum_{k=1}^3 \sigma_k a_k$.

In particular, the 2D Dirac oscillator is rewritten as a Dirac operator for the Landau electron

$$\sqrt{2} H_\mu^{(2)} = -i\alpha_1 \nabla_1 - i\alpha_2 \nabla_2 + \sqrt{2}\mu\beta, \tag{16}$$

where $\alpha_1 = -\sigma_2, \alpha_2 = \sigma_1, \beta = \sigma_3$ and where

$$\nabla_1 = \frac{\partial}{\partial q_1} - iq_2, \quad \nabla_2 = \frac{\partial}{\partial q_2} + iq_1. \tag{17}$$

The commutation relation between ∇_1 and ∇_2 is $[\nabla_1, \nabla_2] = 2i$. The 2D Dirac oscillator of the form (16) is used in [7] as a model Hamiltonian for quantum Hall effect.

The 2D Dirac oscillator admits the $SO(2)$ or the rotational symmetry,

$$[H_\mu^{(2)}, J] = 0, \quad J = L + \frac{1}{2}\sigma_3, \tag{18}$$

$$L = -i\big(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\big) = i(a_1 a_2^\dagger - a_2 a_1^\dagger),$$

and further the $U(1)$ symmetry,

$$[H_\mu^{(2)}, \widetilde{N}] = 0, \quad \widetilde{N} = N + \frac{1}{2}\sigma_3, \tag{19}$$

$$N = a_1^\dagger a_1 + a_2^\dagger a_2.$$

In addition, the J and $\widetilde{N}$ commute

$$[\widetilde{N}, J] = 0. \tag{20}$$

By the effective use of these symmetries, the eigenvalues of the 2D Dirac oscillator can be found; the bulk eigenstates are

$$E_n^\pm = \pm\sqrt{\mu^2 + 2(n+1)}, \quad n = 0, 1, 2, \ldots., \tag{21}$$

and the edge eigenstate is

$$E^{\rm edg} = -\mu. \tag{22}$$

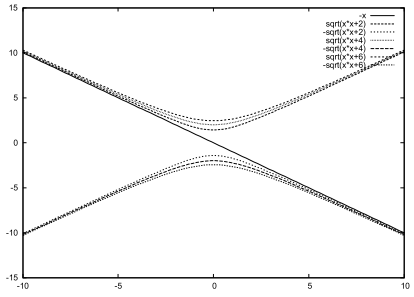

Figure 4: The edge-state eigenvalue and the bulk-state eigenvalues with $n = 0, 1, 2$ are given for the 2D Dirac oscillator. This figure gives an elementary band rearrangement.

4 The 2D Kramers-Dirac oscillator

4.1 The definition of the 2D Kramers-Dirac oscillator

The 2D Dirac oscillator does not admit the time-reversal symmetry,

$$i\sigma_2 \overline{H_\mu^{(2)}}(-i\sigma_2) \neq H_\mu^{(2)}. \tag{23}$$

We wish to extend the 2D Dirac oscillator so that it may admit the time-reversal symmetry. Any complex number is expressed as a 2×2 real matrix, and so are the complex operators $a_1 - ia_2$ and $a_1^\dagger + ia_2^\dagger$, where a_j and $a_j^\dagger$ are real operators. Hence, we may represent the present complex operators in the matrix form,

$$a_1 - ia_2 \mapsto \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}, \quad a_1^\dagger + ia_2^\dagger \mapsto \begin{pmatrix} a_1^\dagger & -a_2^\dagger \\ a_2^\dagger & a_1^\dagger \end{pmatrix}, \tag{24}$$

respectively. Then, we obtain the following Hamiltonian as a Dirac operator which is linear in the momentum and the position variables and admits the time reversal symmetry,

