Nonpersistence of periodic orbits, homoclinic orbits, first integrals and commutative vector fields in perturbed systems

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Abstract. Determination of whether periodic orbits, homoclinic orbits, first integrals or commutative vector fields may persist under perturbations is one of the most important problems in the field of dynamical systems. In this paper, we give several theorems on necessary conditions for their persistence in general perturbed systems. Moreover, we consider periodic perturbations of one-degree-of-freedom Hamiltonian systems and describe some relationships between our results and the standard Melnikov method for periodic orbits and homoclinic orbits. This is a joint work with Kazuyuki Yagasaki (Kyoto University).

1 Introduction

For continuous dynamical systems, periodic orbits, homoclinic orbits, first integrals and commutative vector fields (continuous symmetries) play important roles. Especially, first integrals and commutative vector fields are closely related to integrability in the meaning of Bogoyavlenskij (we see its definition in Section 2) which is a generalization of the complete integrability for Hamiltonian systems. In most case, their persistences are not trivial.

So we consider continuous dynamical systems with perturbations of the form

$$\dot{x} = X_\varepsilon(x), \quad x \in \mathscr{M}$$

where $\mathscr{M}$ is a smooth manifold, $X_\varepsilon = X^0 + \varepsilon X^1 + O(\varepsilon^2)$ is a smooth vector field on $\mathscr{M}$ depending on ε smoothly and X_0 has a periodic orbit or homoclinic orbit, and a first integral. In this paper, we give several theorems on necessary conditions for their persistence in general perturbed systems. Moreover, we apply our result to periodic perturbations of one-degree-of-freedom Hamiltonian systems under the assumption of the Melnikov method and give a connection of the Melnikov method and persistence of the first integral.

2 Summary of the Known Results

We briefly review the known results on persistence of periodic orbits, homoclinic orbits, first integrals and commutative vector fields in perturbed systems.

2.1 Periodic orbits and homoclinic orbits: Melnikov's method

For brevity, we only present the standard Melnikov method for periodic perturbations of single-degree-of-freedom Hamiltonian systems.

We consider systems of the form

$$\dot{x} = JDH(x) + \varepsilon g(x,t), \quad x \in \mathbb{R}^2, \tag{2.1}$$

where ε is a small parameter such that $0 < \varepsilon \ll 1$, $H : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ are smooth, $g(x,t)$ is T-periodic in t with $T > 0$ a constant, and J is the 2×2 symplectic matrix,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

When $\varepsilon = 0$, Eq. (2.1) becomes the single-degree-of-freedom Hamiltonian system with the Hamiltonian $H(x)$,

$$\dot{x} = JDH(x). \tag{2.2}$$

Let $\theta = t \mod T$ so that $\theta \in \mathbb{S}$, where $\mathbb{S}^1 = \mathbb{R}/T\mathbb{Z}$. We rewrite (2.1) as an autonomous system,

$$\dot{x} = JDH(x) + \varepsilon g(x,\theta), \quad \dot{\theta} = 1. \tag{2.3}$$

We make the following assumption:

(M) The unperturbed system (2.2) possesses a one-parameter family of periodic orbits $q^\alpha(t)$ with period T^α, $\alpha \in (\alpha_1, \alpha_2)$, for some $\alpha_1 < \alpha_2$.

Assume that $\alpha \in (\alpha_1, \alpha_2)$ satisfies $lT^\alpha = mT$ for some relatively prime, positive integers m and n. Then we can regard that Eq. (2.3) has a one-parameter family of mT-periodic orbits $x = q^\alpha(t)$ and $\theta = t + \tau$, $\tau \in \mathbb{S}^1$, when $\varepsilon = 0$. Using the Melnikov method [6,13], we see that if the *subharmonic Melnikov function*

$$M^{m/l}(\tau) := \int_0^{mT} DH(q^\alpha(t)) \cdot g(q^\alpha(t), t + \tau) dt$$

has a simple zero at $\tau = \tau_0 \in \mathbb{S}^1$, then for $\varepsilon > 0$ sufficiently small Eq. (2.3) has a periodic orbit of period mT near $x = q^\alpha(t - \tau_0)$ and $\theta = t \mod T$. In other words, the periodic orbit $(x, \theta) = (q^\alpha(t - \tau_0), t)$, $t \in [0, T)$, persists under the perturbation $\varepsilon g(x, \theta)$ if the subharmonic Melnikov function $M^{m/l}(\tau)$ has a simple zero at $\tau = \tau_0 \in \mathbb{S}^1$.

