Weighted Bergman inner products on subspaces of bounded symmetric domains

Ryosuke Nakahama
Graduate School of Mathematical Sciences, The University of Tokyo

Abstract

Let $H_r(\mathbb{F})$ ($\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$) be the space of Hermitian matrices, and we consider the bounded symmetric domain $D\subset H_r(\mathbb{F})^{\mathbb{C}}$. In this article we present a result on the computation of the weighted Bergman inner product on D of a polynomial on the subspace $H_{r'}(\mathbb{F})\oplus H_{r''}(\mathbb{F})$ and an exponential function on $H_r(\mathbb{F})$. Also, as an application, we present a result on explicit construction of intertwining operators from representations of $Sp(r,\mathbb{R})$ to those of the subgroup $U(r',r'')$.

1 Introduction

Throughout the paper let $\mathbb{F}:=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$, and write

$$H_r=H_r(\mathbb{F}):=\{X\in M_r(\mathbb{F})\mid X=X^*\},$$
$$H_r^+=H_r^+(\mathbb{F}):=\{X\in H_r(\mathbb{F})\mid \text{positive definite}\},$$
$$H_r^{\mathbb{C}}=H_r^{\mathbb{C}}(\mathbb{F}):=H_r(\mathbb{F})\otimes_{\mathbb{R}}\mathbb{C},\qquad M_{p,q}^{\mathbb{C}}=M_{p,q}^{\mathbb{C}}(\mathbb{F}):=M_{p,q}(\mathbb{F})\otimes_{\mathbb{R}}\mathbb{C}.$$

For $X\in H_r^{\mathbb{C}}(\mathbb{F})$, let $\overline{X}$ be the $\mathbb{C}$-conjugate of X with respect to the real form $H_r(\mathbb{F})$, and for $X\in M_{p,q}(\mathbb{F})$, let X^* be the $\mathbb{F}$-conjugate transpose of X. We extend $X\mapsto X^*$ $\mathbb{C}$-linearly to the map $M_{p,q}^{\mathbb{C}}(\mathbb{F})\to M_{q,p}^{\mathbb{C}}(\mathbb{F})$. Also we write $d=\dim_{\mathbb{R}}\mathbb{F}$ so that $d\in\{1,2,4\}$, and write $n=\dim_{\mathbb{R}}H_r(\mathbb{F})=r+\frac{d}{2}r(r-1)$.

Next let $D\subset H_r^{\mathbb{C}}$ be the *bounded symmetric domain* given by

$$D:=\{X\in H_r^{\mathbb{C}}\mid I-X\overline{X}\text{ is positive definite}\}.$$

Then for $f,g\in\mathcal{O}(D)$, the *weighted Bergman inner product* is given by

$$\langle f,g\rangle_\lambda:=C_\lambda\int_D f(X)\overline{g(X)}\det(I-X\overline{X})^{\lambda-\frac{2n}{r}}\,dX,$$

where we set

$$C_\lambda:=\left(\int_D\det(I-X\overline{X})^{\lambda-\frac{2n}{r}}\,dX\right)^{-1}=\frac{1}{\pi^n}\prod_{j=1}^r\frac{\Gamma\left(\lambda-\frac{d}{2}(j-1)\right)}{\Gamma\left(\lambda-\frac{n}{r}-\frac{d}{2}(j-1)\right)}.$$

Here we omit the definition of $\det(I-X\overline{X})$ for $\mathbb{F}=\mathbb{H}$ case. Then this converges for all polynomials if $\lambda>\frac{2n}{r}-1$. Let $\mathcal{H}_\lambda(D)\subset\mathcal{O}(D)$ be the corresponding Hilbert space. Then the universal covering group $\tilde{G}$ of $G = \begin{cases} Sp(r,\mathbb{R}) & (\mathbb{F}=\mathbb{R}) \\ U(r,r) & (\mathbb{F}=\mathbb{C}) \\ SO^*(4r) & (\mathbb{F}=\mathbb{H}) \end{cases}$ acts unitarily on $\mathcal{H}_\lambda(D)$. For example, when $\mathbb{F}=\mathbb{C}$, $G=U(r,r)$, the action is given by, for $\lambda_1+\lambda_2=\lambda$,

$$\tau_{\lambda_1,\lambda_2}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}\right) f(X) := \det(A^*+XB^*)^{-\lambda_1}\det(CX+D)^{-\lambda_2} f\left((AX+B)(CX+D)^{-1}\right).$$

We note that $\det(A^*+XB^*)^{-\lambda_1}\det(CX+D)^{-\lambda_2}$ is not well-defined on $G\times D$ if $\lambda_1,\lambda_2\notin\mathbb{Z}$, but is well-defined on the universal covering space $\tilde{G}\times D$.

Next we consider the decomposition of $\mathcal{P}(H_r^{\mathbb{C}}) := \{\text{holomorphic polynomials on } H_r^{\mathbb{C}}\}$. First we recall the determinant $\det(X)$ on $H_r^{\mathbb{C}}(\mathbb{F})$. This is a polynomial of degree r satisfying

$$\det(gXg^*) = \det(gg^*)\det(X) \qquad (g\in GL(r,\mathbb{F})^{\mathbb{C}},\ X\in H_r^{\mathbb{C}}(\mathbb{F})).$$

When $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, this is the usual determinant, and even for $\mathbb{F}=\mathbb{H}$ case such polynomial also exists. Then for $\mathbf{m}=(m_1,\ldots,m_r)\in\mathbb{Z}^r$ with $m_1\ge\cdots\ge m_r\ge 0$, we define $\Delta_{\mathbf{m}}(X)\in\mathcal{P}(H_r^{\mathbb{C}})$ by

$$\Delta_{\mathbf{m}}(X) := \det(X)^{m_r}\prod_{k=1}^{r-1}\det\left((x_{ij})_{1\le i,j\le k}\right)^{m_k-m_{k+1}},$$

and let

$$\mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}) := \operatorname{span}\left\{\Delta_{\mathbf{m}}(gXg^*)\ \middle|\ g\in GL(r,\mathbb{F})^{\mathbb{C}}\right\}\subset\mathcal{P}(H_r^{\mathbb{C}}).$$

Then we have the following.

Theorem 1.1 (Hua-Kostant-Schmid, [3, Theorem XI.2.4])**.**

$$\mathcal{P}(H_r^{\mathbb{C}}) = \bigoplus_{m_1\ge\cdots\ge m_r\ge 0}\mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}).$$

We define another inner product (the *Fischer inner product*) on $\mathcal{P}(H_r^{\mathbb{C}})$ by

$$\langle f,g\rangle_F := \overline{g}\left(\frac{\partial}{\partial X}\right) f(X)\bigg|_{X=0} = \frac{1}{\pi^n}\int_{H_r^{\mathbb{C}}} f(X)\overline{g(X)}e^{-\operatorname{tr}(X\overline{X})}dX$$

(see [3, Section XI.1]), and for $\lambda\in\mathbb{C}$, $\mathbf{m}=(m_1,\ldots,m_r)\in(\mathbb{Z}_{\ge 0})^r$, we write

$$(\lambda)_{\mathbf{m}} = \prod_{j=1}^{r}\left(\lambda-\frac{d}{2}(j-1)\right)_{m_j}, \qquad (\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1).$$

Then we have the following.

Theorem 1.2 (Faraut-Korányi (1990), [3, Corollary XIII.2.3])**.** *For* $f=\sum_{\mathbf{m}} f_{\mathbf{m}}$, $g=\sum_{\mathbf{m}} g_{\mathbf{m}}\in\bigoplus_{\mathbf{m}}\mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}})=\mathcal{P}(H_r^{\mathbb{C}})$ *we have*

$$\langle f,g\rangle_\lambda = \sum_{m_1\ge\cdots\ge m_r\ge 0}\frac{1}{(\lambda)_{\mathbf{m}}}\langle f_{\mathbf{m}},g_{\mathbf{m}}\rangle_F.$$

Next we recall the reproducing kernel of $\langle\cdot,\cdot\rangle_F$. It is given by $e^{\mathrm{tr}(X\overline{A})}$, that is,

$$\left\langle f(X), e^{\mathrm{tr}(X\overline{A})}\right\rangle_F = f(A) \qquad (f\in\mathcal{P}(H_r^{\mathbb{C}}))$$

holds. Therefore, for $f,g\in\mathcal{P}(H_r^{\mathbb{C}})$ we have

$$\langle f,g\rangle_\lambda = \left\langle f(X), \left\langle g(A), e^{\mathrm{tr}(A\overline{X})}\right\rangle_{F,A}\right\rangle_{\lambda,X} = \left\langle \left\langle f(X), e^{\mathrm{tr}(X\overline{A})}\right\rangle_{\lambda,X}, g(A)\right\rangle_{F,A}.$$

That is, $f\mapsto \left\langle f, e^{\mathrm{tr}(\cdot\overline{A})}\right\rangle_\lambda$ does not lose any information. Moreover, by the result of Faraut-Korányi, we get the following.