$$H_\mu = \begin{pmatrix} \mu\mathbb{1} & \sqrt{2}A \\ \sqrt{2}A^\dagger & -\mu\mathbb{1} \end{pmatrix},\ A = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix},\ A^\dagger = \begin{pmatrix} a_1^\dagger & -a_2^\dagger \\ a_2^\dagger & a_1^\dagger \end{pmatrix}, \tag{25}$$

$$((\mathbb{1} \otimes i\sigma_2)K)H_\mu((\mathbb{1} \otimes i\sigma_2)K)^{-1} = H_\mu, \tag{26}$$

where K denotes the complex conjugation and where the time-reversal operator $(\mathbb{1}\otimes i\sigma_2)K$ satisfies

$$((\mathbb{1} \otimes i\sigma_2)K)^2 = -\mathrm{id}_{\mathbb{C}^2\otimes\mathbb{C}^2}. \tag{27}$$

The H_μ acts on $\mathbb{C}^4 \otimes L^2(\mathbb{R}^2)$. We call the H_μ the 2D Kramers-Dirac oscillator. The adjective "Kramers" comes from the time-reversal symmetry. A reason for the nomenclature "oscillator" will be explained in the non-relativistic limit. If squared, H_μ becomes

$$H_\mu^2 = \begin{pmatrix} \mu^2 + 2(1 + h_2 - \sigma_2 \otimes L) & \\ & \mu^2 + 2(-1 + h_2 - \sigma_2 \otimes L) \end{pmatrix}, \tag{28}$$

where $h_2 = N + 1$ denotes the 2D harmonic oscillator Hamiltonian in the sense of Schrödinger. This equation means that in the non-relativistic limit, H_μ becomes a 2D harmonic oscillator with the spin-orbital coupling $\sigma_2 \otimes L$.

Like the 2D Dirac oscillator, the Kramers-Dirac oscillator admits $SO(2) \times U(1)$ symmetry. In fact, one can easily verify that

$$[H_\mu, J] = 0, \quad J = \begin{pmatrix} \mathbb{1}L & \\ & L + \sigma_2 \end{pmatrix}, \tag{29}$$

$$[H_\mu, \widetilde{N}] = 0, \quad \widetilde{N} = \begin{pmatrix} \mathbb{1}(N + \frac{1}{2}) & \\ & \mathbb{1}(N - \frac{1}{2}) \end{pmatrix}, \tag{30}$$

and further

$$[J, \widetilde{N}] = 0. \tag{31}$$

4.2 Eigenvalue problem

On account of the symmetry, the eigenvalue problem for H_μ is decomposed into subproblems on the simultaneous eigenspaces of J and $\widetilde{N}$. Because of the $SO(2)$ symmetry, the initial eigenvalue problem is broken up into subproblems on the eigenspaces for $J = m$. Introducing the polar coordinates (r, θ), one can easily find eigenvalues and eigenstates of J. Then, on the eigenspace of J, eigenstates for H_μ proves to take the form

$$\Phi = \begin{pmatrix} f(r)\chi_m^+(\theta) \\ g(r)\chi_{m+1}^+(\theta) \end{pmatrix}, \quad \Psi = \begin{pmatrix} f(r)\chi_m^-(\theta) \\ g(r)\chi_{m-1}^-(\theta) \end{pmatrix}, \tag{32}$$

where

$$\chi_m^+(\theta) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{im\theta}, \quad \chi_m^-(\theta) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{im\theta}, \tag{33}$$

and

$$\chi^{+}_{m+1}(\theta) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{i(m+1)\theta}, \quad \chi^{-}_{m-1}(\theta) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i(m-1)\theta}. \tag{34}$$

On expressing the Hamiltonian H_μ in the polar coordinates, the initial eigenvalue equations $H_\mu\Phi = E\Phi$ and $H_\mu\Psi = E\Psi$ with Φ and Ψ given in (32) reduce into radial equations

$$\mu f + \frac{dg}{dr} + rg + \frac{m+1}{r}g = Ef, \tag{35a}$$

$$-\frac{df}{dr} + rf + \frac{m}{r}f - \mu g = Eg, \tag{35b}$$

and

$$\mu f + \frac{dg}{dr} + rg - \frac{m-1}{r}g = Ef, \tag{36a}$$

$$-\frac{df}{dr} + rf - \frac{m}{r}f - \mu g = Eg, \tag{36b}$$

respectively.

