RECENT RESULTS ON INTERSECTION SPACE COHOMOLOGY

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ABSTRACT. In this article, recent results on the theory of intersection spaces and their cohomology groups are reviewed. The focus is on the construction of intersection spaces for non-isolated singularities and stratification depth greater than one as well as on the de Rham, sheaf theoretic and algebraic approaches towards intersection space cohomology. At the end, a list of open problems is provided.

1. INTRODUCTION

This survey article is meant to be an update of the survey [7] of Banagl and Maxim. In that article, the authors give an overview of the construction of intersection spaces for complex projective varieties with isolated singularities and their relation to deformations of singularities. Moreover, they point out why intersection space homology is the correct homology theory for type IIB string theory of conifolds, while Goresky-MacPherson's intersection homology is the correct homology theory for type IIA string theory, making both to a so called mirror-pair for Calabi-Yau threefolds.

In this paper, the focus is on the progress in the intersection space homology theory that was made since Banagl-Maxim's survey paper was published. I will distinguish between the following three approaches.

(1) Construction of actual intersection spaces. In particular, the results of Banagl and Chriestenson in [5] on intersection spaces for depth one spaces with nonisolated singular set, of Klimczak in [21] and Wrazidlo in [25] on generalized intersection spaces with fundamental class and of Agustín-Bobadilla in [1] on intersection space pairs of spaces with stratification depth > 1 are reviewed.
(2) De Rham models for intersection space cohomology. This part contains outlines on the de Rham models of Banagl in [4] and the author in [17] describing intersection space cohomology via a complex of differential forms on the smooth part/blowup of the pseudomanifold.

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Moreover, I review the analytic approach to intersection cohomology using L^2-cohomology of Banagl and Hunsicker in [6].

(3) The third part contains Agustín-Bobadilla's sheaf theoretic approach of [1] and Geske's algebraic approach of Geske in [19].

Conventions and Notation: Throughout the paper, the terms "singular space" and "pseudomanifold" denote a Thom-Mather stratified pseudomanifold, e.g. a Whitney-stratified complex projective variety.

Note, that we follow the notation of Agustín-Bobadilla when we talk about truncation and cotruncation both of spaces and differential form complexes. For example, the notation for the homology truncation in degree k of a space L is $L_{\leq k}$ which would be $L_{<(k+1)}$ in Banagl's notation.

2. Intersection Spaces

Intersection spaces have been first constructed in [2] for singular spaces X with isolated singularities or trivial link bundles. For $X^n = M \cup_{\partial M} \bigsqcup_i \operatorname{cone}(L_i)$, where M is an n-dimensional compact manifold with boundary $\partial M = \bigsqcup_i L_i$ and the L_i are the links of the singularities, the intersecion space is a finite CW complex defined as $I^{\bar{p}}X = M \cup_{\partial M} \operatorname{cone}\left(\bigsqcup_i \operatorname{cone}((L_i)_{\leq \bar{q}(n)})\right)$. As reviewed in [7], the CW complexes $(L_i)_{\leq \bar{q}(n)}$ come with structural maps $(f_i)_{\leq \bar{q}(n)} : (L_i)_{\leq \bar{q}(n)} \to L_i$ and are called the Moore-approximations or homology truncations in degree $\bar{q}(n)$ of the links L_i.

For $X = M \cup_{\partial M = B \times L} (B \times \operatorname{cone}(L))$, with connected singular set B of codimension $b < n$, the intersection space is defined as $I^{\bar{p}}X := M \cup_{\partial M} \operatorname{cone}(B \times L_{\leq \bar{q}(b)})$. The truncation of the link is performed fiberwisely before the product is coned off. This construction gives the basic idea, how to generalize the construction to twisted link bundles.

2.1. Equivariant Moore approximation and intersection spaces.

In [5], Banagl-Chriestenson perform fiberwise truncation of twisted bundles by a systematic machinery, called equivariant Moore approximation, to construct intersection spaces. For a topological group G, a G-space X and any integer k, a degree k equivariant Moore approximation of X is a G-space $X_{\leq k}$ together with a G-equivariant map $f_{\leq k} : X_{\leq k} \to X$, such that the induced map $H_r(f_{\leq k}) : H_r(X_{\leq k}) \to H_r(X)$ on homology is an isomorphism for $r \leq k$ while $H_r(X_{\leq \bar{q}(b)}) = 0$ for $r > k$. There are obstructions on the existence of equivariant Moore approximations and Banagl-Chriestenson give some examples where those exist: Oriented sphere bundles with vanishing Euler class (see [5, Proposition 12.1]), trivial group actions on simply connected CW complexes ([5, Example 3.5]), cellular group actions with vanishing or injective cellular boundary operator ([5, Example 3.6]) and symplectic toric 4-manifolds ([5, Proposition 12.3]).

The equivariant Moore approximations can be used to perform fiberwise truncations of twisted bundles as follows. Let $\pi : E \to B$ be a fiber bundle of closed manifolds with closed fiber L, structure group G and all spaces compatibly oriented. Suppose, that there is a degree k equivariant Moore approximation $f_{\leq k} : L_{\leq k} \to L$. Let $E_P \to B$ be the underlying principal bundle of π, that is $E = E_P \times_G B$. Then, there is a fiber bundle $\pi_{\leq k} : ft_{\leq k}E := E_P \times_G L_{\leq k} \to B$ together with a bundle morphism $F_{\leq k} : ft_{\leq k}E \to E$ induced by the map $f_{\leq k}$ and we call the pair $(ft_{\leq k}E, F_{\leq k})$ a fiberwise k-truncation of the bundle E.