Corollary 1.3. *For* $f=\sum_{\mathbf{m}} f_{\mathbf{m}}\in\bigoplus_{\mathbf{m}}\mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}})=\mathcal{P}(H_r^{\mathbb{C}})$ *we have*

$$\left\langle f, e^{\mathrm{tr}(\cdot\overline{A})}\right\rangle_\lambda = \sum_{m_1\ge\cdots\ge m_r\ge 0}\frac{1}{(\lambda)_{\mathbf{m}}} f_{\mathbf{m}}(A).$$

Now let $r=r'+r''$, and define $\mathrm{Proj}_1: H_r^{\mathbb{C}}\to H_{r'}^{\mathbb{C}}$, $\mathrm{Proj}_2: H_r^{\mathbb{C}}\to H_{r''}^{\mathbb{C}}$ by

$$\mathrm{Proj}_1\begin{pmatrix} x & y\\ y^* & z\end{pmatrix} := x, \qquad \mathrm{Proj}_2\begin{pmatrix} x & y\\ y^* & z\end{pmatrix} := z.$$

Then again by the Hua-Kostant-Schmid theorem we have

$$\mathcal{P}(H_{r'}^{\mathbb{C}}) = \bigoplus_{k_1\ge\cdots\ge k_{r'}\ge 0}\mathcal{P}_{\mathbf{k}}(H_{r'}^{\mathbb{C}}), \qquad \mathcal{P}(H_{r''}^{\mathbb{C}}) = \bigoplus_{l_1\ge\cdots\ge l_{r''}\ge 0}\mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}}).$$

The aim of this paper is to compute

$$\left\langle f_1(\mathrm{Proj}_1(\cdot)) f_2(\mathrm{Proj}_2(\cdot)), e^{\mathrm{tr}(\cdot\overline{A})}\right\rangle_\lambda \qquad (f_1\in\mathcal{P}_{\mathbf{k}}(H_{r'}^{\mathbb{C}}),\ f_2\in\mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})).$$

This is computable if $\mathbf{k}=(k,\dots,k)$ so that $\mathcal{P}_{(k,\dots,k)}(H_{r'}^{\mathbb{C}})=\mathbb{C}\det(x)^k$. In the rest of this paper we assume this.

2 Main result and applications

We fix some notations. For $X,A\in H_r^{\mathbb{C}}$, we write $X=\begin{pmatrix} x & y\\ y^* & z\end{pmatrix}$, $A=\begin{pmatrix} a & b\\ b^* & c\end{pmatrix}$ with $x,a\in H_{r'}^{\mathbb{C}}$, $y,b\in M_{r',r''}^{\mathbb{C}}$, $z,c\in H_{r''}^{\mathbb{C}}$. Next, for $k\in\mathbb{Z}_{\ge 0}$ we write $\underline{k}_r := (\underbrace{k,\dots,k}_{r})$. Also, we identify $\mathbf{l}=(l_1,\dots,l_{r''})\in\mathbb{Z}^{r''}$ and $(l_1,\dots,l_{r''},0,\dots,0)\in\mathbb{Z}^r$, so that

$$(\lambda)_{\underline{k}_r+\mathbf{l}} = (\lambda)_{(k+l_1,\dots,k+l_{r''},k,\dots,k)} = \prod_{j=1}^{r''}\left(\lambda-\frac{d}{2}(j-1)\right)_{k+l_j}\prod_{j=r''+1}^{r}\left(\lambda-\frac{d}{2}(j-1)\right)_k$$

holds. Next, for $\mathbf{l}\in\mathbb{Z}^{r''}$ with $l_1\ge\ldots\ge l_{r''}\ge 0$ and for $f(z)\in\mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$, we define $\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c)\in\mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}})\otimes\mathcal{P}_{\mathbf{n}}(H_{r''}^{\mathbb{C}})$ by

$$f(z+c) = \sum_{m_1\ge\cdots\ge m_{r''}\ge 0}\ \sum_{n_1\ge\cdots\ge n_{r''}\ge 0}\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c).$$

This is a finite sum since $\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c)=0$ if $|\mathbf{m}|+|\mathbf{n}|\ne|\mathbf{l}|$. Then the main result is as follows.

Theorem 2.1. *Let* $\operatorname{Re}\lambda > \frac{2n}{r}-1$, $k \in \mathbb{Z}_{\geq 0}$, $\mathbf{l} \in \mathbb{Z}^{r''}$, $l_1 \geq \cdots \geq l_{r''} \geq 0$. *Then for* $f \in \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$, *we have*

$$\begin{aligned}
&\left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\overline{A})} \right\rangle_{\lambda,X} \\
&= \frac{\det(a)^k}{(\lambda)_{\underline{k}_r+\mathbf{l}}} \det(c-b^*a^{-1}b)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \det(c-b^*a^{-1}b)^{\lambda+k-\frac{n}{r}} f(c) && \text{(A)} \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}} \det(a)^k \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b-c, c\right) && \text{(B)} \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}} \det(a)^k \sum_{\mathbf{n}} \frac{\left(\lambda+k-\frac{d}{2}r'\right)_{\mathbf{n}}}{\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b, c-b^*a^{-1}b\right) && \text{(C)} \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{l}}} \det(a)^k \sum_{\mathbf{m},\mathbf{n}} (-k)_{\mathbf{m}} \left(\lambda+k-\frac{d}{2}r'\right)_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b, c\right). && \text{(D)}
\end{aligned}$$

The proof is given in Sections 4 and 5. Especially, by (D) we have

$$\begin{aligned}
&\left\| \det(x)^k f(z) \right\|_{\lambda,X}^2 = \left\langle \det(x)^k f(z), \det(x)^k f(z) \right\rangle_{\lambda,X} \\
&= \left\langle \left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\overline{A})} \right\rangle_{\lambda,X}, \det(a)^k f(c) \right\rangle_{F,A} \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{l}}} \sum_{\mathbf{m},\mathbf{n}} (-k)_{\mathbf{m}} \left(\lambda+k-\frac{d}{2}r'\right)_{\mathbf{n}} \left\langle \det(a)^k \tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b, c\right), \det(a)^k f(c) \right\rangle_{F,A} \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{l}}} (-k)_{\underline{0}_{r''}} \left(\lambda+k-\frac{d}{2}r'\right)_{\mathbf{l}} \left\langle \det(a)^k f(c), \det(a)^k f(c) \right\rangle_{F,A} \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}+\mathbf{l}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{l}}} \left\| \det(a)^k f(c) \right\|_{F,A}^2.
\end{aligned}$$

Corollary 2.2. *Let* $\operatorname{Re}\lambda > \frac{2n}{r}-1$, $k \in \mathbb{Z}_{\geq 0}$, $\mathbf{l} \in \mathbb{Z}^{r''}$, $l_1 \geq \cdots \geq l_{r''} \geq 0$. *Then for* $f \in \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$, *we have*

$$\left\| \det(x)^k f(z) \right\|_{\lambda,X}^2 = \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}+\mathbf{l}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{l}}} \left\| \det(a)^k f(c) \right\|_{F,A}^2.$$

Now we assume $\mathbf{l} = (l, \ldots, l)$, $f(z) = \det(z)^l$. We define $\Psi^{\mathbf{m}}(z) \in \mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}})$ by

$$\sum_{m_1 \geq \cdots \geq m_{r''} \geq 0} \Psi^{\mathbf{m}}(z) = e^{\operatorname{tr}(z)}$$

When $r''=1$ we have $\Psi^m(z) = \frac{1}{m!}z^m$. Also, for $\mathbf{m} = (m_1, \ldots, m_{r''})$, we write $\mathbf{m}^\vee :=$