We now take advantage of the $U(1)$ symmetry to restrict the functional form of f and g to radial wave functions for the 2D Schrödinger harmonic oscillator. As is well known, the radial wave functions in question and the associated eigenvalues are given by

$$R_{n,|m|}(r) = \sqrt{\frac{2n!}{\Gamma(n+|m|+1)}}\, r^{|m|} e^{-r^2/2} L^{|m|}_n(r^2), \quad \varepsilon = 2n + |m| + 1, \tag{37}$$

respectively, where n is the radial quantum number and where $L^{\alpha}_n(z)$ are the Laguerre polynomials defined to be

$$L^{\alpha}_n(z) = \frac{1}{n!} e^z z^{-\alpha} \frac{d^n}{dz^n}(e^{-z} z^{\alpha+n}), \quad \alpha > -1. \tag{38}$$

A straightforward calculation with the $U(1)$ symmetry provides us with

$$\Phi^{+}_{n,m} = \begin{pmatrix} c_1 R_{n,|m|}(r)\chi^{+}_m(\theta) \\ c_2 R_{n,|m+1|}(r)\chi^{+}_{m+1}(\theta) \end{pmatrix} \quad \text{for} \quad m \geq 0, \tag{39a}$$

$$\Phi^{-}_{n,m} = \begin{pmatrix} c_1 R_{n,|m|}(r)\chi^{+}_m(\theta) \\ c_2 R_{n+1,|m+1|}(r)\chi^{+}_{m+1}(\theta) \end{pmatrix} \quad \text{for} \quad m \leq -1, \tag{39b}$$

and

$$\Psi^{+}_{n,m} = \begin{pmatrix} c'_1 R_{n,|m|}(r)\chi^{-}_m(\theta) \\ c'_2 R_{n+1,|m-1|}(r)\chi^{-}_{m-1}(\theta) \end{pmatrix} \quad \text{for} \quad m \geq 1, \tag{40a}$$

$$\Psi^{-}_{n,m} = \begin{pmatrix} c'_1 R_{n,|m|}(r)\chi^{-}_m(\theta) \\ c'_2 R_{n,|m-1|}(r)\chi^{-}_{m-1}(\theta) \end{pmatrix} \quad \text{for} \quad m \leq 0. \tag{40b}$$

Our last task is to determine the constants c_j and c'_j (or the ratio of them). If c_1 or c_2 (rep. c'_1 or c'_2) vanishes, Eqs. (35) and (36) are solved to give the eigenvalue and the associated eigenstates of the form

$$\Phi_{\rm edg} = \begin{pmatrix} 0 \\ r^{-(m+1)} e^{-r^2/2} \chi^+_{m+1}(\theta) \end{pmatrix}, \quad E^{\rm edg} = -\mu \quad \text{for} \quad m+1 \le 0, \tag{41}$$

$$\Psi_{\rm edg} = \begin{pmatrix} 0 \\ r^{m-1} e^{-r^2/2} \chi^-_{m-1}(\theta) \end{pmatrix}, \quad E^{\rm edg} = -\mu \quad \text{for} \quad m-1 \ge 0. \tag{42}$$