Banagl-Chriestenson use the fiberwise truncation to define intersection spaces for pseudomanifolds X of depth one with singular set a closed smooth manifold B of codimension b and twisted link bundle. The Thom-Mather control data give rise to a decomposition $X = M \cup_{\partial M} T$, with M a compact manifold with boundary $\partial M = E$, the total space of the (twisted) link bundle $\pi : E \to B$ with fiber L, and T a tubular neighbourhood of the singular stratum $B \subset X$. Let $\bar{p}$ and $\bar{q}$ be dual perversities. If a degree $\bar{q}(b)$ equivariant Moore approximation of the link L exists, the intersection space is defined as $I^{\bar{p}}X := M \cup_{\partial M} \operatorname{cone} ft_{\leq \bar{q}(b)}E = \operatorname{cone} \tau_{\leq \bar{q}(b)}$, where $\tau_{\leq \bar{q}(b)} : ft_{\leq \bar{q}(b)}E \to M$ is the composition of $F_{\leq \bar{q}(b)}$ with the inclusion $E = \partial M \hookrightarrow M$. Note, that there are obstructions on Poincaré duality for intersection spaces of complementary perversities. Banagl-Chriestenson introduce a set of so called "local duality obstructions", that is a set of certain cup products in the $(n-1)$st cohomology group of E. To describe these local duality obstructions, let $ft_{>\bar{q}(b)}E$ be the homotopy pushout of the diagram $B \xleftarrow{\pi_{\leq \bar{q}(b)}} ft_{\leq \bar{q}(b)}E \xrightarrow{F_{\leq \bar{q}(b)}} E$ (see [5, Definition 2.1]). It has the structure of a fiber bundle $\pi_{>\bar{q}(b)} : ft_{>\bar{q}(b)}E \to B$ and comes with a vertex section $\sigma : B \to ft_{>\bar{q}(b)}E$ and a bundle morphism $c_{>\bar{q}(b)} : E \to ft_{>\bar{q}(b)}E$. Banagl and Chriestenson denote by $Q_{>\bar{q}(b)}E = \operatorname{cone}(\sigma)$ the mapping cone of σ, which contains $ft_{>\bar{q}(b)}E$ as a subspace, embedded as $\xi_{>\bar{q}(b)} : ft_{>\bar{q}(b)}E \hookrightarrow Q_{>\bar{q}(b)}E$. Let $C_{>\bar{q}(b)} : E \to Q_{>\bar{q}(b)}E$ be the composition of $c_{>\bar{q}(b)}$ and $\xi_{>\bar{q}(b)}$. If both degree $\bar{q}(b)$ and $\bar{p}(b)$ equivariant Moore approximations of the fiber L of the bundle E exist, the local duality obstructions of $\pi : E \to B$ in degree i are then defined as the following subset of $H^{n-1}(E)$.

$$\left\{ C^*_{>\bar{q}(b)}(x) \cup C^*_{>\bar{p}(b)}(y) \,|\, x \in \widetilde{H}^i(Q_{>\bar{q}(b)}E),\ y \in \widetilde{H}^{n-1-i}(Q_{>\bar{p}(b)}E) \right\}.$$

Theorem 2.1.1. [5, Theorem 9.5] *Let X^n be a compact oriented two strata pseudomanifold of dimension n with singular set B of codimension b with link bundle $L \to E \to B$. Let $\bar{p}$ and $\bar{q}$ be complementary perversities. If the degree $\bar{q}(b)$ and $\bar{p}(b)$ equivariant Moore approximations of L exist and the local duality obstructions of the link bundle $E \to B$ vanish in all degrees, then there is a global Poincaré duality isomorphism*

$$\widetilde{H}^r(I^{\bar{p}}X) \cong \widetilde{H}_{n-r}(I^{\bar{q}}X).$$

Finally, generalizing the same statement for product link bundles and isolated singularities (see [2, Theorem 2.28]), Banagl-Chriestenson show that for a Witt space X such that an appropriate equivariant Moore approximation of the link exists, the intersection form of the intersection space can be chosen to be symmetric and such that its signature equals the Goresky-MacPherson-Siegel signature of intersection homology ([5, Corollary 11.4]).

2.2. **Intersection spaces with fundamental class.** In [21], Klimczak expands the concept of intersection spaces for pseudomanifolds with isolated singularities to tackle the problem, that for Banagl's intersection spaces Poincaré duality only manifests as equality of the Betti numbers of complementary degrees, and not, as for manifolds, as an isomorphism between homology/cohomology groups induced by a cap product with a fundamental class. By exploiting the rational Hurewicz theorem, he shows that for simply connected links, one can glue an n-cell near the singular set to obtain a Poincaré duality space that is homeomorphic to Banagl's intersection space plus an n-cell. This gluing produces a fundamental homology class, which means that cap product with this class is an isomorphism.

In [25], Wrazidlo applies Klimczak's ideas to depth one pseudomanifolds with nonisolated singular set as in [5] that satisfy the Witt conditions $H_{\dim(L_i)/2}(L_i) = 0$ for the links L_i. For these spaces, gluing the n-cell near the singular stratum is obstructed by a condition on the rational Hurewicz homomorphism. As shown in [25, Theorem 6.3], this obstruction is strongly related to Banagl and Chriestenson's local duality obstructions. Both are equivalent if the dimensions of the bases and links of the link bundles are related in a certain way.

2.3. **Agustín-Bobadilla's intersection space pairs.** In [1], Agustín and Bobadilla provide a method to generalize the construction of intersection spaces to pseudomanifolds of arbitrary stratification depth. Their idea is to modify the pair $(X, \mathrm{Sing}(X))$ inductively to produce a sequence of intersection space pairs. As the procedure advances, tubular neighbourhoods of strata of increasing codimension are replaced by fiberwise cones on fiberwise homology truncations of their link bundles. In each step, the construction is obstructed by the existence of a fiberwise truncation of the respective link bundle. It is not unique and follows the scheme of obstruction theory: The choices made at each step might obstruct the following steps in the inductive construction. If it is possible to make choices such that the procedure terminates, Agustín and Bobadilla say that "the intersection space pair exists". Their construction is different from the ones for depth one spaces described in the previous sections: Since they want to sheafify the intersection space construction, that is give a constructible sheaf complex on the pseudomanifold with hypercohomology the intersection space cohomology, Agustín-Bobadilla need special homotopy models for the pseudomanifold that contain the intersection spaces as subspaces. Their intersection spaces are only homotopy equivalent to those of Banagl, in general, which can be seen in Example 2.3.1.

Since the notation in the general setting is rather involved and technical, Agustín-Bobadilla's construction is reviewed explicitly for the example of the three strata pseudomanifold $X = \mathrm{cone}(\mathrm{cone}(T^2))$. Note, that this singular space can be restratified to a pseudomanifold with two strata, but since intersection spaces depend on the explicit stratification of a pseudomanifold, this is not a problem. The notation used is borrowed from [1]. Since the link bundles are trivial in this example, the obstructions for the existence of an intersection space pair vanish.

Example 2.3.1. *Let* $X = \mathrm{cone}(\mathrm{cone}(T^2))$ *with filtration* $X = X_4 \supset X_1 = \mathrm{cone}(c) \supset X_0 = \{C\}$, *with* $c =$ *the cone point of the inner cone and* $C =$ *the cone point of the final cone.*
As perversity, we take the upper middle perversity $\bar{p} = \bar{m}$. *Let* $\underline{m}$ *denote the lower middle perversity, which is dual to* $\bar{m}$. *In the first part of the example, the intermediate intersection space pair* $(I_3^{\bar{m}}X, X_1)$ *is constructed. The subscript "3" refers to the fact that the pair is derived from the previous step (that is from the pseudomanifold* X *itself) by replacing the link bundle of the stratum of codimension* 3 *by its fiberwise truncation in degree* $\underline{m}(3)$.