We next assume the following instead of assumption (M).

(M') The unperturbed system (2.2) possesses a hyperbolic saddle point p connected to itself by a homoclinic orbit $q^{\mathrm{h}}(t)$.

Under the assumption (M') we can regard that Eq. (2.3) has a hyperbolic periodic orbit $(x, \theta) = (p, t)$ connected to itself by a one-parameter family of homoclinic orbits $x = q^{\mathrm{h}}(t)$ and $\theta = t + \tau \mod T$, $\tau \in \mathbb{S}^1$, when $\varepsilon = 0$. We easily show that there exists a hyperbolic periodic orbit near $x = p$ and $\theta = t \mod T$ (see [6,13] for the proof). Using the Melnikov method [6,7,13], we see that if the *homoclinic Melnikov function*

$$M(\tau) := \int_{-\infty}^{\infty} DH(q^{\mathrm{h}}(t)) \cdot g(q^{\mathrm{h}}(t), t + \tau) dt$$

has a simple zero, then for $\varepsilon > 0$ sufficiently small, there exists a transverse homoclinic orbit of (2.1). In other words, the homoclinic orbit $x = q^\alpha(t - \tau_0)$ and $\theta = t \mod T$ persists under the perturbation $\varepsilon g(x, \theta)$ if the homoclinic Melnikov function $M(\tau)$ has a simple zero at $\tau = \tau_0 \in \mathbb{S}^1$. By the Smale-Birkhoff theorem [6, 13], the existence of transverse homoclinic orbits to hyperbolic periodic orbits implies that chaotic behavior occurs.

2.2 First integrals and commutative vector fields

Integrability for autonomous systems due to Bogoyavlenskij [4] means that the systems have an adequate amounts of first integrals and commutative vector fields.

Definition 2.1 (Bogoyavlenskij)**.** *Let $\mathscr{M}$ be a n-dimensional smooth manifold and X be a smooth vector field on $\mathscr{M}$. Consider an autonomous n-dimensional system*

$$\dot{x} = X(x), \quad x \in \mathscr{M}. \tag{2.4}$$

Equation (2.4) *is called* integrable in the meaning of Bogoyavlenskij *if there exist q vector fields $X_1(:= X), X_2, \ldots, X_q$ and $n - q$ scalar-valued functions $F_1, \ldots, F_{n-q}$ such that*

(i) *$X_1, \ldots, X_q$ are linearly independent almost everywhere and commute with each other, i.e., $[X_j, X_k] := 0$ for $j, k = 1, \ldots, q$ where $[\cdot, \cdot]$ is the Lie bracket;*

(ii) *$dF_1, \ldots, dF_{n-q}$ are linearly independent almost everywhere and $F_1, \ldots, F_{n-q}$ are first integrals of $X_1, \ldots, X_q$, i.e., $dF_k(X_j) = 0$ for $j = 1, \ldots, q$ and $k = 1, \ldots, n - q$.*

If $X_1, X_2, \ldots, X_q$ and $F_1, \ldots, F_{n-q}$ are analytic, then Eq. (2.4) *is said to be* analytically integrable.

Definition 2.1 is regarded as a generalization of complete integrability for Hamiltonian systems. The statement similar to that of the Liouville-Arnold theorem [1] also holds for integrable systems in the meaning of Bogoyavlenskij: if Eq. (2.4) is integrable and a connected component of the level set $F^{-1}(c)$ with $F := (F_1, \ldots, F_{n-q})$ is regular and compact for $c \in \mathbb{R}^{n-q}$, then it can be transformed to a linear flow on the q-dimensional torus $\mathbb{T}^q$ [4].

It is well known that Poincaré proved nonintegrability of the restricted three body problem [12]. In his work, he proved nonintegrability of analytic nearly integrable Hamiltonian systems under some assumptions. Here we only emphasize that his result means that first integrals and commutative (Hamiltonian) vector fields do not persist generally.

2.3 Relationships between the Melnikov method and integrability

In addition to facts in Subsection 2.1 and 2.2, there are some relationships between integrability and the Melnikov function for perturbed systems.