We turn to the case where neither c_1 nor c_2 (resp. c'_1 nor c'_2) vanishes. In respective cases, the coefficients c_k, c'_k, $k = 1, 2$, in (39) and (40) are determined by using (35) and (36) together with the recurrence formulae [8, 9],

$$\Big(\frac{d}{dr} + r + \frac{|m|}{r}\Big) R_{n,|m|}(r) = 2\sqrt{n+|m|}\, R_{n,|m|-1}(r), \tag{43a}$$

$$\Big(\frac{d}{dr} - r - \frac{|m|}{r}\Big) R_{n,|m|}(r) = -2\sqrt{n+|m|+1}\, R_{n,|m|+1}(r), \tag{43b}$$

$$\Big(\frac{d}{dr} - r + \frac{|m|}{r}\Big) R_{n,|m|}(r) = 2\sqrt{n+1}\, R_{n+1,|m|-1}(r), \tag{43c}$$

$$\Big(\frac{d}{dr} + r - \frac{|m|}{r}\Big) R_{n,|m|}(r) = -2\sqrt{n}\, R_{n-1,|m|+1}(r). \tag{43d}$$

In fact, (35) and (36) together with these recurrence formulae give rise to algebraic eigenvalue problems for unknown constants c_k, c'_k, $k = 1, 2$, which are easily solved. Thus, we obtain the following eigenvalues and associated eigenstates,

$$E^{\pm}_{n;m\ge 0} = \pm\sqrt{\mu^2 + 4(n+|m|+1)}, \quad m \ge 0, \tag{44}$$

$$\Phi^{\pm}_{n;m\ge 0} = \begin{pmatrix} 2\sqrt{n+|m|+1}\, R_{n,|m|}(r)\chi^+_m(\theta) \\ (\pm\sqrt{\mu^2+4(n+|m|+1)} - \mu) R_{n,|m+1|}(r)\chi^+_{m+1}(\theta) \end{pmatrix}, \ m \ge 0, \tag{45}$$

$$E^{\pm}_{n\,(m\le -1)} = \pm\sqrt{\mu^2 + 4(n+1)}, \quad m \le -1, \tag{46}$$

$$\Phi^{\pm}_{n\,(m\le -1)} = \begin{pmatrix} -2\sqrt{n+1}\, R_{n,|m|}(r)\chi^+_m(\theta) \\ (\pm\sqrt{\mu^2+4(n+1)} - \mu) R_{n+1,|m+1|}(r)\chi^+_{m+1}(\theta) \end{pmatrix}, \ m \le -1, \tag{47}$$

$$E^{\pm}_{n\,(m\ge 1)} = \pm\sqrt{\mu^2 + 4(n+1)}, \quad m \ge 1, \tag{48}$$

$$\Psi^{\pm}_{n\,(m\ge 1)} = \begin{pmatrix} -2\sqrt{n+1}\, R_{n,|m|}(r)\chi^-_m(\theta) \\ (\pm\sqrt{\mu^2+4(n+1)} - \mu) R_{n+1,|m-1|}(r)\chi^-_{m-1}(\theta) \end{pmatrix}, \ m \ge 1, \tag{49}$$

$$E^{\pm}_{n;m\le 0} = \pm\sqrt{\mu^2 + 4(n+|m|+1)}, \quad m \le 0, \tag{50}$$

$$\Psi^{\pm}_{n;m\le 0} = \begin{pmatrix} 2\sqrt{n+|m|+1}\, R_{n,|m|}(r)\chi^-_m(\theta) \\ (\pm\sqrt{\mu^2+4(n+|m|+1)} - \mu) R_{n,|m-1|}(r)\chi^-_{m-1}(\theta) \end{pmatrix}, \ m \le 0. \tag{51}$$

4.3 Kramers pairs

Under the time-reversal operator, the spin-orbital angular momentum operator J changes the sign,

$$((\mathbb{1}\otimes i\sigma_2)K)J((\mathbb{1}\otimes i\sigma_2)K)^{-1} = -J, \tag{52}$$

which means that the eigenvalue m of J changes the sign, and accordingly, one has the inversion of m; $m \to -m$. The eigenstates so far obtained are shown to be related by

$$(\mathbb{1}\otimes i\sigma_2)K\Phi_{\rm edg} = \Psi_{\rm edg}, \tag{53}$$

$$(\mathbb{1}\otimes i\sigma_2)K\Phi^{\pm}_{n;m\geq 0} = \Psi^{\pm}_{n;-|m|\leq 0}, \tag{54}$$

$$(\mathbb{1}\otimes i\sigma_2)K\Phi^{\pm}_{n\,(m\leq -1)} = \Psi^{\pm}_{n\,(m\geq 1)}. \tag{55}$$