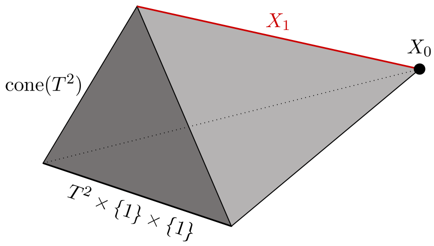


FIGURE 1. $\mathrm{cone}(\mathrm{cone}(T^2))$

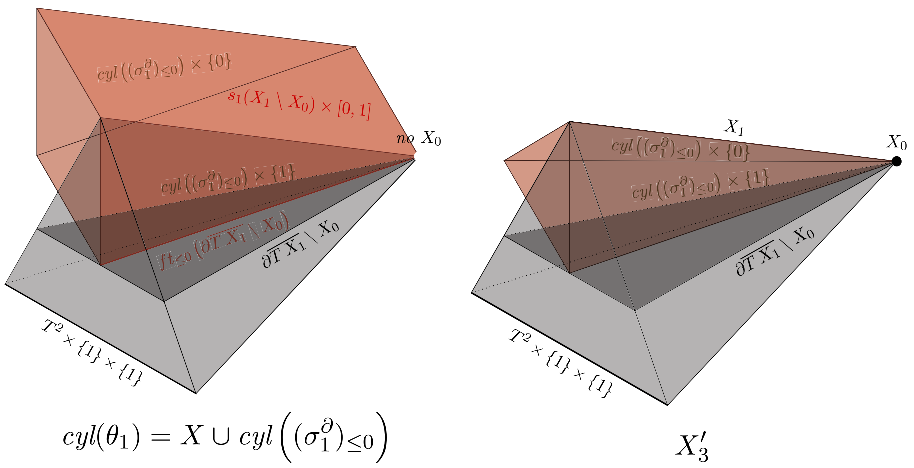


FIGURE 2. Construction of the intermediate intersection space

To construct the intermediate intersection space $I_3^{\bar{m}}X$, *we first derive a homotopy model* $X_3' \simeq X$ *of* X *that contains* $I_3^{\bar{m}}X$ *as a subspace. In Figure 2, the construction of* X_3' *is illustrated. The link bundle of the intermediate stratum* $X_1 \setminus X_0 \cong (0,1)$ *is the trivial bundle* $\sigma_1^\partial : (\partial \overline{T X_1}) \setminus X_0 \cong T^2 \times (0,1) \to (0,1)$. *The cutoff value for the link* T^2 *is* $\underline{m}(3) = 0$. *As explained in* [2, Example 2.2.8], *a suitable homology truncation of* T^2 *in this degree is* $T_{\le 0} = \{P\} \hookrightarrow T^2$, *with* P *a zero cell of* T^2. *Since the bundle is trivial, a* *suitable fiberwise truncation of* σ_1^∂ *in degree* 0 *is* $(\sigma_1^\partial)_{\leq 0} : \{P\} \times (0,1) \to (0,1)$ *with bundle morphism* $\phi_1^\partial : \{P\} \times (0,1) \hookrightarrow T^2 \times (0,1)$. *To construct* X_3', *one adds* $cyl\left((\sigma_1^\partial)_{\leq 0}\right) \times [0,1]$ *to* X, *as shown in the first picture of Figure 2. The additional dimension for the cylinder is depicted pointing into the page. The cylinder on the intermediate stratum* $X_1 \setminus X_0$ *is contained in this space as* $s_1(X_1 \setminus X_0) \times [0,1]$ *via a vertex section* $s_1 : X_1 \setminus X_0 \hookrightarrow cyl\left((\sigma_1^\partial)_{\leq 0}\right)$.

Collapsing the sets $s_1(x) \times [0,1],\ x \in X_1 \setminus X_0$ *to points and taking the union with the bottom stratum* X_0 *yields the homotopy model* X_3', *illustrated in the second picture of Figure 2. Shrinking the cylinder* $cyl\left(cyl\left((\sigma_1^\partial)_{\leq 0}\right)\right)$ *induces a map* $\pi_3 : X_3' \to X$. *It becomes a homotopy equivalence if one endows* X_3' *with the topology generated by all open sets on* $X_3' \setminus X_0$ *and all the sets* ${\pi_3'}^{-1}(U)$ *for* $U \subset X$ *open. The intermediate intersection space, illustrated in the first picture of Figure 3, is the following subspace of* X_3', *where* $ft_{\leq 0}\left(\partial \overline{TX_1} \setminus X_0\right) \times [0,1] = cyl(\phi_1^\partial) \subset cyl\left(cyl\left((\sigma_1^\partial)_{\leq 0}\right)\right)$ *in unison with Agustín-Bobadilla's notation.*

$$I_3^{\bar{m}} X = (X \setminus \overline{TX_1}) \cup cyl(\phi_1^\partial) \cup cyl\left((\sigma_1^\partial)_{\leq 0}\right) \times \{0\}\,.$$

To construct the final intersection space pair, one has to truncate the

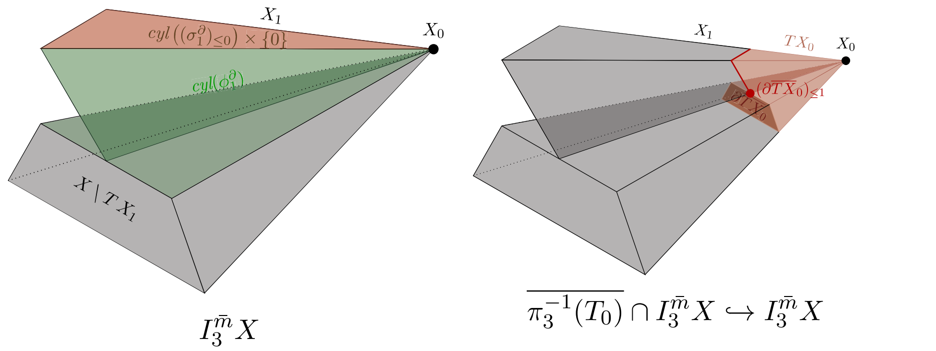
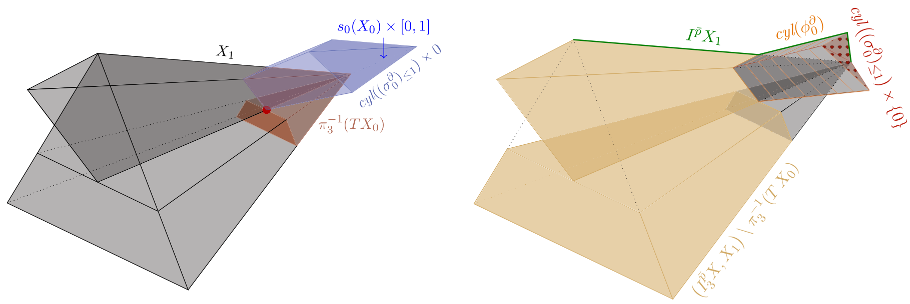