$(m_{r''},\dots,m_1)$. Then by [3, Proposition XII.1.3] we have

$$\begin{aligned}
f(z+c) &= \det(z+c)^l = \det(c)^l \det\left(I+\sqrt{c}^{-1}z\sqrt{c}^{-1}\right)^l \\
&= \det(c)^l \sum_{m_1\ge\cdots\ge m_{r''}\ge 0} (-1)^{|\mathbf{m}|}(-l)_{\mathbf{m}}\Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}z\sqrt{c}^{-1}\right) \\
&\in \bigoplus_{m_1\ge\cdots\ge m_{r''}\ge 0} \mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}})_z \otimes \mathcal{P}_{\underline{l}_{r''}-\mathbf{m}^\vee}(H_{r''}^{\mathbb{C}})_w,
\end{aligned}$$

and therefore $\tilde{f}_{\mathbf{m},\,\underline{l}_{r''}-\mathbf{m}^\vee}(z,c) = (-1)^{|\mathbf{m}|}(-l)_{\mathbf{m}}\det(c)^l\Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}z\sqrt{c}^{-1}\right)$ holds. When $r''=1$ this coincides with the usual binomial expansion

$$(z+c)^l = c^l\sum_{m=0}^{l}\frac{(-1)^m(-l)_m}{m!}\left(\frac{z}{c}\right)^m = \sum_{m=0}^{l}\frac{(-1)^m(-l)_m}{m!}z^m c^{l-m}.$$

Therefore by (D) we have

$$\begin{aligned}
&\left\langle \det(x)^k\det(z)^l, e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X} \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{l}}}\det(a)^k\sum_{\mathbf{m},\mathbf{n}}(-k)_{\mathbf{m}}\left(\lambda+k-\frac{d}{2}r'\right)_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b,c\right) \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\underline{l}_{r''}}\left(\lambda-\frac{d}{2}r'\right)_{\underline{l}_{r''}}}\det(a)^k\sum_{\mathbf{m}}(-k)_{\mathbf{m}}\left(\lambda+k-\frac{d}{2}r'\right)_{\underline{l}_{r''}-\mathbf{m}^\vee} \\
&\qquad\times(-1)^{|\mathbf{m}|}(-l)_{\mathbf{m}}\det(c)^l\Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}\left(\lambda+k-\frac{d}{2}r'\right)_{\underline{l}_{r''}}}{(\lambda)_{\underline{k}_r+\underline{l}_{r''}}\left(\lambda-\frac{d}{2}r'\right)_{\underline{l}_{r''}}}\det(a)^k\det(c)^l\sum_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}(-l)_{\mathbf{m}}}{\left(-\lambda-k-l+\frac{d}{2}r'+\frac{n''}{r''}\right)_{\mathbf{m}}} \\
&\qquad\times\Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\
&= \frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k+l}_{r''}}\det(a)^k\det(c)^l}{(\lambda)_{\underline{k}_r+\underline{l}_{r''}}\left(\lambda-\frac{d}{2}r'\right)_{\underline{l}_{r''}}}\sum_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}(-l)_{\mathbf{m}}}{\left(-\lambda-k-l+\frac{n}{r}\right)_{\mathbf{m}}}\Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\
&= \frac{\left(\lambda+l-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\underline{l}_{r''}}}\det(a)^k\det(c)^l\sum_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}(-l)_{\mathbf{m}}}{\left(-\lambda-k-l+\frac{n}{r}\right)_{\mathbf{m}}}\Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\
&= \frac{\left(\lambda+l-\frac{d}{2}r'\right)_{\underline{k}_{r''}}\left(\lambda+k-\frac{d}{2}r''\right)_{\underline{l}_{r'}}}{(\lambda)_{\underline{k+l}_r}}\det(a)^k\det(c)^l \\
&\qquad\times\sum_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}(-l)_{\mathbf{m}}}{\left(-\lambda-k-l+\frac{n}{r}\right)_{\mathbf{m}}}\Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right).
\end{aligned}$$

Now we write

$$\sum_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}}\Psi^{\mathbf{m}}(z) =: {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};z\right).$$

When $r''=1$, this is the usual hypergeometric function $\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!}z^m$. Then the above result is summarized as follows.

Corollary 2.3. *Let* $\operatorname{Re}\lambda > \frac{2n}{r}-1$, $k,l\in\mathbb{Z}_{\ge 0}$. *Then we have*

$$\left\langle \det(x)^k\det(z)^l, e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X} = \frac{\left(\lambda+l-\frac{d}{2}r'\right)_{\underline{k}_{r''}}\left(\lambda+k-\frac{d}{2}r''\right)_{\underline{l}_{r'}}}{(\lambda)_{\underline{k+l}_r}}$$
$$\times \det(a)^k\det(c)^l{}_2F_1\left(\begin{matrix}-k,-l\\ -\lambda-k-l+\frac{n}{r}\end{matrix};\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right).$$

Remark 2.4. *This* ${}_2F_1$ *coincides with a special case of Heckman-Opdam's hypergeometric functions of type* $BC_{r''}$ *(Beerends-Opdam, 1993 [1]).*

This work is motivated by branching laws for the symmetric pairs

$$(G,G_1):=\begin{cases}(Sp(r,\mathbb{R}),U(r',r'')) & (\mathbb{F}=\mathbb{R}),\\ (U(r,r),U(r',r'')\times U(r'',r')) & (\mathbb{F}=\mathbb{C}),\\ (SO^*(4r),U(2r',2r'')) & (\mathbb{F}=\mathbb{H}).\end{cases}$$

We look at the example $\mathbb{F}=\mathbb{C}$. In this case, as representations of $U(r')\times U(r')$ and $U(r'')\times U(r'')$ we have

$$\mathcal{P}(H_{r'}^{\mathbb{C}})=\bigoplus_{k_1\ge\cdots\ge k_{r'}\ge 0}\mathcal{P}_{\mathbf{k}}(H_{r'}^{\mathbb{C}})\simeq\bigoplus_{k_1\ge\cdots\ge k_{r'}\ge 0}V_{\mathbf{k}}^{(r')\vee}\boxtimes V_{\mathbf{k}}^{(r')},$$
$$\mathcal{P}(H_{r''}^{\mathbb{C}})=\bigoplus_{l_1\ge\cdots\ge l_{r''}\ge 0}\mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})\simeq\bigoplus_{l_1\ge\cdots\ge l_{r''}\ge 0}V_{\mathbf{l}}^{(r'')\vee}\boxtimes V_{\mathbf{l}}^{(r'')}.$$

Here we omit the definition of notations. According to this decomposition, it is known that the restriction of the unitary representation $\mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r})$ of $\tilde{U}(r,r)$ to the subgroup $\tilde{U}(r',r'')\times\tilde{U}(r'',r')$ is decomposed as

$$\mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r})|_{U(r',r'')\times U(r'',r')}$$
$$\simeq\sum_{\substack{k_1\ge\cdots\ge k_{r'}\ge 0\\ l_1\ge\cdots\ge l_{r''}\ge 0}}^{\oplus}\mathcal{H}_{\lambda_1+\lambda_2}(D_{r',r''},V_{\mathbf{k}}^{(r')\vee}\boxtimes V_{\mathbf{l}}^{(r'')})\,\hat{\boxtimes}\,\mathcal{H}_{\lambda_1+\lambda_2}(D_{r'',r'},V_{\mathbf{l}}^{(r'')\vee}\boxtimes V_{\mathbf{k}}^{(r')})$$

(see [6, Theorem 8.3]). Now we take following functions

$$f_1(x)\in\left(\mathcal{P}_{\mathbf{k}}(H_{r'}^{\mathbb{C}})\otimes\overline{V_{\mathbf{k}}^{(r')\vee}\boxtimes V_{\mathbf{k}}^{(r')}}\right)^{U(r')\times U(r')},$$
$$f_2(z)\in\left(\mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})\otimes\overline{V_{\mathbf{l}}^{(r'')\vee}\boxtimes V_{\mathbf{l}}^{(r'')}}\right)^{U(r'')\times U(r'')}$$

which are unique up to scalar multiple, and define a vector-valued polynomial $F(A)\in\mathcal{P}(H_r^{\mathbb{C}},V_{\mathbf{k}}^{(r')\vee}\boxtimes V_{\mathbf{k}}^{(r')}\boxtimes V_{\mathbf{l}}^{(r'')\vee}\boxtimes V_{\mathbf{l}}^{(r'')})$ by

$$F(A):=\left\langle e^{\operatorname{tr}(XA)},f_1(x)f_2(z)\right\rangle_{\lambda,X}.$$

Then the following holds.