Those Kramers pairs related by $(\mathbb{1}\otimes i\sigma_2)K$ belong to the eigenspaces associated, respectively, with

$$E^{\rm edg} = -\mu, \quad \{E^{\pm}_{n;m>0}, E^{\pm}_{n;m\leq 0}\}, \quad \{E^{\pm}_{n(m\leq -1)}, E^{\pm}_{n(m\geq 1)}\}.$$

The bulk-state eigenvalues so far obtained are structured, as is shown in Fig. 5. The pairs (m, n) assigning the same eigenvalue determined by $n+|m| = k$ (see (44) and (50)) or $n = k$ (see (46) and (48)) are sitting on the line $n+|m| = k$ or the line $n = k$. The edge-state eigenvalue may be assigned by $n + 1 = 0$, as is seen from (46) and (48). The figure is symmetric under the reflection $m \mapsto -m$, which is a consequence of the time-reversal symmetry of the Hamiltonian. The eigenvalues $E^{\pm}_{n(m\leq -1)} = E^{\pm}_{n(m\geq 1)}$ and $E^{\rm edg} = -\mu$ are infinitely degenerate, but the eigenvalue $E^{\pm}_{n;m\leq 0} = E^{\pm}_{n;m>0}$ is finitely degenerate with multiplicity $2k + 1$ if $n + |m| = k$. Though the band rearrangement for the 2D Kramers-Dirac oscillator looks similar to that of the 2D Dirac oscillator, the band structure is quite different from one another owing to the time-reversal symmetry.

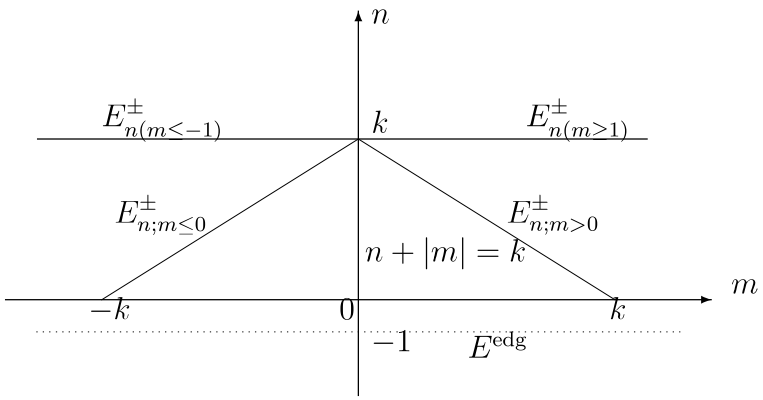

Figure 5: The eigenvalues of the 2D Kramers-Dirac oscillator are structured, depending on the radial and the angular quantum numbers, n and m.

5 A relativistic view of the 2D Kramers-Dirac oscillator

While we have defined the 2D Kramers-Dirac oscillator in (25), a question may arise as to dimensionality. The space-time for the 2D Kramers-Dirac oscillator is $\mathbb{R}^{1+2}$, but the size of the gamma matrices we have used is 4×4. For the 2D Dirac oscillator, the gamma matrices are of size 2×2. The identity component of the Lorentz group acting on $\mathbb{R}^{1+2}$ is $SO_0(1,2)$, and its covering group is $SU(1,1)$, the former of which is usually described in the form of real matrix of size 3×3 and the latter in the form of complex matrix of size 2×2. For the 2D Kramers-Dirac oscillator, the group $SU(1,1)$ should be represented in the 4×4 matrix form. Another physical question is this: The 2D Dirac oscillator is put in the form (16) or in the form of minimal coupling with a gauge potential for a magnetic flux density. For a physical reason or on account of the time-reversal symmetry, the 2D Kramers-Dirac oscillator is not put in the form of minimal coupling. We then have to describe the 2D Kramers-Dirac oscillator in the form of non-minimal coupling.