FIGURE 3. The intermediate intersection space $I_3^{\bar{m}} X$ and the truncation of the link of X_0

link bundle of X_0 *in the pair* $(I_3^{\bar{m}} X, X_1)$ *in degree* $\underline{m}(4) = 1$. *The total space of that bundle is* $\partial \overline{\pi_3^{-1}(TX_0)} \cap I_3^{\bar{m}} X \cong T^2 \times [\frac{1}{2}, 1) \cup_{\{P\} \times \{\frac{1}{2}\}} I$, *with* I *a closed interval. Exploiting* [2, Example 2.2.8] *once more, the pair* $(\partial I_3^{\bar{m}} \overline{TX_0})_{\leq 1} := \left(T^2_{\leq 1} \times \left\{\frac{1}{2}\right\} \cup I, pt\right) \subset \partial \overline{\pi_3^{-1}(TX_0)}$ *is a suitable choice for the desired truncation. It is illustrated as the red line and the small red circle in the second picture of Figure 3. The truncated bundle projection is denoted by* $(\sigma_0^\partial)_{\leq 1} : (\partial I_3^{\bar{m}} \overline{TX_0})_{\leq 1} \to X_0$ *and the accompanying bundle morphism by* $\phi_0^\partial : (\partial I_3^{\bar{m}} \overline{TX_0})_{\leq 1} \to \partial \overline{\pi_3^{-1} TX_0} \cap (I_3^{\bar{m}} X, X_1)$. *To construct the homotopy model* X_4' *of* X *that contains the final intersection space pair, one takes the union of* X_3' *and* $cyl((\sigma_0^\partial)_{\leq 1}) \times [0,1]$ *as shown in the first picture of Figure 4. This space is called* $cyl(\theta_0)$ *in Agustín-Bobadilla's notation and contains the* *subspace* $X_0 \times [0,1]$, *embedded via the vertex section* $s_0 : X_0 \to cyl\left((\sigma_0^\partial)_{\leq 1}\right)$. *Collapsing* $s_0(X_0) \times [0,1] \subset cyl(\theta_0)$ *to a point gives the homotopy model* $X_4' \simeq X_3' \simeq X$. *The final intersection space* $I^{\bar{m}}X$, *shown in the second picture of Figure 4, is the union of* $I_3^{\bar{m}}X \setminus \pi_3^{-1}(TX_0)$ *(yellow part) with the subspaces* $cyl(\phi_0^\partial) = (\partial\overline{\pi_3^{-1}TX_0})_{\leq 1} \times [0,1]$ *(orange hatched surface) and* $cyl\left((\sigma_0^\partial)_{\leq 1}\right)$ *(red dotted surface) of* $cyl(\theta_0)$.

$$I^{\bar{m}}X := (I_3^{\bar{m}}X \setminus \pi_3^{-1}TX0) \cup cyl(\phi_0^\partial) \cup cyl\left((\sigma_0^\partial)_{\leq 1}\right) \subset X_4'$$

The subspace $I^{\bar{m}}X_1 \subset I^{\bar{m}}X$ *is illustrated by the green line in the picture.*



$cyl(\theta_0) = X_3' \cup (cyl((\sigma_0^\partial)_{\leq 1}) \times [0,1])$ *The pair* $(I^{\bar{m}}X, I^{\bar{m}}X_1)$

FIGURE 4. The final intersection space pair $(I^{\bar{m}}X, I^{\bar{m}}X_1)$

At the end of the example, it is worth pointing out that the method introduced by Banagl in [3] *to construct interesction spaces for some classes with stratification depth two is applicable in this setting. To perform Banagl's construction, one removes from* X *tubular neighbourhoods* TX_1 *of* $X_1 \setminus X_0$ *and* TX_0 *of* X_0 *to get a manifold* $M \cong T^2 \times (0,1]^2$ *with two boundary parts that both look like* $T^2 \times (0,1]$. *One then takes suitable truncations of the boundary parts in the respective degrees and takes the mapping cone of the inclusion of the union of these truncations. The resulting intersection space is illustrated in Figure 5. Note, that in contrast to Agustín-Bobadilla's construction, the singular set* $Sing(X) = X_1$ *has no one dimensional heritage in this intersection space.*

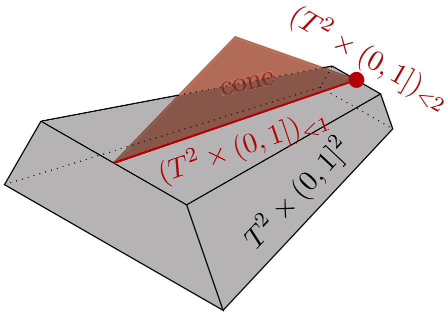


FIGURE 5. Banagl's $I^{\bar{m}}X$

To end this section, it should be pointed out that there is no Poincaré duality theorem for Agustín-Bobadilla's intersection space pairs, yet. This issue will be discussed in more detail in Section 4.1.

3. Intersection space cohomology with differential forms

3.1. **Pseudomanifold of depth one.** In [4], Banagl introduces an approach to intersection space cohomology via differential forms. It is comparable to the description of intersection cohomology via differential forms of Goresky-MacPherson-Brylinski, presented in [12].

A depth one pseudomanifold is said to have geometrically flat link bundles, if for each singular stratum $\Sigma \subset X$, the fiber L_Σ of the link bundle $p : \partial T_\Sigma \to \Sigma$, where T_Σ is a tubular neighbourhood of Σ in X, is a closed manifold that can be endowed with a Riemannian metric such that the transition functions of the bundle are locally trivial and the structure group of the bundle is contained in the isometries of L_Σ. For example, each trivializable bundle is geometrically flat. For such spaces, Banagl defines the complex of fiberwisely cotruncated multiplicatively structured forms $ft_{>k}\Omega^\bullet_{\mathcal{MS}}(\partial T_\Sigma) \subset \Omega^\bullet(\partial T_\Sigma)$. First, the cotruncation $\tau_{>k}\Omega^\bullet(L_\Sigma) \subset \Omega^\bullet(L_\Sigma)$ in degree k is defined as

$$\tau_{>k}\Omega^\bullet(L_\Sigma) := \cdots \to 0 \to \ker \delta_g \to \Omega^{k+2}(L_\Sigma) \to \Omega^{k+3}(L_\Sigma) \to \dots$$

with δ_g the dual of the boundary operator d with respect to the inner product induced by the Riemannian metric g on L_Σ. By the Hodge Decomposition Theorem, the cohomology of this complex coincides with the cohomology of L_Σ in degrees greater than k, while it vanishes in other degrees.