Theorem 2.5 ([11, Theorem 3.10 (1)]). *The following map intertwines the $\tilde{U}(r', r'') \times \tilde{U}(r'', r')$-action.*

$$\begin{aligned}\mathcal{F} : \mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r}) \to \mathcal{H}_{\lambda_1+\lambda_2}(D_{r',r''}, V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{l}}^{(r'')}) \hat{\boxtimes} \mathcal{H}_{\lambda_1+\lambda_2}(D_{r'',r'}, V_{\mathbf{l}}^{(r'')\vee} \boxtimes V_{\mathbf{k}}^{(r')}),\\ \mathcal{F}\varphi(y_1, y_2) = F\left(\frac{\partial}{\partial X}\right)\varphi(X)\bigg|_{x=z=0} \qquad \left(X = \begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix}\right).\end{aligned}$$

Especially, if $\mathbf{k} = (k, \ldots, k)$, $\mathbf{l} = (l, \ldots, l)$, then up to scalar multiple we have

$$F\begin{pmatrix} a & b_1 \\ b_2 & c \end{pmatrix} = \det(a)^k \det(c)^l {}_2F_1\left(\begin{matrix} -k, -l \\ -\lambda - k - l + \frac{n}{r} \end{matrix}; \sqrt{c}^{-1} b_2 a^{-1} b_1 \sqrt{c}^{-1}\right),$$

and therefore the following holds.

Corollary 2.6. *The following map intertwines the $\tilde{U}(r', r'') \times \tilde{U}(r'', r')$-action.*

$$\begin{aligned}\mathcal{F} : \mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r}) \to \mathcal{H}_{(\lambda_1+k)+(\lambda_2+l)}(D_{r',r''}) \hat{\boxtimes} \mathcal{H}_{(\lambda_1+l)+(\lambda_2+k)}(D_{r'',r'}),\\ \mathcal{F}\varphi(y_1, y_2) = F\left(\frac{\partial}{\partial X}\right)\varphi(X)\bigg|_{x=z=0} \qquad \left(X = \begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix}\right).\end{aligned}$$

Remark 2.7. *There are previous works on differential symmetry breaking operators by, e.g.,*

- *Cohen (1975) [2]:* $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}), SL(2, \mathbb{R}))$,
- *Peng-Zhang (2004) [12]:* $(G \times G, G)$ *(G: Hermitian),*
- *Juhl (2009) [5]:* $(SO_0(1, n), SO_0(1, n-1))$,
- *Ibukiyama-Kuzumaki-Ochiai (2012) [4]:* $(Sp(2n, \mathbb{R}), Sp(n, \mathbb{R}) \times Sp(n, \mathbb{R}))$,
- *Kobayashi-Ørsted-Somberg-Souček (2015) [8]:* $(SO_0(p, q), SO_0(p, q-1))$,
- *Kobayashi-Pevzner (2016) [9, 10]:* $(SO_0(2, n), SO_0(2, n-1))$, $(Sp(n, \mathbb{R}), Sp(n-1, \mathbb{R}) \times Sp(1, \mathbb{R}))$, $(U(n, 1) \times U(n, 1), U(n, 1))$,
- *Kobayashi-Kubo-Pevzner (2016) [7]:* $(O(1, n), O(1, n-1))$ *(vector-valued case).*

The result in this paper deals with $(Sp(r, \mathbb{R}), U(r', r''))$, $(U(r, r), U(r', r'') \times U(r'', r'))$ *and* $(SO^*(4r), U(2r', 2r''))$ *cases.*

We have another motivation. Although the weighted Bergman inner product

$$\langle f, g\rangle_\lambda = C_\lambda \int_D f(X)\overline{g(X)} \det(I - X\overline{X})^{\lambda - \frac{2n}{r}} dX$$

originally converges only when $\operatorname{Re}\lambda > \frac{2n}{r} - 1$, this is meromorphically continued for all $\lambda \in \mathbb{C}$ since

$$\langle f, g\rangle_\lambda = \sum_{m_1 \geq \cdots \geq m_r \geq 0} \frac{1}{\prod_{j=1}^r \left(\lambda - \frac{d}{2}(j-1)\right)_{m_j}} \langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_F$$

holds, and is positive definite if $\lambda > \frac{d}{2}(r-1)$. Moreover, $\mathcal{O}_\lambda(D)_{\tilde{K}} = \mathcal{P}(H_r^{\mathbb{C}})$ is reducible if and only if λ is a pole, and if $\lambda \in \frac{d}{2}(j-1) - \mathbb{Z}_{\ge 0}$ holds, then

$$M_j(\lambda) := \bigoplus_{\substack{m_1 \ge \cdots \ge m_r \ge 0 \\ m_j \le \frac{d}{2}(j-1)-\lambda}} \mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}) \qquad (j = 1, 2, \ldots, r)$$

are non-trivial submodules. Also, when $\lambda = \frac{d}{2}(j-1)$ $(j = 1, 2, \ldots, r)$, this pairing becomes positive definite if it is restricted on the submodule $M_j\left(\frac{d}{2}(j-1)\right)$, and makes it infinitesimally unitary. For example, let $\mathbb{F} = \mathbb{C}$ so that $d = 2$. Then $\mathcal{O}_\lambda(D)_{\tilde{K}}$ is reducible if and only if $\lambda \in \mathbb{Z}_{\le r-1}$ holds, and has a sequence of $(\mathfrak{u}(r,r), \tilde{U}(r) \times \tilde{U}(r))$-submodules

$$\mathcal{O}_\lambda(D)_{\tilde{K}} \supset M_r(\lambda) \supset M_{r-1}(\lambda) \supset \cdots \supset M_{\max\{\lambda,0\}+1}(\lambda) \supset \{0\}.$$

$\mathcal{O}_\lambda(D)_{\tilde{K}}$ for $\lambda > r-1$ and $M_j(j-1)$ $(j = 1, 2, \ldots, r)$ are infinitesimally unitary.

In which submodule a polynomial is contained is determined by the order of poles of $\langle f, g\rangle_\lambda$. That is, for $f \in \mathcal{P}(H_r^{\mathbb{C}})$ and for $\mathbf{p} \in \mathbb{Z}^r$, $p_1 \ge \cdots \ge p_r \ge 0$, if

$$(\lambda)_{\mathbf{p}} \left\langle f, e^{\mathrm{tr}(\cdot \overline{A})} \right\rangle_\lambda = \prod_{j=1}^{r} \left(\lambda - \frac{d}{2}(j-1)\right)_{p_j} \left\langle f, e^{\mathrm{tr}(\cdot \overline{A})} \right\rangle_\lambda$$

is holomorphic for all $\lambda \in \mathbb{C}$, then for $j = 1, \ldots, r$ we can show that $f \in M_j(\lambda)$ holds for $\lambda \in \frac{d}{2}(j-1) - p_j - \mathbb{Z}_{\ge 0}$.

Now, for $k \in \mathbb{Z}_{\ge 0}$, $\mathbf{l} \in \mathbb{Z}^{r''}$, $l_1 \ge \cdots \ge l_{r''} \ge 0$, we define $p(k, \mathbf{l}) \in \mathbb{Z}^r$ by

$$p(k,\mathbf{l}) := \begin{cases} \Bigl(k+l_1, \ldots, k+l_{r''}, \overbrace{k, \ldots, k}^{r'-r''}, \min\{k, l_1\}, \ldots, \min\{k, l_{r''}\}\Bigr) & (r' \ge r''), \\ \Bigl(k+l_1, \ldots, k+l_{r'}, \min\{k+l_{r'+1}, l_1\}, \ldots, \min\{k+l_{r''}, l_{r''-r'}\}, \\ \qquad \min\{k, l_{r''-r'+1}\}, \ldots, \min\{k, l_{r''}\}\Bigr) & (r' < r''). \end{cases}$$

Then the following holds.

Corollary 2.8. *For $f \in \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$,*

$$(\lambda)_{p(k,\mathbf{l})} \left\langle \det(x)^k f(z), e^{\mathrm{tr}(X\overline{A})} \right\rangle_{\lambda, X}$$

is entirely holomorphic with respect to $\lambda \in \mathbb{C}$.

From this we can determine the submodules containing $\mathcal{P}_{\underline{k}_{r'}}(H_{r'}^{\mathbb{C}}) \otimes \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$.