As Moser stated in his monograph [11], the horseshoe map does not possess a real analytic first integral. This means that, for (2.1) with assumption (M'), if the Melnikov function has a simple zero, then (2.1) is real analytically nonintegrable.

In [8], Morales showed that, for (2.1) under some conditions with the assumption (M'), if the differential Galois group for the variational equation around the unperturbed homoclinic orbit is commutative, then the Melnikov function is identically zero. We remark that if an extended system of (2.3) is integrable near homoclinic orbit, then the identity component of the differential Galois group is commutative, by Morales-Ramis theory [9], [10].

Our main results in Section 3 can be regarded as a primitive case of these facts.

3 Main results

Let $\mathscr{M}$ be a n-dimensional, paracompact, oriented, and smooth real manifold. Consider following system in $\mathscr{M}$:

$$\dot{x} = X_\varepsilon(x) \tag{3.1}$$

where X_ε is a smooth vector field such that $X_\varepsilon = X^0 + \varepsilon X^1 + O(\varepsilon^2)$.

3.1 Periodic orbits

We take following assumptions:

(A1) X^0 has a T-periodic orbit $\gamma(t)$ where $T > 0$ is a constant,

(A2) X^0 has a non-constant smooth first integral F.

Let Γ be a trajectory defined by $x = \gamma(t)$ and

$$\mathscr{I}_{F,\gamma} := \int_0^T dF(X^1)(\gamma(t))dt.$$

Theorem 3.1. *Assume (A1) and (A2) for* (3.1)*. If X_ε has a smooth first integral F_ε near Γ depending on ε smoothly such that $F_0 = F$, then $\mathscr{I}_{F,\gamma} = 0$ holds.*

Theorem 3.2. *Assume (A1) and (A2) for* (3.1)*. If X_ε has a T_ε-periodic orbit γ_ε depending on ε smoothly such that $T_0 = T$ and $\gamma_0 = \gamma$, then $\mathscr{I}_{F,\gamma} = 0$ holds.*

In other words, if $\mathscr{I}_{F,\gamma} \neq 0$, then F and γ does not persist in the perturbed system.

Next, we impose an additional assumption:

(A3) X^0 has a smooth commutative vector field Z.

Let

$$\mathscr{J}_{F,Z,\gamma} := \int_0^T dF([X^1, Z])(\gamma(t))dt$$

where $[\cdot,\cdot]$ is Lie bracket.

Theorem 3.3. *Assume (A1), (A2) and (A3) for* (3.1)*. If X_ε has a smooth commutative vector field Z_ε near Γ depending on ε smoothly such that $Z_0 = Z$, then $\mathscr{J}_{F,Z,\gamma} = 0$ holds.*

3.2 Homoclinic orbits of periodic orbits

We give similar results for homoclinic orbits of periodic orbits. Instead of the assumption (A1), we consider following assumption:

(A1') X^0 has a homoclinic orbit $\gamma^{\mathrm{h}}(t)$ of a periodic orbit $\gamma^{\mathrm{p}}(t)$.

In this subsection we always assume (A1') and (A2).

As in Subsection 3.1, we want to define a specific integral but in this case the situation is complicated. When $\gamma^{\mathrm{p}}(t)$ is an equilibrium, we denote $x_0 = \gamma^p$ and define a formal integral

$$\tilde{\mathscr{I}}_{F,\gamma^{\mathrm{h}},x_0} := \int_{-\infty}^{\infty} dF(X^1)(\gamma^{\mathrm{h}}(t))dt. \tag{3.2}$$

When $\gamma^{\mathrm{p}}(t)$ is not an equilibrium, we define $\tilde{\mathscr{I}}_{F,\gamma^{\mathrm{h}},x_0}$ as follows. Let $\Gamma^{\mathrm{p}} = \{\gamma^{\mathrm{p}}(t) : t \in \mathbb{R}\}$ and $\Gamma^{\mathrm{h}} = \{\gamma^{\mathrm{h}}(t) : t \in \mathbb{R}\} \cup \Gamma^{\mathrm{p}}$. Fix $x_0 \in \{\gamma^{\mathrm{p}}(t) : t \in \mathbb{R}\}$ and take $n-1$ dimensional supersurface S such that $x_0 \in S$ and $S \pitchfork \Gamma^{\mathrm{p}}$. Take a sufficiently small neighberhood V_{x_0} of x_0. We set a Poincaré section $\Sigma_{x_0} = S \cap V_{x_0}$. If $|t|$ is sufficiently large, since $\gamma^{\mathrm{h}}(t)$ is a homoclinic orbit of a periodic orbit $\gamma^{\mathrm{p}}(t)$, by its continuity, $\gamma^{\mathrm{h}}(t)$ and Σ_{x_0} cross. We denote the positive (respectively negative) time of the i-th intersection by $T_i^{x_0}$ (respectively $T_{-i}^{x_0}$). In this setting, (A1) means that