Let $\{U_\alpha\}_{\alpha\in I}$ be a trivializing atlas of the bundle $p : \partial T_\Sigma \to \Sigma$. Then, a differential form $\omega \in \Omega^\bullet(\partial T_\Sigma)$ is called multiplicatively structured and fiberwisely cotruncated in degree k, i.e. $\omega \in ft_{>k}\Omega^\bullet_{\mathcal{MS}}(\partial T_\Sigma)$, if and only if $\forall \alpha \in I$ it holds that $\omega|_{p^{-1}(U_\alpha)} = \phi^*_\alpha \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j$, for some $\eta_j \in \Omega^\bullet(U_\alpha)$, $\gamma_j \in \tau_{>k}\Omega^\bullet(L_\Sigma)$. The featured maps are the trivialization $\phi_\alpha : U_\alpha \times L_\Sigma \xrightarrow{\cong} p^{-1}(U_\alpha)$, and the two projections $\pi_1 : U_\alpha \times L_\Sigma \to U_\alpha$ and $\pi_2 : U_\alpha \times L_\Sigma \to L_\Sigma$. A form $\omega \in \Omega^\bullet(X_{reg})$ is contained in the intersection form complex $\Omega I^\bullet_{\bar{p}}(X)$, if for any singular stratum $\Sigma \subset \mathrm{Sing}(X)$ of codimension b with tubular neighbourhood T_Σ, it holds that $\omega|_{T_\Sigma \cap X_{reg}} = \pi^*\eta$ with $\eta \in ft_{>\bar{q}(b)}\Omega^\bullet_{\mathcal{MS}}(\partial T_\Sigma)$, where $\pi : T_\Sigma \cap X_{reg} \cong \partial T_\Sigma \times (0,1) \to \partial T_\Sigma$ is the projection. Banagl proves that the cohomology groups of intersection form complexes for complementary perversities satisfy Poincaré duality in the classical sense.

Theorem 3.1.1. [4, Theorem 8.2] *Let $\bar{p}$ and $\bar{q}$ be complementary perversities and X^n an n-dimensional compact and oriented depth one pseudomanifold without boundary. Then, integration of wedge products of forms induces a nondegenerate bilinear form*

$$\int : H^r(\Omega I^\bullet_{\bar{p}}(X)) \times H^{n-r}(\Omega I^\bullet_{\bar{q}}(X)) \to \mathbb{R}, \quad ([\omega], [\eta]) \mapsto \int_{X_{reg}} \omega \wedge \eta.$$

Banagl also proves that for pseudomanifolds with isolated singularities X, the de Rham cohomology groups $H^r(\Omega I^\bullet_{\bar{p}}(X))$ are isomorphic as vector spaces to the reduced cohomology groups $\widetilde{H}^r(I^{\bar{p}}X; \mathbb{R})$ of the intersection space $I^{\bar{p}}X$ with real coefficients, see [4, Theorem 9.11]. This statement is generalized in [23], where Schlöder and the author use pullback constructions in the category of DGAs to show that the de Rham theorem can be lifted to the cohomology rings. Moreover, in [16], the author generalized Banagl's de Rham Theorem 3.1.1 to depth one pseudomanifolds with product link bundles $\Sigma \times L_\Sigma$.

3.2. **L^2-description of intersection space cohomology.** In [6], Banagl and Hunsicker give a Hodge theoretic description of intersection space cohomology via extended weighted L^2-harmonic forms. Their approach is in the spirit of Cheeger's approach to Poincaré duality on singular spaces. In [13, 14, 15], Cheeger works with L^2-cohomology with respect to conical metrics on the regular part of a pseudomanifold and proves Poincaré duality. He shows that for pseudomanifolds with only even dimensional strata (that statement was later generalized to Witt spaces) the space of L^2-harmonic forms is isomorphic to the linear dual of Goresky-MacPherson's middle perversity intersection homology.

Banagl and Hunsicker work with depth one pseudomanifolds with (connected) singular stratum Σ and product link bundle $\Sigma \times L$. They find a Riemannian metric g on the regular part X_{reg} of X, which is very different from Cheeger's conical metric, and a special space of L^2-harmonic forms that is isomorphic to the de Rham cohomology groups $H^\bullet(\Omega I^\bullet_{\bar{p}}(X))$ and, by [16], therefore also to the reduced singular cohomology groups $\widetilde{H}^\bullet(I^{\bar{p}}X)$ of the intersection spaces. The type of metric they use is called product type fibered scattering metric, the space contains all extended weighted L^2-harmonic forms. For a weight c and a metric g, a differential form $\omega \in L^2_g\Omega^\bullet(X_{reg})$ is c-weighted, i.e. $\omega \in x^c L^2_g \Omega^\bullet_g(X_{reg})$, if $\int_{X_{reg}} \|x^{-c}\omega\|_g \, d\mathrm{vol}_g < \infty$, where $\|\cdot\|_g$ denotes the pointwise metric on the space $\Omega^\bullet(X_{reg})$ induced by the metric g. The space $x^c L^2_g \Omega^\bullet_g(X_{reg})$ can be completed to a Hilbert space with respect to the inner product $\langle \alpha, \beta\rangle_c := \int_{X_{reg}} \alpha \wedge x^{-2c} *_g \beta$. Let $\delta_{g,c}$ denote the formal adjoint of the de Rham boundary operator d with respect to this inner product and $D_{g,c} := d + \delta_{g,c}$. Then, a form $\omega \in L^2_g\Omega^\bullet_g(X_{reg})$ is extended c-weighted L^2-harmonic, i.e. $\omega \in \mathcal{H}^\bullet_{ext}(X_{reg}, g, c)$, if ω is $(c-\epsilon)$-weighted for all $\epsilon > 0$ and $D_{g,c}\omega = 0$. The Hodge description of intersection cohomology via extended weighted L^2-harmonic forms is then given by the following theorem. To prove this theorem, Banagl and Hunsicker use conifold transitions, that were outlined in [7].

Theorem 3.2.1. [6, Theorem 1.1] *Let X be a pseudomanifold of depth one with singular stratum $\Sigma \subset X$ of codimension b that admits a product link bundle $\Sigma \times L$. Let g_{fs} be an associated product type fibered scattering metric on the regular part $X_{reg} = X \setminus \Sigma$. Then, there is an isomorphism*

$$H^\bullet(\Omega I^\bullet_{\bar{p}}(X)) \cong \mathcal{H}^\bullet_{ext}\left(X_{reg}, g_{fs}, \frac{b}{2} - 1 - \bar{p}(b)\right).$$