3 Proof of the corollary on poles

In this section we prove Corollary 2.8. First we prove this for $r' \ge r''$ case. Then by (B) of Theorem 2.1 we have

$$\begin{aligned} \left\langle \det(x)^k f(z), e^{\mathrm{tr}(X\overline{A})} \right\rangle_{\lambda, X} &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r + 1}} \det(a)^k \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}} \left(b^* a^{-1} b - c, c\right) \\ &= \frac{1}{(\lambda)_{\underline{k}_{r'} + 1}} \det(a)^k \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}} \left(b^* a^{-1} b - c, c\right). \end{aligned}$$

Now we recall that for $f(z)\in\mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$ we write

$$f(z+c)=\sum_{\mathbf{m},\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c)\in\bigoplus_{\mathbf{m},\mathbf{n}}\mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}})\otimes\mathcal{P}_{\mathbf{n}}(H_{r''}^{\mathbb{C}}).$$

Then we can show that $\sum_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c)\neq 0$ implies $m_j\le l_j$, and $(-k)_{\mathbf{m}}\neq 0$ implies $m_j\le k$. Now for $\lambda\in\mathbb{C}$, $\mathbf{s}\in\mathbb{C}^r$, $\mathbf{m}\in(\mathbb{Z}_{\ge 0})^r$ we write

$$(\lambda+\mathbf{s})_{\mathbf{m}}:=\prod_{j=1}^{r}\left(\lambda-\frac{d}{2}(j-1)+s_j\right)_{m_j}.$$

Then we have

$$\begin{aligned}
&\left\langle\det(x)^k f(z),e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X}\\
&=\frac{1}{(\lambda)_{\underline{k}_{r'}+\mathbf{l}}}\det(a)^k\sum_{\mathbf{m}\le\min\{\underline{k}_{r''},\mathbf{l}\}}\frac{(-k)_{\mathbf{m}}}{\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{m}}}\sum_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b-c,c\right)\\
&=\frac{1}{(\lambda)_{\underline{k}_{r'}+\mathbf{l}}\left(\lambda-\frac{d}{2}r'\right)_{\min\{\underline{k}_{r''},\mathbf{l}\}}}\det(a)^k\\
&\qquad\times\sum_{\mathbf{m}\le\min\{\underline{k}_{r''},\mathbf{l}\}}(-k)_{\mathbf{m}}\left(\lambda-\frac{d}{2}r'+\mathbf{m}\right)_{\min\{\underline{k}_{r''},\mathbf{l}\}-\mathbf{m}}\sum_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b-c,c\right)\\
&=\frac{\det(a)^k}{(\lambda)_{p(k,\mathbf{l})}}\sum_{\mathbf{m}\le\min\{\underline{k}_{r''},\mathbf{l}\}}(-k)_{\mathbf{m}}\left(\lambda-\frac{d}{2}r'+\mathbf{m}\right)_{\min\{\underline{k}_{r''},\mathbf{l}\}-\mathbf{m}}\sum_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b-c,c\right),
\end{aligned}$$

and therefore $(\lambda)_{p(k,\mathbf{l})}\left\langle\det(x)^k f(z),e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X}$ is holomorphic.

Next we prove the corollary for $r'<r''$ case. In this case, since $\sum_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}(b^*a^{-1}b-c,c)$ in Theorem 2.1 (B) are not linearly independent, we use (C), so that

$$\begin{aligned}
&\left\langle\det(x)^k f(z),e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X}\\
&=\frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+\mathbf{l}}}\det(a)^k\sum_{\mathbf{n}}\frac{\left(\lambda+k-\frac{d}{2}r'\right)_{\mathbf{n}}}{\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{n}}}\sum_{\mathbf{m}}\tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b,c-b^*a^{-1}b\right)\\
&=\frac{\det(a)^k}{(\lambda)_{\underline{k}_r+\mathbf{l}}}\sum_{\mathbf{n}}\frac{\left(\lambda-\frac{d}{2}r'\right)_{\underline{k}_{r''}+\mathbf{n}}}{\left(\lambda-\frac{d}{2}r'\right)_{\mathbf{n}}}\sum_{\mathbf{m}}\tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b,c-b^*a^{-1}b\right)\\
&=\frac{\det(a)^k}{(\lambda)_{\underline{k}_r+\mathbf{l}}}\sum_{\mathbf{n}}\left(\lambda-\frac{d}{2}r'+\mathbf{n}\right)_{\underline{k}_{r''}}\sum_{\mathbf{m}}\tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b,c-b^*a^{-1}b\right).
\end{aligned}$$

Now for $z,c\in H_{r''}^{\mathbb{C}}$, if $\operatorname{rank}z\le r'$ then we can show that $\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c)\neq 0$ holds only if $m_{r'+1}=m_{r''+2}=\cdots=m_r=0$, and moreover $\sum_{\mathbf{m}}\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c)\neq 0$ implies

$$\begin{cases}l_{r'+j}\le n_j\le l_j & (1\le j\le r''-r'),\\ 0\le n_j\le l_j & (r''-r'<j\le r'').\end{cases}$$

Now we write $\mathbf{l}' := (l_1, \dots, l_{r'})$, $\mathbf{l}'' := (l_{r'+1}, \dots, l_{r''}, 0, \dots, 0)$. Then we can show that if $\mathbf{l}'' \le \mathbf{n} \le \mathbf{l}$ then

$$\left(\lambda - \frac{d}{2}r' + \mathbf{n}\right)_{\underline{k}_{r''}} = \prod_{j=1}^{r''} \left(\lambda - \frac{d}{2}(r'+j-1) + n_j\right)_k$$

is divisible by

$$\prod_{\substack{1\le j\le r'' \\ k+l_{r'+j}>l_j}} \left(\lambda - \frac{d}{2}(r'+j-1) + l_j\right)_{k+l_{r'+j}-l_j} = \left(\lambda - \frac{d}{2}r' + \mathbf{l}\right)_{\max\{\underline{k}_{r''}+\mathbf{l}''-\mathbf{l},\underline{0}_{r''}\}}.$$

Therefore we have

$$\begin{aligned}
&\left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X} \\
&= \frac{\det(a)^k}{(\lambda)_{\underline{k}_r+\mathbf{l}}} \sum_{\mathbf{l}''\le\mathbf{n}\le\mathbf{l}} \left(\lambda - \frac{d}{2}r' + \mathbf{n}\right)_{\underline{k}_{r''}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b, c - b^*a^{-1}b\right) \\
&= \frac{\left(\lambda - \frac{d}{2}r' + \mathbf{l}\right)_{\max\{\underline{k}_{r''}+\mathbf{l}''-\mathbf{l},\underline{0}_{r''}\}}}{(\lambda)_{\underline{k}_{r'}+\mathbf{l}'}\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}+\mathbf{l}''}} \det(a)^k \\
&\quad \times \sum_{\mathbf{l}''\le\mathbf{n}\le\mathbf{l}} \underbrace{\frac{\left(\lambda - \frac{d}{2}r' + \mathbf{n}\right)_{\underline{k}_{r''}}}{\left(\lambda - \frac{d}{2}r' + \mathbf{l}\right)_{\max\{\underline{k}_{r''}+\mathbf{l}''-\mathbf{l},\underline{0}_{r''}\}}}}_{\in\mathbb{C}[\lambda]} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m},\mathbf{n}}\left(b^*a^{-1}b, c - b^*a^{-1}b\right) \\
&\in \frac{1}{(\lambda)_{\underline{k}_{r'}+\mathbf{l}'}\left(\lambda - \frac{d}{2}r'\right)_{\min\{\underline{k}_{r''}+\mathbf{l}'',\mathbf{l}\}}} \mathbb{C}[\lambda]\otimes\mathcal{P}(H_r^{\mathbb{C}}) = \frac{1}{(\lambda)_{p(k,\mathbf{l})}}\mathbb{C}[\lambda]\otimes\mathcal{P}(H_r^{\mathbb{C}}),
\end{aligned}$$

and hence $(\lambda)_{p(k,\mathbf{l})}\left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X}$ is holomorphic. This completes the proof of Corollary 2.8.

4 Proof of the 1st identity of the main result

In this section we prove the 1st identity of Theorem 2.1. For the proof we use the *Laplace transform*. We recall that the Laplace transform $\mathcal{L} : L^2(H_r^+) \to \mathcal{O}(H_r^+ + \sqrt{-1}H_r)$ and its inverse are given by

$$\begin{aligned}
(\mathcal{L}f)(A) &:= \int_{H_r^+} e^{-\operatorname{tr}(AX)} f(A)\, dA && (X \in H_r^+ + \sqrt{-1}H_r), \\
(\mathcal{L}^{-1}g)(X) &:= \frac{1}{(2\pi\sqrt{-1})^n} \int_{Z+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} g(X)\, dX && (A \in H_r^+).
\end{aligned}$$

Here the 2nd formula does not depend on the choice of $Z \in H_r^+$. Next, for $\alpha \in \mathbb{C}$, $\mathbf{s} \in \mathbb{C}^r$, let

$$\Gamma_r(\alpha + \mathbf{s}) := (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(\alpha + s_j - \frac{d}{2}(j-1)\right),$$
$$\Gamma_r(\alpha) := \Gamma_r(\alpha + \underline{0}_r),$$

so that $\dfrac{\Gamma_r(\alpha+\mathbf{m})}{\Gamma_r(\alpha)} = (\alpha)_{\mathbf{m}}$ holds for $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$. Then the following is known.