$$\lim_{k\to\pm\infty} \gamma^{\mathrm{h}}(T_k^{x_0}) = x_0. \tag{3.3}$$

Then we define a formal integral

$$\tilde{\mathscr{I}}_{F,\gamma^{\mathrm{h}},x_0} := \lim_{k,l\to+\infty} \int_{T_{-k}^{x_0}}^{T_l^{x_0}} dF(X^1)(\gamma^{\mathrm{h}}(t))dt. \tag{3.4}$$

Theorem 3.4. *Assume (A1') and (A2) for* (3.1)*. If X_ε has a smooth first integral F_ε near Γ^{h} depending on ε smoothly such that $F_0 = F$, then $\tilde{\mathscr{I}}_{F,\gamma^{\mathrm{h}},x_0}$ converges to* 0*.*

Theorem 3.5. *Assume (A1') and (A2) for* (3.1)*. Suppose that there exists a periodic orbit $\gamma_\varepsilon^{\mathrm{p}}$ of X_ε depending on ε smoothly such that $\gamma_0^{\mathrm{p}} = \gamma^{\mathrm{p}}$. If X_ε has a homoclinic orbit $\gamma_\varepsilon^{\mathrm{h}}$ to γ^{p} depending on ε smoothly such that $\gamma_0^{\mathrm{h}} = \gamma^{\mathrm{h}}$, then $\tilde{\mathscr{I}}_{F,\gamma^{\mathrm{h}},x_0}$ converges to* 0*.*

Now we impose the additional assumption (A3). When γ^p is an equilibrium, define

$$\tilde{\mathscr{J}}_{F,\gamma^{\mathrm{h}},x_0} := \lim_{k,l\to+\infty} \int_{T_{-k}^{x_0}}^{T_l^{x_0}} dF([X^1, Z])(\gamma^{\mathrm{h}}(t))dt. \tag{3.5}$$

and when γ^p is not an equilibrium, define

$$\tilde{\mathscr{J}}_{F,\gamma^{\mathrm{h}},x_0} := \int_{-\infty}^{\infty} dF([X^1, Z])(\gamma^{\mathrm{h}}(t))dt. \tag{3.6}$$

Theorem 3.6. *Assume (A1'), (A2) and (A3) for* (3.1)*. If X_ε has a smooth commutative vector field Z_ε near Γ^h depending on ε smoothly such that $Z_0 = Z$, then $\tilde{\mathscr{J}}_{F,Z,\gamma}$ converges to* 0*.*

4 Ideas of the proofs

We have Theorem 3.1 and 3.2 by calculating $\int_\gamma dF_\varepsilon$ and $\int_{\gamma_\varepsilon} dF$ respectively. Similarly, we can prove Theorem 3.3 and 3.4.

Here we give the sketch of the proof of Theorem 3.3. At first, we construct a first integral corresponding to given commutative vector field by using the cotangent lift trick [3]. It is well-known that any cotangent bundle has a symplectic form induced by Liouville form [2] and we denote it by Ω_0. Let X be a smooth vector field on $\mathscr{M}$ and h_X be a function on $T^*\mathscr{M}$ defined by

$$h_X(x,p) = \langle p, X(x)\rangle \tag{4.1}$$

for $(x,p) \in T^*\mathscr{M}$ where $\langle , \rangle$ is a natural pairing. Then the *cotangent lift* of X, denoted by $\hat{X}$, is the Hamiltonian vector field of the Hamiltonian h_X with the symplectic form Ω_0. In the local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$, with the frame $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial p_1}, ..., \frac{\partial}{\partial p_n}$, the differential equation given by $\hat{X}$ is expressed as

$$\frac{dx}{dt} = X(x) \left(= \frac{\partial h_X}{\partial p} \right) \tag{4.2}$$

$$\frac{dp}{dt} = -\frac{\partial X(x)}{\partial x}^{\mathrm{T}} p \left(= -\frac{\partial h_X}{\partial x} \right). \tag{4.3}$$

In [3]:proposition 2, they use following fact.