3.3. **Pseudomanifolds of depth two.** In [17], the author extends the intersection form complex $\Omega I^\bullet_{\bar p}$ to depth two pseudomanifolds with filtration $X = X_n \supset X_{n-b} \supset X_0 = \{x_0\}$, three strata $X \setminus X_{n-b}$, $X_{n-b} \setminus X_0$ and X_0 and geometrically flat link bundle for the middle stratum $X_{n-b} \setminus X_0$. $\Omega I^\bullet_{\bar p}$ is defined on the blowup $\bar X$ of X that is constructed by removing first half a tubular neighbourhood TX_0 of $X_0 \subset X$ from X and then half a tubular neighbourhood TX_{n-b} of $X_{n-b} \setminus X_0$. The leftover is a compact manifold with corners $\bar X$, with compact boundary parts E, W, glued along their common boundary. The halves of the tubular neighbourhoods that are not removed from X induce compatible collars $c_E : E \times [0,1) \hookrightarrow \bar X$ and $c_W : W \times [0,1) \hookrightarrow \bar X$. Compatibility means, that the restriction of c_W to $\partial W \times [0,1)$ is a collar of $\partial W = \partial E$ in E and vice versa. These collars are illustrated for the example $X = \mathrm{cone}(\mathrm{cone}(T^2))$ from above in Figure 6. Note, that W is the blowup of the link L_0 of X_0, which is itself singular (with two strata). In the example $X = \mathrm{cone}(\mathrm{cone}(T^2))$, $W = T^2 \times (0,1] = \overline{\mathrm{cone}(T^2)}$. To define the intersection forms complex $\Omega I^\bullet_{\bar p}(X)$, we need to cotruncate the intersection form complex $\Omega I^\bullet_{\bar p}(L_0)$ in degree $\bar q(n)$. This can be done by choosing a complement $\mathfrak{C}^{\bar q(n)+1}$ of the image of the boundary operator $\operatorname{im} d \in \Omega^{\bar q(n)+1}(W)$ and set

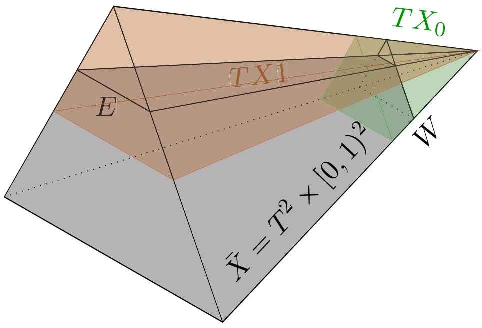


FIGURE 6. The blowup of the 3-strata space $X = \mathrm{cone}(\mathrm{cone}(T^2))$

$$\tau_{>\bar q(n)}\Omega I^\bullet_{\bar p}(L_0) := \dots \to \mathfrak{C}^{\bar q(n)+1} \to \Omega I^{\bar q(n)+2}_{\bar p}(L_0) \to \dots$$

With $ft_{>\bar q(b)}\Omega^\bullet_{\mathcal{MS}}(E)$ defined as in Section 3.1 and $\pi_E : E \times [0,1) \to E$, $\pi_W : W \times [0,1) \to W$ the first factor projections, the complex $\Omega I^\bullet_{\bar p}(X)$ in the depth two setting is then defined as follows.

$$\Omega I^\bullet_{\bar p}(X) := \Big\{ \omega \in \Omega^\bullet(\bar X) \, : c_E^* \omega = \pi_E^* \eta, \; \eta \in ft_{>\bar q(b)}\Omega^\bullet_{\mathcal{MS}}(E) \text{ and } c_W^*\omega = \pi_W^*\zeta, \; \zeta \in \tau_{>\bar q(n)}\Omega I^\bullet_{\bar p}(L_0) \Big\}$$

The cohomology groups of $\Omega I^\bullet_{\bar p}(X)$ satisfy Poincaré duality in the same sense as in the depth one setting.

Theorem 3.3.1. [17, Theorem 7.4.1] *For complementary perversities $\bar p$ and $\bar q$, integration induces nondegenerate bilinear forms*

$$\int : HI^r_{\bar p}(\bar X) \times HI^{n-r}_{\bar q}(\bar X) \to \mathbb{R}, \quad \big([\omega],[\eta]\big) \mapsto \int_{\bar X} \omega \wedge \eta.$$

4. Other approaches to intersection space homology

4.1. **Intersection space cohomology via sheaf theory.** The sheaf theoretic approach to intersection cohomology, motivated by Deligne and introduced by Goresky and MacPherson in [20], is very powerful. Not only has it been used to prove Poincaré duality and topological invariance of intersection cohomology for topological pseudomanifolds, it has also led to a proof of the Kazhdan-Lusztig conjecture via $\mathcal{D}$-modules, relating representation theory and intersection cohomology. The axiomatic definition has another advantage: It makes it easy to check, whether a new approach computes intersection cohomology or not.

With these promising results in the back of one's mind, an analogous sheaf theoretical description for intersection space cohomology is desirable. In [4, Section 6], Banagl shows that his intersection form complex $\Omega I^\bullet_{\bar{p}}$ gives rise to a complex of soft sheaves on X with global hypercohomology the de Rham cohomology $H^\bullet(\Omega I^\bullet_{\bar{p}}(X))$. Agustín-Bobadilla follow a more axiomatic approach in their paper. Based on their iterative construction of intersection space pairs, they derive a constructible complex of sheaves $\mathbf{IS}^\bullet_{\bar{p}}$ and show that its global hypercohomology is the cohomology of the intersection space pair. Moreover, in [1, Section 6], they introduce a set of properties, called the $\mathbf{IS}^\bullet_{\bar{p}}$-properties in the following, acting as an analogue to the axioms for intersection cohomology of [20, Section 3.3]. A sheaf complex satisfying these properties will be called an $\mathbf{IS}^\bullet_{\bar{p}}$-complex. There are three main differences from intersection cohomology:

(1) Except for the case of isolated singularities, an $\mathbf{IS}^\bullet_{\bar{p}}$-complex cannot be a perverse sheaf complex.
(2) The $\mathbf{IS}^\bullet_{\bar{p}}$-properties do not fix an $\mathbf{IS}^\bullet_{\bar{p}}$-complex of sheaves up to quasi-isomorphism.
(3) In unison with the other approaches to intersection space cohomology, there does not always have to be an $\mathbf{IS}^\bullet_{\bar{p}}$-complex.