Theorem 4.1 (Gindikin (1964), [3, Lemma XI.2.3])**.** *Let* $\operatorname{Re}\lambda + m_r > \frac{2n}{r} - 1$. *For* $f \in \mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}})$, $A, Z \in H_r^+$, $X, W \in H_r^+ + \sqrt{-1}H_r$, *we have*

$$\int_{H_r^+} e^{-\operatorname{tr}(A(X+W))} f(A) \det(A)^{\lambda - \frac{n}{r}} dA = \Gamma_r(\lambda+\mathbf{m}) f((X+W)^{-1}) \det(X+W)^{-\lambda},$$
$$\frac{\Gamma_r(\lambda+\mathbf{m})}{(2\pi\sqrt{-1})^n} \int_{Z+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} f((X+W)^{-1}) \det(X+W)^{-\lambda} dX = e^{-\operatorname{tr}(AW)} f(A) \det(A)^{\lambda-\frac{n}{r}}.$$

In the 2nd formula we set $W = 0$. Then we have

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \det(A)^{-\lambda+\frac{n}{r}} \int_{Z+\sqrt{-1}H_r} f(X^{-1}) \det(X)^{-\lambda} dX = \frac{1}{(\lambda)_{\mathbf{m}}} f(A).$$

Then comparing this with the Faraut-Korányi's result

$$\left\langle f, e^{\operatorname{tr}(\cdot \overline{A})} \right\rangle_\lambda = \frac{1}{(\lambda)_{\mathbf{m}}} f(A),$$

we get the following.

Lemma 4.2. *For* $\operatorname{Re}\lambda > \frac{2n}{r} - 1$, $f \in \mathcal{P}(H_r^{\mathbb{C}})$, $A \in H_r^+$, *we have*

$$\left\langle f, e^{\operatorname{tr}(\cdot \overline{A})} \right\rangle_\lambda = \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \det(A)^{-\lambda+\frac{n}{r}} \int_{Z+\sqrt{-1}H_r} f(X^{-1}) \det(X)^{-\lambda} dX.$$

This does not depend on the choice of $Z \in H_r^+$.

Now we start the proof of the 1st equality. For $X = \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in H_r^{\mathbb{C}}(\mathbb{F})$, we write $\operatorname{Proj}_1 \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} = x \in H_{r'}^{\mathbb{C}}(\mathbb{F})$, $\operatorname{Proj}_2 \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} = z \in H_{r''}^{\mathbb{C}}(\mathbb{F})$. Then by the above lemma, for $\operatorname{Re}\lambda > \frac{2n}{r} - 1$, $A \in H_r^+$, we have

$$\begin{aligned}&\left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\overline{A})} \right\rangle_{\lambda,X} = \left\langle \det(\operatorname{Proj}_1(X))^k f(\operatorname{Proj}_2(X)), e^{\operatorname{tr}(X\overline{A})} \right\rangle_{\lambda,X} \\ &= \det(A)^{-\lambda+\frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} \det\left(\operatorname{Proj}_1(X^{-1})\right)^k f\left(\operatorname{Proj}_2(X^{-1})\right) \det(X)^{-\lambda} dX.\end{aligned}$$

Then since for $X = \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in H_r^{\mathbb{C}}(\mathbb{F})$ we have

$$X^{-1} = \begin{pmatrix} (x - yz^{-1}y^*)^{-1} & -(x-yz^{-1}y^*)^{-1} y z^{-1} \\ -z^{-1}y^*(x-yz^{-1}y^*)^{-1} & (z-y^*x^{-1}y)^{-1} \end{pmatrix},$$
$$\det(X) = \det(z - y^*x^{-1}y)\det(x) = \det(x - yz^{-1}y^*)\det(z),$$

we have

$$\begin{aligned}
&\left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X} \\
&= \det(A)^{-\lambda+\frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} \det\left((x-yz^{-1}y^*)^{-1}\right)^k \\
&\qquad\qquad \times f\left((z-y^*x^{-1}y)^{-1}\right) \det(z-y^*x^{-1}y)^{-\lambda} \det(x)^{-\lambda} dX \\
&= \det(A)^{-\lambda+\frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} f\left((z-y^*x^{-1}y)^{-1}\right) \\
&\qquad\qquad \times \det(z-y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} \det(z)^k dX \\
&= \det(A)^{-\lambda+\frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\qquad \times f\left((z-y^*x^{-1}y)^{-1}\right) \det(z-y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} \det(z)^k dxdydz \\
&= \det(A)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\qquad\qquad \times f\left((z-y^*x^{-1}y)^{-1}\right) \det(z-y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} dxdydz.
\end{aligned}$$

Since $f \in \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$, by using the inverse Laplace formula

$$\frac{\Gamma_{r''}(\mu+\mathbf{l})}{(2\pi\sqrt{-1})^{n''}} \int_{I+\sqrt{-1}H_{r''}} e^{\operatorname{tr}(cz)} f((z+w)^{-1}) \det(z+w)^{-\mu} dz = e^{-\operatorname{tr}(cw)} f(c) \det(c)^{\mu-\frac{n''}{r''}}$$

twice, for $x \in I+\sqrt{-1}H_{r'}$, $y \in \sqrt{-1}M_{r',r''}$, $c \in H_{r''}^+$ we get

$$\begin{aligned}
&\frac{1}{(2\pi\sqrt{-1})^{n''}} \int_{I+\sqrt{-1}H_{r''}} e^{\operatorname{tr}(cz)} f\left((z-y^*x^{-1}y)^{-1}\right) \det(z-y^*x^{-1}y)^{-\lambda-k} dz \\
&= \frac{1}{\Gamma_{r''}(\lambda+k+\mathbf{l})} e^{\operatorname{tr}(cy^*x^{-1}y)} f(c) \det(c)^{\lambda+k-\frac{n''}{r''}} \\
&= \frac{1}{(\lambda+k)_{\mathbf{l}}} \frac{1}{\Gamma_{r''}(\lambda+k)} e^{\operatorname{tr}(cy^*x^{-1}y)} \det(c)^{\lambda+k-\frac{n''}{r''}} f(c) \\
&= \frac{1}{(\lambda+k)_{\mathbf{l}}} \frac{1}{(2\pi\sqrt{-1})^{n''}} \int_{I+\sqrt{-1}H_{r''}} e^{\operatorname{tr}(cz)} \det(z-y^*x^{-1}y)^{-\lambda-k} dz\, f(c).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\overline{A})}\right\rangle_{\lambda,X} \\
&= \det(A)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\qquad\qquad \times f\left((z-y^*x^{-1}y)^{-1}\right) \det(z-y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} dxdydz \\
&= \frac{1}{(\lambda+k)_{\mathbf{l}}} \det(A)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\qquad\qquad \times \det(z-y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} dxdydz\, f(c)
\end{aligned}$$

$$= \frac{1}{(\lambda+k)_{\mathbf{1}}} \det(A)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} \det(X)^{-\lambda-k} dX\, f(c)$$
$$= \frac{1}{(\lambda+k)_{\mathbf{1}}} \det(A)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{\Gamma_r(\lambda+k)} \det(A)^{\lambda+k-\frac{n}{r}} f(c)$$
$$= \frac{1}{(\lambda)_{\underline{k}_r}} \frac{1}{(\lambda+k)_{\mathbf{1}}} \det(c-b^*a^{-1}b)^{-\lambda+\frac{n}{r}} \det(a)^{-\lambda+\frac{n}{r}}$$
$$\times \det\left(\frac{\partial}{\partial c}\right)^k \det(c-b^*a^{-1}b)^{\lambda+k-\frac{n}{r}} \det(a)^{\lambda+k-\frac{n}{r}} f(c)$$
$$= \frac{\det(a)^k}{(\lambda)_{\underline{k}_r+\mathbf{1}}} \det(c-b^*a^{-1}b)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \det(c-b^*a^{-1}b)^{\lambda+k-\frac{n}{r}} f(c).$$

This computation is originally done for $A \in H_r^+(\mathbb{F})$, but since the both sides are holomorphic, this equality holds for all $A \in H_r^{\mathbb{C}}(\mathbb{F})$. This proves the 1st equality.