In order to put the Dirac equation for H_μ in the form of non-minimal coupling, we start with the time-dependent Dirac equation $i\frac{\partial \Phi}{\partial t} = H_\mu \Phi$, which can be shown to be rewritten as

$$(i\gamma^\alpha \frac{\partial}{\partial q_\alpha} + \frac{1}{2}\sigma^{\alpha\beta}F_{\alpha\beta} - \mu I)\Phi = 0, \quad \alpha, \beta = 0, 1, 2, \tag{56}$$

where $q_0 = t$ and I denotes the 4×4 identity matrix, and

$$\gamma^0 = \sigma_3 \otimes \mathbb{1}, \quad \gamma^1 = i\sigma_1 \otimes \mathbb{1}, \quad \gamma^2 = -i\sigma_2 \otimes \sigma_2, \tag{57}$$

$$\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta], \tag{58}$$

and where $(F_{\alpha\beta})$ is the electro-magnetic field determined by

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix}, \quad E_1 = q_1, \quad E_2 = q_2, \quad B = 0. \tag{59}$$

Equation (56) shows that the 2D Kramers-Dirac oscillator is an operator for a charged particle in an electric field.

We turn to a representation of $SU(1,1)$ in the 4×4 matrix form. In terms of

$$\xi_0 = \frac{1}{2}\begin{pmatrix} i\sigma_2 & \\ & -i\sigma_2 \end{pmatrix}, \ \xi_1 = \frac{1}{2}\begin{pmatrix} & -\sigma_2 \\ -\sigma_2 & \end{pmatrix}, \ \xi_2 = \frac{1}{2}\begin{pmatrix} & i\mathbb{1} \\ -i\mathbb{1} & \end{pmatrix}, \tag{60}$$

the group $SU(1,1)$ proves to be realized as

$$\exp(\theta\xi_0)\exp(\tau\xi_2)\exp(\phi\xi_0) = \begin{pmatrix} e^{\frac{i}{2}(\theta+\phi)\sigma_2}\cosh\frac{\tau}{2} & ie^{\frac{i}{2}(\theta-\phi)\sigma_2}\sinh\frac{\tau}{2} \\ -ie^{-\frac{i}{2}(\theta-\phi)\sigma_2}\sinh\frac{\tau}{2} & e^{-\frac{i}{2}(\theta+\phi)\sigma_2}\cosh\frac{\tau}{2} \end{pmatrix}. \tag{61}$$

This group is shown to be a covering group of $SO_0(1,2)$. Let

$$\mathcal{M} = \{X = \sum_\alpha q_\alpha \gamma^\alpha,\ (q_\alpha) \in \mathbb{R}^3\} \cong \mathbb{R}^{1+2}, \tag{62}$$

for which the Lorentz structure is given by

$$\frac{1}{4}\mathrm{tr}(X^T X) = q_0^2 - q_1^2 - q_2^2. \tag{63}$$

Then, the adjoint action by $g \in SU(1,1)$ of the form (61) gives rise to the Lorentz group,

$$g\gamma^\alpha g^{-1} = \sum \gamma^\beta h_\beta{}^\alpha, \quad (h_\beta{}^\alpha) \in SO_0(1,2), \tag{64}$$

where $SO_0(1,2)$ denotes the identity component of $SO(1,2)$. The Lorenz covariance of (56) can be verified in a similar manner.