If the intersection space pair exists, then the above complex $\mathbf{IS}^\bullet_{\bar{p}}$ is an $\mathbf{IS}^\bullet_{\bar{p}}$-complex. The sheaf theoretic approach is more general, though (see e.g. [1, Section 9.1]). Agustín-Bobadilla give necessary and sufficient conditions on the existence of an $\mathbf{IS}^\bullet_{\bar{p}}$-sheaf complex. Their construction is inductive, starting on the regular stratum. In the k-th step, an $\mathbf{IS}^\bullet_{\bar{p}}$-complex $(\mathbf{IS}^\bullet_{\bar{p}})_{k-1}$ on $U_k := X \setminus X_{n-k}$ can be extended to a complex satisfying the axioms on U_{k+1} if and only if the following distinguished triangle in the derived category splits.

$$\tau_{\leq \bar{q}(k)} j_k^* i_{k*} (\mathbf{IS}^\bullet_{\bar{p}})_{k-1} \longrightarrow j_k^* i_{k*} (\mathbf{IS}^\bullet_{\bar{p}})_{k-1} \longrightarrow \tau_{>\bar{q}(k)} j_k^* i_{k*} (\mathbf{IS}^\bullet_{\bar{p}})_{k-1} \overset{[+1]}{\longrightarrow}$$

The maps $i_k : U_k \hookrightarrow U_{k+1}$ and $j_k : X_{n-k-1} \setminus X_{n-k-2} \hookrightarrow U_k$ are (open and closed) inclusions. If this triangle splits, one has to choose such a splitting to proceed. The obstruction at each step might, as for the construction of the intersection space pairs, depend on all the previous choices. If X is an algebraic variety, Agustín-Bobadilla show that the construction can be lifted to the category of mixed Hodge modules on X: If an $\mathbf{IS}^\bullet_{\bar{p}}$-complex exists, it is a mixed Hodge module, i.e. its global hypercohomology groups have a mixed Hodge structure (see [1, Theorem 8.3]).

Finding an answer for the Poincaré-duality question, raised in Section 2.3, does not become easy when using the sheaf theoretic approach. Though the Verdier dual of an $\mathbf{IS}^\bullet_{\bar{p}}$-complex is an $\mathbf{IS}^\bullet_{\bar{q}}$-complex ([1, Theorem 10.1]), this does not imply global Poincaré duality. Agustín-Bobadilla have a partial answer for a two strata space X^d of dimension d with singular set X_{d-k}. If an intersection space for a given perversity $\bar{p}$ exists, the intersection space complexes $\mathbf{IS}^\bullet_{\bar{p}}$ are parametrized by the vector space

$$E_{\bar{p}} := \hom\left(\tau_{>\bar{q}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}|_{U_k}, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*} \mathbb{Q}_{U_k}\right)$$

(see [1, Corollary 7.6]). An element $\mathbf{IS}^\bullet_{\bar{p}} \in E_{\bar{p}}$ is called a $GIS_{\bar{p}}$ or *general intersection space complex* of X with perversity $\bar{p}$, if the hypercohomology groups of $\mathbf{IS}^\bullet_{\bar{p}}$ are minimal compared to the hypercohomology groups of all complexes in $E_{\bar{p}}$, that is $\dim\left(\mathbb{H}^i(X, \mathbf{IS}^\bullet_{\bar{p}}\right) = \min_{\mathbf{S}^\bullet \in E_{\bar{p}}} \mathbb{H}^i(X, \mathbf{S}^\bullet)$. Provided the existence of such a $GIS_{\bar{p}}$ $\mathbf{IS}^\bullet_{\bar{p}}$, Agustín-Bobadilla prove that the Verdier dual $\mathbf{IS}^\bullet_{\bar{q}} := \mathcal{D}\mathbf{IS}^\bullet_{\bar{p}}[-d]$ is a $GIS_{\bar{q}}$, with $\bar{q}$ the dual perversity of $\bar{p}$, and that there is an isomorphism

$$\mathbb{H}^i(X, \mathbf{IS}^\bullet_{\bar{p}}) \cong \hom\left(\mathbb{H}^{d-i}(X, \mathbf{IS}^\bullet_{\bar{q}}), \mathbb{R}\right)$$

of $\mathbb{Q}$-vector spaces for all i, see [1, Theorem 10.6]. It is not clear, what the conditions are that determine the existence of such a $GIS_{\bar{p}}$-complex and how it can be constructed. It is interesting to know, whether the intersection space of Banagl-Chriestenson in the setting of [5] or the intersection form complex $\Omega I^\bullet_{\bar{p}}$ of [4] give rise to $GIS_{\bar{p}}$-complexes. Because of the iterative nature of Agustín-Bobadilla's construction, a generalization of their Poincaré duality theorem might need a sequence of complexes with minimality conditions for the hypercohomology groups at each step of the construction. This seems involved and it might be a good idea to check, whether the complexes $\widetilde{\Omega I}^\bullet_{\bar{p}}$ and $\Omega I^\bullet_{\bar{p}}$ of [17] satisfy such minimality conditions, first.

4.2. **Geske's algebraic intersection spaces.** In [19], Geske establishes a different approach to intersection space homology. He uses a local to global technique to construct a chain complex, the homology of which generalizes intersection space homology and satisfies Poincaré duality over complementary perversities, at least if certain duality obstructions vanish. His approach is applicable to all compact orientable Whitney stratified pseudomanifolds that are contained in a real/complex analytic manifold. In particular, that class contains all complex projective varieties. His starting point is the following observation: The (fiberwise) homology truncations of [2, 5] are related to the intersection homology groups of the tubular neighbourhoods T of the singular strata. For spaces with isolated singularities, with link truncation $f : \partial T_{\leq \bar{q}(n)} \to \partial T$, the composition $H_\bullet(\partial T_{\leq \bar{q}(n)}) \xrightarrow{f_*}$

$H_\bullet(\partial T) \to IH^{\bar p}_\bullet(T)$ is an isomorphism. For the pseudomanifolds considered in [5], see Section 2.1, with singular set Σ of codimension b, the composition $H_\bullet(ft_{\leq \bar q(b)}\partial T) \to H_\bullet(\partial T) \to IH^{\bar p}_\bullet(T)$ of $(F_{\leq \bar q(b)})_*$ and the induced map of the inclusion $\partial T \hookrightarrow T$ is also an isomorphism, see [5, Proposition 6.5].

For a more general compact, orientable Whitney stratified pseudomanifold X, contained in a real/complex analytic manifold, the theory of subanalytic sets gives rise to a tubular neighbourhood T of the singular set $\operatorname{Sing}(X)$. If the stratification depth is greater than one, this tubular neighbourhood is not related to the tubular neighbourhoods induced by the Thom-Mather control data, a priori. For depth one pseudomanifolds, both notions are similar. Geske points out, that one of the obstructions to the existence of an intersection space is the surjectivity of the map $H_\bullet(\partial T) \to IH^{\bar p}_\bullet(T)$ on homology. In many cases, this map is not surjective, so Geske replaces $IH^{\bar p}_\bullet(T)$ by $\operatorname{im}\left(H_\bullet(\partial T) \to IH^{\bar p}_\bullet(T)\right)$. He introduces the notion of a $\bar p$ algebraic intersection approximation for T, which is a pair $(A_\bullet, f_\bullet)$, consisting of a chain complex $A_\bullet$ and a chain map $f_\bullet : A_\bullet \to C_\bullet(\partial T)$ such that the composition

$$H_\bullet(A_\bullet) \xrightarrow{f_*} H_\bullet(\partial T) \to \operatorname{im}\left(H_\bullet(\partial T) \to IH^{\bar p}_\bullet(T)\right) \tag{1}$$

on homology is an isomorphism (with field coefficients for all homology groups). Such $\bar p$ algebraic intersection approximations always exist, see [19, Proposition 4.5]. For two complementary perversities $\bar p$ and $\bar q$ and corresponding algebraic intersection approximations $(A^{\bar p}_\bullet, f^{\bar p}_\bullet)$ and $(A^{\bar q}_\bullet, f^{\bar q}_\bullet)$, there is always a local duality isomorphism $D : H_r(\operatorname{cone} f^{\bar p}_\bullet) \xrightarrow{\cong} H^{n-r-1}(A^{\bar q}_\bullet)$. To lift that duality isomorphism to the algebraic intersection space cohomology groups, the following diagram must be commutative.