5 Proof of the 2nd, 3rd and 4th identities of the main result

In this section we prove the 2nd, 3rd and 4th identities of Theorem 2.1. We write $b^*a^{-1}b =: z \in H_{r''}^{\mathbb{C}}$ and $\lambda - \frac{d}{2}r' =: \mu$, so that $\lambda - \frac{n}{r} = \lambda - \frac{d}{2}r' - \frac{n''}{r''} = \mu - \frac{n''}{r''}$ holds. Then our aim is to prove, for $f \in \mathcal{P}_{\mathbf{1}}(H_{r''}^{\mathbb{C}})$,

$$\det(c-z)^{-\mu+\frac{n''}{r''}} \det\left(\frac{\partial}{\partial c}\right)^k \det(c-z)^{\mu+k-\frac{n''}{r''}} f(c) \tag{A'}$$
$$= (\mu)_{\underline{k}_{r''}} \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{(\mu)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z-c, c) \tag{B'}$$
$$= (\mu)_{\underline{k}_{r''}} \sum_{\mathbf{n}} \frac{(\mu+k)_{\mathbf{n}}}{(\mu)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z, c-z) \tag{C'}$$
$$= \frac{(\mu)_{\underline{k}_{r''}}}{(\mu)_{\mathbf{1}}} \sum_{\mathbf{m},\mathbf{n}} (-k)_{\mathbf{m}} (\mu+k)_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z, c). \tag{D'}$$

Since each side is holomorphic with respect to z, c, it is enough to prove this when $c - z \in H_{r''}^+$. To do this, we recall the *Riesz distribution.* For $\operatorname{Re}\alpha > \frac{n''}{r''} - 1$, let

$$R_\alpha(w) := \frac{1}{\Gamma_{r''}(\alpha)} \det(w)^{\alpha-\frac{n''}{r''}} 1_{H_{r''}^+} \in \mathcal{S}'(H_{r''}).$$

Then $R_\alpha(w) \in \mathcal{S}'(H_{r''})$ is analytically continued for all $\alpha \in \mathbb{C}$ as distributions, and for $\alpha = -k \in -\mathbb{Z}_{\geq 0}$,

$$R_{-k}(w) = \det\left(\frac{\partial}{\partial w}\right)^k \delta_0$$

holds (see [3, Theorem VII.2.2]). This $R_\alpha(w)$ is called the Riesz distribution. Especially, when $c-z\in H_{r''}^+$, the formula (A'),

$$\begin{aligned}
&\det(c-z)^{-\mu+\frac{n''}{r''}}\det\left(\frac{\partial}{\partial c}\right)^k\det(c-z)^{\mu+k-\frac{n''}{r''}}f(c)\\
&=\det(c-z)^{-\mu+\frac{n''}{r''}}\det\left(\frac{\partial}{\partial w}\right)^k\det(c-z+w)^{\mu+k-\frac{n''}{r''}}f(c+w)\bigg|_{w=0}\\
&=\det(c-z)^{-\mu+\frac{n''}{r''}}\det\left(-\frac{\partial}{\partial w}\right)^k\det(c-z-w)^{\mu+k-\frac{n''}{r''}}f(c-w)\bigg|_{w=0}\\
&=\det(c-z)^{-\mu+\frac{n''}{r''}}\int_{w\in(c-z-H_{r''}^+)}\left(\det\left(\frac{\partial}{\partial w}\right)^k\delta_0\right)\det(c-z-w)^{\mu+k-\frac{n''}{r''}}f(c-w)\,dw\\
&=\det(c-z)^{-\mu+\frac{n''}{r''}}\int_{w\in(c-z-H_{r''}^+)}R_{-k}(w)\det(c-z-w)^{\mu+k-\frac{n''}{r''}}f(c-w)\,dw
\end{aligned}$$

is viewed as the analytic continuation of

$$\frac{\det(c-z)^{-\mu+\frac{n''}{r''}}}{\Gamma_{r''}(-k)}\int_{H_{r''}^+\cap(c-z-H_{r''}^+)}\det(w)^{-k-\frac{n''}{r''}}\det(c-z-w)^{\mu+k-\frac{n''}{r''}}f(c-w)\,dw.$$

Therefore, by putting $-k=:\alpha$, $\mu+k=:\beta$ so that $(\mu)_{\underline{k}_{r''}}=\frac{\Gamma_{r''}(\mu+k)}{\Gamma_{r''}(\mu)}=\frac{\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta)}$, it is enough to prove

$$\det(c-z)^{-\alpha-\beta+\frac{n''}{r''}}\int_{H_{r''}^+\cap(c-z-H_{r''}^+)}\det(w)^{\alpha-\frac{n''}{r''}}\det(c-z-w)^{\beta-\frac{n''}{r''}}f(c-w)\,dw \tag{A''}$$

$$=\frac{\Gamma_{r''}(\alpha)\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta)}\sum_{\mathbf{m}}\frac{(\alpha)_{\mathbf{m}}}{(\alpha+\beta)_{\mathbf{m}}}\sum_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}(z-c,c) \tag{B''}$$

$$=\frac{\Gamma_{r''}(\alpha)\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta)}\sum_{\mathbf{n}}\frac{(\beta)_{\mathbf{n}}}{(\alpha+\beta)_{\mathbf{n}}}\sum_{\mathbf{m}}\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c-z) \tag{C''}$$

$$=\frac{\Gamma_{r''}(\alpha)\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta)(\alpha+\beta)_{\mathbf{l}}}\sum_{\mathbf{m},\mathbf{n}}(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{n}}\tilde{f}_{\mathbf{m},\mathbf{n}}(z,c) \tag{D''}$$

for $\operatorname{Re}\alpha,\operatorname{Re}\beta>\frac{n''}{r''}-1$. We note that the formula (A") is written in a more symmetric form

$$\det(c-z)^{-\alpha-\beta+\frac{n''}{r''}}\int_{(z+H_{r''}^+)\cap(c-H_{r''}^+)}\det(w-z)^{\alpha-\frac{n''}{r''}}\det(c-w)^{\beta-\frac{n''}{r''}}f(c+z-w)\,dw,$$

but we do not use this expression for the proof.

To prove these equalities, for $\mathbf{l},\mathbf{m},\mathbf{n}\in\mathbb{Z}^{r''}$ with $l_1\ge\cdots\ge l_{r''}\ge 0$, $m_1\ge\cdots\ge m_{r''}\ge 0$, $n_1\ge\cdots\ge n_{r''}\ge 0$, we define vector spaces $\mathcal{P}^-_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''}\oplus H^{\mathbb{C}}_{r''})$, $\mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''}\oplus H^{\mathbb{C}}_{r''})\subset\mathcal{P}(H^{\mathbb{C}}_{r''})\otimes\mathcal{P}(H^{\mathbb{C}}_{r''})$ as

$$\begin{aligned}
\mathcal{P}^-_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''}\oplus H^{\mathbb{C}}_{r''})&:=\{f(z,c)\in\mathcal{P}_{\mathbf{m}}(H^{\mathbb{C}}_{r''})\otimes\mathcal{P}_{\mathbf{n}}(H^{\mathbb{C}}_{r''})\mid f(z,z)=0\},\\
\mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''}\oplus H^{\mathbb{C}}_{r''})&:=\{f(z,c)\in\mathcal{P}_{\mathbf{m}}(H^{\mathbb{C}}_{r''})\otimes\mathcal{P}_{\mathbf{n}}(H^{\mathbb{C}}_{r''})\mid f(z,z)\in\mathcal{P}_{\mathbf{l}}(H^{\mathbb{C}}_{r''})\}\\
&\qquad\cap\mathcal{P}^-_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''}\oplus H^{\mathbb{C}}_{r''})^\perp.
\end{aligned}$$

Then the following holds.