In addition, we make a remark on $SU(1,1)$ of the form (61). The group $SO^*(2n)$ is defined to be

$$SO^*(2n) = \{g \in GL(2n,\mathbb{C}) |\, g^T J_{2n} \overline{g} = J_{2n},\ g^T g = I_{2n}\}, \tag{65}$$

where

$$J_{2n} = \begin{pmatrix} J_2 & & \\ & \ddots & \\ & & J_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

The Lie algebra $\mathfrak{so}^*(2n)$ is then defined to be

$$\mathfrak{so}^*(2n) = \{X \in \mathfrak{gl}(2n,\mathbb{C}) |\, J_{2n} X = \overline{X} J_{2n},\ X^T + X = 0\}. \tag{66}$$

The Lie algebra $\mathfrak{so}(1,2) \cong \mathfrak{su}(1,1)$ concerning the 2D Kramers-Dirac oscillator, which is spanned by ξ_0, ξ_1, ξ_2, is a subalgebra of $\mathfrak{so}^*(4)$, where $\mathfrak{so}^*(4)$ is decomposed into

$$\mathfrak{so}^*(4) = \mathfrak{so}(1,2) \oplus \mathfrak{so}(3). \tag{67}$$

In the rest of this section, we touch upon a physical application. According to the reference [10], the quantum spin Hall phase is a time reversal invariant electronic state with a bulk electronic band gap that supports the transfer of charge and spin in gapless edge states. From this point of view, the 2D Kramers-Dirac oscillator is qualified as a model Hamiltonian for the quantum spin Hall effect, like the 2D Dirac oscillator which serves as a model for quantum Hall effect [7].

6 Corresponding semi-quantum systems

For a quantum system of one-degree-of-freedom, a Bose coherent state is defined to be a normalized eigenstate of the annihilation operator,

$$a|z\rangle = z|z\rangle, \quad a|0\rangle = 0, \quad z \in \mathbb{C},$$

which are known as the most classical-like states (by Schrödinger). Taking the expectation values, one has

$$\langle z|a|z\rangle = z, \quad \langle z|a^\dagger|z\rangle = \overline{z}.$$

For a quantum system of two-degrees-of-freedom, the definition of coherent sates are easily extended. Taking expectation values, one obtains a semi-quantum system corresponding to the 2D Kramers-Dirac oscillator (within a multiplication $\sqrt{2}$),

$$K_\mu = \begin{pmatrix} \mu\mathbb{1} & C \\ C^\dagger & -\mu\mathbb{1} \end{pmatrix}, \quad C = \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix}, \quad C^\dagger = \begin{pmatrix} \overline{z}_1 & -\overline{z}_2 \\ \overline{z}_2 & \overline{z}_1 \end{pmatrix}. \tag{68}$$

However, this does not admit the time-reversal symmetry.

6.1 A semi-quantum Hamiltonian with time-reversal symmetry

A relevant semi-quantum system admitting the time-reversal symmetry is given by

$$K'_\mu = \begin{pmatrix} \mu\mathbb{1} & W \\ W^\dagger & -\mu\mathbb{1} \end{pmatrix}, \quad W = \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}, \quad W^\dagger = \begin{pmatrix} \overline{z}_1 & -z_2 \\ \overline{z}_2 & z_1 \end{pmatrix}. \tag{69}$$

In fact, one verifies that

$$(\mathbb{1} \otimes i\sigma_2)\overline{K'}_\mu(\mathbb{1} \otimes (-i\sigma_2)) = K'_\mu. \tag{70}$$

On introducing new coordinates k_j, $j = 1, \ldots, 4$, by

$$z_1 = k_4 - ik_3, \quad z_2 = -k_2 - ik_1, \tag{71}$$

the K'_μ is expressed as

$$K'_\mu = \begin{pmatrix} \mu\mathbb{1} & k_4\mathbb{1} - i\boldsymbol{k}\cdot\boldsymbol{\sigma} \\ k_4\mathbb{1} + i\boldsymbol{k}\cdot\boldsymbol{\sigma} & -\mu\mathbb{1} \end{pmatrix}, \tag{72}$$

where the gamma matrices given in the last line of (8) are used. The eigenvalues of K'_μ are

$$\lambda_\pm = \pm\sqrt{\mu^2 + k^2}, \quad k^2 = |\vec{k}|^2, \quad \vec{k} = (k_j),\ j = 1, 2, 3, 4, \tag{73}$$

each of which is doubly degenerate because of the time-reversal symmetry.