$$\begin{array}{ccc} H^{n-r-1}(\partial T) & \longrightarrow & H^{n-r-1}(A^{\bar q}_\bullet) \\ \cong\uparrow & & \cong\uparrow D \\ H_r(\partial T) & \longrightarrow & H_r(\operatorname{cone}(f^{\bar p}_\bullet)) \end{array}$$

Geske calls that the local duality obstructions, which makes sense since these obstructions are equivalent to Banagl-Chriestienson's local duality obstruction for depth one pseudomanifolds. This follows from [5, Proposition 6.9] and [19, Proposition 4.7]. For Witt spaces of even dimension with $\bar p = \bar q = \bar m = \underline{m}$ the middle perversity, the algebraic intersection approximations to T can be chosen such that this obstruction vanishes, see [19, Theorem 4.9]. Globally, Geske defines the algebraic intersection space $I_{f^{\bar p}_\bullet}X$ with respect to the perversity $\bar p$ as the algebraic mapping cone of the composition $A^{\bar p}_\bullet \to C_\bullet(\partial T) \to C_\bullet(X \setminus \mathring{T})$ with $\mathring{T} = T \setminus \partial T$ of $f^{\bar p}_\bullet$ followed by a subcomplex inclusion. By a purely algebraic argument, he can then lift the local duality isomorphism to a global Poincaré duality isomorphism if the local duality obstructions vanish.

Theorem 4.2.1. [19, Theorem 5.1] *If the local duality obstructions vanish for* $(A^{\bar{p}}_{\bullet}, f^{\bar{p}}_{\bullet})$ *and* $(A^{\bar{q}}_{\bullet}, f^{\bar{q}}_{\bullet})$*, there is a non-canonical Poincaré duality isomorphism*

$$D : H_r(I_{f^{\bar{p}}_{\bullet}}X) \xrightarrow{\cong} H^{n-r}(I_{f^{\bar{q}}_{\bullet}}X).$$

Given the differences between his and Agustín-Bobadilla's approaches to generalizing intersection space homology to pseudomanifolds of stratification depth greater than one, Geske calculated the homology groups of both approaches for the example of a projective cone of an irreducible degree three nodal hypersurface with one isolated singularity. He finds that the homology of the intersection space pair vanishes, while the homology of the algebraic intersection space does not, see [19, Section 6]. That means that the two theories do not compute the same homology.

The duality isomorphism of Theorem 4.2.1 is constructed by choosing sections

$$s^{\bar{p}}_r : \mathrm{im}\left(H_r(I_{f^{\bar{p}}_{\bullet}}X) \to H_r(X - \mathring{T}, \partial T)\right) \to H_r(I_{f^{\bar{p}}_{\bullet}}X) \quad \text{and}$$
$$r^i_{\bar{q}} : \mathrm{im}\left(H^i(I_{f^{\bar{q}}_{\bullet}}X) \to H^i(X \setminus \mathring{T})\right) \to H^i(I_{f^{\bar{q}}_{\bullet}}X).$$

Geske proves in [19, Theorem 6.5], that for a Witt space X of even dimension $n = 2m$, there are choices of these sections for $\bar{p} = \bar{q} = \bar{m} = \underline{m}$ such that the signature induced by the Poincaré duality isomorphism is exactly the Goresky-MacPherson-Siegel signature for intersection homology, see [24].

5. Open Questions

The end of this survey article is a list of open questions concerning intersection space cohomology theory.

5.1. **Previously stated questions.** In [7, Section 5], Banagl and Maxim conclude their paper with four open questions. The first one, asking for a sheaf theoretic description of intersection space cohomology, was answered in [1]. As was pointed out in Section 4.1, this characterization is not similar to Goresky-MacPherson's axiomatic description, though. Banagl-Maxim's third question, asking for a canonical mixed Hodge structure on intersection space cohomology of a complex projective variety, was also answered in [1, Section 8]. Note, that since Agustín-Bobadilla's $\mathbf{IS}^{\bullet}_{\bar{p}}$-sheaf complex depends on choices and is not canonical, thus, the same is true for the mixed Hodge structure on the global hypercohomology. Note also, that there is an alternative approach to mixed Hodge structures for isolated singularities by Klimczak, covered in [22]. Their other two question, asking for (weak and hard) Lefschetz theorems for intersection space cohomology and for a generalization on their results of the relation with smooth deformations of singularities, have not been answered yet.

5.2. **Additional open questions.**

(1) To which choices in the splittings of the relevant distinguished triangles of Agustín-Bobadilla corresponds the sheafification of the author's intersection form complex $\Omega I^\bullet_{\bar{p}}$ for depth two pseudomanifolds (sheafified similar to Banagl's construction in [4, Section 6])?
(2) Is there a de Rham theorem for depth two pseudomanifolds, relating the de Rham cohomology groups of $\Omega I^\bullet_{\bar{p}}$ and the cohomology of Agustín-Bobadilla's intersection space pairs or is $\Omega I^\bullet_{\bar{p}}$ related to Geske's approach? (Geske's and Agustín-Bobadilla's approach seem to be different as mentioned in Section 4.2)
(3) Do the approaches to intersection space cohomology of [4, 5, 17, 19] give rise to $GIS_{\bar{p}}$-complexes in the sense of Agustín-Bobadilla (see the end of Section 4.1)?
(4) As mentioned, the de Rham approaches of Banagl and the author seem to be related to the de Rham approach to intersection homology of [12]. But there are various other de Rham approaches to intersection homology, see [8], [9], [10, 18] or [11], which motivates the following question: Are there de Rham approaches to intersection space cohomology, comparable to the alternative de Rham descriptions of intersection cohomology? In particular, is there an approach related to Brasselet-Legrand's β-bounded forms of [11], where the poles of the forms on the singular strata are controlled? Does Geske's observation, taking into account the dualized version of (1) on cohomology, help to construct such a complex?
(5) Can Klimczak's and Wrazidlo's fundamental class constructions for depth one spaces, inducing a Poincaré duality isomorphism via cap product, be generalized to the pseudomanifolds considered by Agustín-Bobadilla in [1]? If so, what are the obstructions for the gluing processes and how many cells have to been glued to the space?

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