Lemma 5.1. *(1) Let* $\operatorname{Re}\alpha, \operatorname{Re}\beta > \frac{n''}{r''} - 1$*. If* $f(z,c) \in \mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})$*, then for* $z \in H^{+}_{r''}$ *we have*

$$\det(z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(z-w)^{\beta-\frac{n''}{r''}} f(w, z-w)\, dw \\ = \frac{\Gamma_{r''}(\alpha+\mathbf{m})\Gamma_{r''}(\beta+\mathbf{n})}{\Gamma_{r''}(\alpha+\beta+\mathbf{l})} f(z,z).$$

(2) Let $\operatorname{Re}\alpha, \operatorname{Re}\beta > \frac{n''}{r''} - 1$*. If* $f(z,c) \in \mathcal{P}^{-}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})$*, then for* $z \in H^{+}_{r''}$ *we have*

$$\det(z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(z-w)^{\beta-\frac{n''}{r''}} f(w, z-w)\, dw = 0.$$

When $f(z,c) \in \mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})$ is a highest weight vector under the diagonal action of $GL(r'',\mathbb{F})$, this is proved by a similar way as in [3, Theorem VII.1.7 (ii)]. Then by $GL(r'',\mathbb{F})$-equivariance, this holds for all $f(z,c) \in \mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})$.

By this lemma, (A”) = (B”) is proved as

$$\begin{aligned}
&\det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} f(c-w)\, dw \\
&= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \sum_{\mathbf{m},\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(-w,c)\, dw \\
&= \sum_{\mathbf{m}} \frac{\Gamma_{r''}(\alpha+\mathbf{m})\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta+\mathbf{m})} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(-(c-z),c) \\
&= \frac{\Gamma_{r''}(\alpha)\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta)} \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{(\alpha+\beta)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z-c,c),
\end{aligned}$$

and (A”) = (C”) is proved as

$$\begin{aligned}
&\det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} f(c-w)\, dw \\
&= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} f(z+c-z-w)\, dw \\
&= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \sum_{\mathbf{m},\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z,c-z-w)\, dw \\
&= \sum_{\mathbf{n}} \frac{\Gamma_{r''}(\alpha)\Gamma_{r''}(\beta+\mathbf{n})}{\Gamma_{r''}(\alpha+\beta+\mathbf{n})} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z,c-z) \\
&= \frac{\Gamma_{r''}(\alpha)\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta)} \sum_{\mathbf{n}} \frac{(\beta)_{\mathbf{n}}}{(\alpha+\beta)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z,c-z).
\end{aligned}$$

On the other hand, the proof of (A”) = (D”) is more complicated. First, by the $GL(r'',\mathbb{F})$-equivariance, it suffices to prove for one non-zero $f(z) \in \mathcal{P}_{\mathbf{l}}(H^{\mathbb{C}}_{r''})$. Here we set $f(z) = \Psi^{\mathbf{l}}(z)$,

$$\Psi^{\mathbf{l}}(z) \in \mathcal{P}_{\mathbf{l}}(H^{\mathbb{C}}_{r''}), \qquad \sum_{l_1\ge\cdots\ge l_{r''}\ge 0} \Psi^{\mathbf{l}}(z) = e^{\operatorname{tr}(z)}.$$

Then we sum up the formula (A”) with respect to $\mathbf{l}$. Then we get

$$\begin{aligned}
&\sum_{\mathbf{l}} \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \Psi^{\mathbf{l}}(c-w)\,dw \\
&= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c-w)} dw \\
&= \det(c-z)^{-\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(\sqrt{c-z}^{-1} w \sqrt{c-z}^{-1})^{\alpha-\frac{n''}{r''}} \\
&\qquad\qquad \times \det(I-\sqrt{c-z}^{-1} w \sqrt{c-z}^{-1})^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c-w)} dw \\
&= \int_{H^+_{r''}\cap(I-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c)-\mathrm{tr}(\sqrt{c-z}w\sqrt{c-z})} dw \\
&= \int_{H^+_{r''}\cap(I-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c)-\mathrm{tr}(w(c-z))} dw \\
&= \int_{H^+_{r''}\cap(I-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(wz)+\mathrm{tr}((I-w)c)} dw.
\end{aligned}$$

Now we expand $e^{\mathrm{tr}(xy)+\mathrm{tr}(zw)}$ as

$$\begin{aligned}
e^{\mathrm{tr}(xy)+\mathrm{tr}(zw)} &= \sum_{\mathbf{m},\mathbf{n}} \left(\sum_{\mathbf{l}} K^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(x,z;y,w) + K^{-}_{\mathbf{m},\mathbf{n}}(x,z;y,w) \right) \\
&\in \bigoplus_{\mathbf{m},\mathbf{n}} \left(\bigoplus_{\mathbf{l}} \left(\mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})_{x,z} \otimes \mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})_{y,w} \right) \right. \\
&\qquad \left. \oplus \left(\mathcal{P}^{-}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})_{x,z} \otimes \mathcal{P}^{-}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})_{y,w} \right) \right).
\end{aligned}$$

Then $\tilde{\Psi}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(z,c) = K^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(z,c;I,I) = K^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(I,I;z,c)$ holds. Therefore, by Lemma 5.1 we get

$$\begin{aligned}
&\sum_{\mathbf{l}} \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H^+_{r''}\cap(c-z-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \Psi^{\mathbf{l}}(c-w)\,dw \\
&= \int_{H^+_{r''}\cap(I-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(wz)+\mathrm{tr}((I-w)c)} dw \\
&= \int_{H^+_{r''}\cap(I-H^+_{r''})} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} \\
&\qquad\qquad \times \sum_{\mathbf{m},\mathbf{n}} \left(\sum_{\mathbf{l}} K^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(w,I-w;z,c) + K^{-}_{\mathbf{m},\mathbf{n}}(w,I-w;z,c) \right) dw \\
&= \sum_{\mathbf{l},\mathbf{m},\mathbf{n}} \frac{\Gamma_{r''}(\alpha+\mathbf{m})\Gamma_{r''}(\beta+\mathbf{n})}{\Gamma_{r''}(\alpha+\beta+\mathbf{l})} K^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(I,I;z,c) \\
&= \sum_{\mathbf{l}} \frac{\Gamma_{r''}(\alpha)\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha+\beta)(\alpha+\beta)_{\mathbf{l}}} \sum_{\mathbf{m},\mathbf{n}} (\alpha)_{\mathbf{m}}(\beta)_{\mathbf{n}} \tilde{\Psi}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(z,c).
\end{aligned}$$

Then projecting both sides to $\bigoplus_{\mathbf{m},\mathbf{n}} \mathcal{P}^{\mathbf{l}}_{\mathbf{m},\mathbf{n}}(H^{\mathbb{C}}_{r''} \oplus H^{\mathbb{C}}_{r''})$, we get (A”) = (D”).

References

[1] R.J. Beerends and E.M. Opdam, *Certain hypergeometric series related to the root system BC*. Trans. Amer. Math. Soc. **339** (1993), no. 2, 581–609.

[2] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters.* Math. Ann. **217** (1975), no. 3, 271–285.

[3] J. Faraut and A. Korányi, *Analysis on symmetric cones.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.

[4] T. Ibukiyama, T. Kuzumaki and H. Ochiai, *Holonomic systems of Gegenbauer type polynomials of matrix arguments related with Siegel modular forms.* J. Math. Soc. Japan **64** (2012), no. 1, 273–316.

[5] A. Juhl, *Families of conformally covariant differential operators, Q-curvature and holography.* Progress in Mathematics, 275. Birkhäuser Verlag, Basel, 2009.

[6] T. Kobayashi, *Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs.* Representation theory and automorphic forms, 45–109, Progr. Math., 255, Birkhäuser Boston, Boston, MA, 2008.

[7] T. Kobayashi, T. Kubo and M. Pevzner, *Conformal symmetry breaking operators for differential forms on spheres.* Lecture Notes in Math. 2170, Springer, Singapore, 2016, ix+192 pp.

[8] T. Kobayashi, B, Ørsted, P. Somberg and V. Souček, *Branching laws for Verma modules and applications in parabolic geometry. I.* Adv. Math. **285** (2015), 1796–1852.

[9] T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators: I. Genreal theory and F-method.* Selecta Math. (N.S.) **22** (2016), no. 2, 801–845.

[10] T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators: II. Rankin-Cohen Operators for Symmetric Pairs.* Selecta Math. (N.S.) **22** (2016), no. 2, 847–911.

[11] R. Nakahama, *Construction of intertwining operators between holomorphic discrete series representations.* SIGMA Symmetry Integrability Geom. Methods Appl. **15** (2019), 036, 101 pages.

[12] L. Peng and G. Zhang, *Tensor products of holomorphic representations and bilinear differential operators.* J. Funct. Anal. **210** (2004), no. 1, 171–192.

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba Meguro-ku Tokyo 153-8914
JAPAN
E-mail address: `nakahama@ms.u-tokyo.ac.jp`