For $\mu \neq 0$, the eigen-vector bundles $L^\pm$ associated with $\lambda_\pm$ are defined over $\mathbb{R}^4$. Our interest centers on L^+ in what follows. The projection onto the eigenspace associated with λ_+ is given by

$$P_+ = \frac{1}{2\lambda_+}\begin{pmatrix} (\lambda_+ + \mu)\mathbb{1} & k_4\mathbb{1} - i\boldsymbol{k}\cdot\boldsymbol{\sigma} \\ k_4\mathbb{1} + i\boldsymbol{k}\cdot\boldsymbol{\sigma} & (\lambda_+ - \mu)\mathbb{1} \end{pmatrix} = \frac{1}{2\lambda_+}(\lambda_+ I + K'_\mu). \tag{74}$$

The structure group of the vector bundle L^+ is found to be

$$U = \frac{1}{|\vec{k}|}\begin{pmatrix} -k_4 + ik_3 & ik_1 + k_2 \\ ik_1 - k_2 & -k_4 - ik_3 \end{pmatrix} = \frac{1}{|\vec{k}|}(-k_4\mathbb{1} + i\boldsymbol{k}\cdot\boldsymbol{\sigma}). \tag{75}$$

The connection A and the curvature F are defined through the covariant derivation $d^{\nabla} = P_+ d$. Then, the formal second Chern number is defined to be

$$c_2(L^+) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F). \tag{76}$$

We are interested in change in the Chern number against the parameter, which is defined and calculate as

$$\begin{aligned} &\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F)\Big|_{\mu>0} - \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F)\Big|_{\mu<0} \\ =&\frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU), \end{aligned} \tag{77}$$

where U is the structure group given in (75). Eq. (77) means that the change in Chern number or a delta-Chern is equal to the mapping degree of

$$U:\ S^3 \to SU(2) \cong S^3, \tag{78}$$

which is evaluated as 1, so that one has the delta-Chern, $c_2(L^+)|_{\mu>0} - c_2(L^+)|_{\mu<0} = 1$.

6.2 Bulk-edge correspondence

The 2D Kramers-Dirac oscillator has infinitely degenerate edge-state eigenvalue $E^{\mathrm{edg}} = -\mu$, which is responsible for band rearrangement. We associate with it the degenerate spectral flow -1, which is defined to be the net number of sets of degenerate eigenvalues passing through zero in the positive direction as the parameter runs.

We note that the delta-Chern, $c_2(L^+)|_{\mu>0} - c_2(L^+)|_{\mu<0}$, depends on the orientation of $\mathbb{C}^2$. Setting $z_1 = q_1 + ip_1, z_2 = q_2 + ip_2$, one has

$$dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2 = dk_1 \wedge dk_2 \wedge dk_3 \wedge dk_4.$$

However, we may take the positive volume form as $dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2$, then the orientation is inverted,

$$dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 = -dk_1 \wedge dk_2 \wedge dk_3 \wedge dk_4.$$

If this volume form is taken as positive, then the delta-Chern changes the sign to be expressed as

$$c_2(L^+)|_{\mu>0} - c_2(L^+)|_{\mu<0} = -1. \tag{79}$$

The degenerate spectral flow and the delta-Chern are now in exact correspondence. Since the eigenvalues (73) of the semi-quantum Hamiltonian may be considered as corresponding to the bulk-state eigenvalues of the 2D Kramers-Dirac oscillator, the correspondence between the degenerate spectral flow and the delta-Chern may be looked on as the bulk-edge correspondence.

In conclusion, we remark that the eigenvalues and the eigenstates of H_μ have already been given in [11] without reference to the spectral flow and the second Chern class.

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