Lemma 4.1. *Assume Z is a commutative vector field of X. Then h_Z is a first integral of the cotangent lift $\hat{X}$ of X.*

Next, we find a periodic orbit in the lifted system. Let X be a smooth vector field on $\mathscr{M}$and Γ be an integral curve given by a non-stationary particular solution $x = \phi(t)$ of X. An immersion $i : \Gamma \to \mathscr{M}$ induces a vector bundle $T_\Gamma := i^*TM$. Then we get a connection of the vector bundle T_Γ:

$$\nabla s := \mathcal{L}_X Y|_\Gamma \tag{4.4}$$

where Y is any smooth vector field extension of the section s of the bundle T_Γ. Then $\nabla s = 0$ is said to be the variational equation of X along Γ [2,5]. Moreover, for the dual connection ∇^*, $\nabla^*\alpha = 0$ is said to be the *adjoint variational equation* of X along Γ. Locally, $\nabla^*\alpha = 0$ can be written as

$$\frac{d\eta}{dt} = -\left(\frac{\partial X}{\partial \phi}(\phi(t)) \right)^{\mathrm{T}} \eta. \tag{4.5}$$

Lemma 4.2. *Let Γ be an integral curve of the vector field X and $\nabla^*\alpha = 0$ be its adjoint variational equation. If X has a first integral F, then $\alpha = dF|_\Gamma$ is a horizontal section of ∇^*.*

We remark that (4.3) is the same as (4.5) when $x = \phi(t)$. So under the assumptions of Theorem 3.3, the lifted system $\hat{X}_0$ of X_0 has a periodic orbit $(\gamma(t), dF(\gamma(t)))$ by Lemma 4.2 and a first integral h_Z by Lemma 4.1. If the lifted system $\hat{X}_\varepsilon$ has a commutative vector field Z_ε such that $Z_\varepsilon = Z + O(\varepsilon)$, $\hat{X}_\varepsilon$ has a first integral h_{Z_ε} such that $h_{Z_\varepsilon} = h_Z + O(\varepsilon)$ and we can apply Theorem 3.1 to the lifted system.

5 Some relation with Melnikov methods

Finally, we remark relationships between our main results and Melnikov method.

We return to (2.3) and make the assumption (M). By the setting and the assumption (M), we have a first integral H and a orbit $(q^\alpha(t), t)$ of the unperturbed system of (2.3). Moreover, when the resonance condition $lT^\alpha = mT$ ($l, m \in \mathbb{N}$ are relatevely prime) holds, $\hat{\gamma}_\tau^{m/l}(t) := (q^\alpha(t), t+\tau)$ is mT-periodic orbit of unperturbed system of (2.3) for all $\tau \in [0, T]$. So we can apply Theorem 3.1 to the system (2.3) with a first integral $H(x)$ and mT-periodic orbit $\hat{\gamma}_\tau^{m/l}(t)$. Then the integral in Theorem 3.1 is

$$\mathscr{I}_{H,\hat{\gamma}_\tau^{m/l}} = \int_0^{mT} DH(q^\alpha(t)) \cdot g(q^\alpha(t), t+\tau)dt$$

and this coinsides with the subharmonic Melnikov function $M^{m/l}(\tau)$. So we get following Theorem.

Theorem 5.1. *Under the resonance condition $lT^\alpha = mT$ ($l, m \in \mathbb{N}$ are relatevely prime) and the assumption (M), if* (2.3) *has a smooth first integral F_ε depending on ε smoothly such that $F_0 = F$, then $M^{m/l}(\tau)$ must be identically zero.*

As in the subharmonic case, we get similar statement for the cese of homoclinic orbits.

Theorem 5.2. *Under the assumption (M'), if* (2.3) *has a smooth first integral F_ε depending on ε smoothly such that $F_0 = F$, then $M(\tau)$ must be identically zero.*

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