Martin's Maximum and the Diagonal Reflection Principle*

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Abstract

We prove that Martin's Maximum does not imply the Diagonal Reflection Principle for stationary subsets of $[\omega_2]^\omega$.

1 Introduction

In Foreman-Magidor-Shelah [5], it was shown that Martin's Maximum MM implies the following stationary reflection principle, which is called the Weak Reflection Principle:

$\mathsf{WRP} \equiv$ For any cardinal $\lambda \geq \omega_2$ and any stationary $X \subseteq [\lambda]^\omega$, there is $R \in [\lambda]^{\omega_1}$ with $R \supseteq \omega_1$ such that $X \cap [R]^\omega$ is stationary in $[R]^\omega$.

WRP is known to have many interesting cosequences such as Chang's Conjecture (Foreman-Magidor-Shelah [5]), the presaturation of the non-stationary ideal over ω_1 (Feng-Magidor [4]), $2^\omega \leq \omega_2$ (folklore) and the Singular Cardinal Hypothesis (Shelah [12]).

As for stationary reflection principles, simultaneous reflection is often discussed. Larson [10] proved that MM also implies the following simultaneous reflection principle of ω_1-many stationary sets:

$\mathsf{WRP}_{\omega_1} \equiv$ For any cardinal $\lambda \geq \omega_2$ and any sequence $\langle X_\xi \mid \xi < \omega_1 \rangle$ of stationary subsets of $[\lambda]^\omega$, there is $R \in [\lambda]^{\omega_1}$ with $R \supseteq \omega_1$ such that $X_\xi \cap [R]^\omega$ is stationary in $[R]^\omega$ for all $\xi < \omega_1$.

Cox [2] formulated the following strengthening of WRP_{ω_1}, which is called the Diagonal Reflection Principle:

$\mathsf{DRP} \equiv$ For any cardinal $\lambda \geq \omega_2$ and any sequence $\langle X_\alpha \mid \alpha < \lambda \rangle$ of stationary subsets of $[\lambda]^\omega$, there is $R \in [\lambda]^{\omega_1}$ with $R \supseteq \omega_1$ such that $X_\alpha \cap [R]^\omega$ is stationary in $[R]^\omega$ for all $\alpha \in R$.

Recently, Fuchino-Ottenbreit-Sakai [6] proved that a variation of DRP is equivalent to some variation of the downward Löwenheim-Skolem theorem of the stationary logic. Cox [2] also introduced the following weakning of DRP, where $X \subseteq [\lambda]^\omega$ is said to be *projectively stationary* if the set $\{x \in X \mid x \cap \omega_1 \in S\}$ is stationary in $[\lambda]^\omega$ for any stationary $S \subseteq \omega_1$:

$\mathsf{wDRP} \equiv$ For any cardinal $\lambda \geq \omega_2$ and any sequence $\langle X_\alpha \mid \alpha < \lambda \rangle$ of projectively stationary subsets of $[\lambda]^\omega$, there is $R \subseteq [\lambda]^{\omega_1}$ with $R \supseteq \omega_1$ such that $X_\alpha \cap [R]^\omega$ is stationary in $[R]^\omega$ for all $\alpha \in R$.

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Cox [2] proved that MM implies wDRP, but it remained open whether MM implies DRP. In this paper, we prove that MM does not imply DRP. In fact, we prove slightly more.

To state our result, we recall $+$-versions of the forcing axiom. For a class Γ of forcing notions and a cardinal $\mu \leq \omega_1$, $\mathsf{MA}^{+\mu}(\Gamma)$ is the following statement:

$\mathsf{MA}^{+\mu}(\Gamma) \equiv$ For any $\mathbb{P} \in \Gamma$, any sequence $\langle D_\xi \mid \xi < \omega_1 \rangle$ of dense subsets of $\mathbb{P}$ and any sequence $\langle \dot{S}_\eta \mid \eta < \mu \rangle$ of $\mathbb{P}$-names of stationary subsets of ω_1, there is a filter $g \subseteq \mathbb{P}$ such that

(i) $g \cap D_\xi \neq \emptyset$ for any $\xi < \omega_1$,

(ii) $\dot{S}_\eta^g = \{\alpha < \omega_1 \mid \exists p \in g,\ p \Vdash_{\mathbb{P}} \text{“}\alpha \in \dot{S}_\eta\text{”}\}$ is stationary in ω_1 for all $\eta < \mu$.

Let $\mathsf{MA}^{+\mu}(\sigma\text{-closed})$ denote $\mathsf{MA}^{+\mu}(\Gamma)$ for the class Γ of all σ-closed forcing notions. Also, let $\mathsf{MM}^{+\mu}$ denote $\mathsf{MA}^{+\mu}(\Gamma)$ for the class Γ of all ω_1-stationary preserving forcing notions. It is well-known that $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ holds if a supercompact cardinal is Lévy collapsed to ω_2 and that $\mathsf{MM}^{+\omega_1}$ holds in the standard model of MM constructed in Foreman-Magidor-Shelah [5].

Cox [2] proved that $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ implies DRP. So $\mathsf{MM}^{+\omega_1}$ implies DRP. In this paper, we prove that $\mathsf{MM}^{+\omega}$ does not imply DRP:

Main Theorem. *Assume $\mathsf{MM}^{+\omega}$ holds. Then there is a forcing extension in which $\mathsf{MM}^{+\omega}$ remains to hold, but DRP fails at $[\omega_2]^\omega$.*

Our proof of the Main Theorem is based on the proof of the classical result, due to Beaudoin [1] and Magidor, that the Proper Forcing Axiom does not imply the reflection of stationary subsets of the set $\{\alpha \in \omega_2 \mid \mathrm{cof}(\alpha) = \omega\}$. Similar arguments are used in König-Yoshinobu [8], Yoshinobu [13], [14] and Cox [3], to separate reflection principles from strong forcing axioms.

We will prove the Main Theorem in Section 3. In Section 2, we will present our notation and basic facts used in this paper.

2 Preliminaries

Here we present our notation and basic facts. See Jech [7] for those which are not mentioned here.

First, we recall the notion of stationary sets in $[W]^\omega$. Let W be a set with $\omega_1 \subseteq W$. $Z \subseteq [W]^\omega$ is said to be *club* in $[W]^\omega$ if Z is $\subseteq$-cofinal in $[W]^\omega$, and $\bigcup_{n\in\omega} x_n \in Z$ for any $\subseteq$-increasing sequence $\langle x_n \mid n < \omega \rangle$ of elements of Z. $X \subseteq [W]^\omega$ is said to be *stationary* in $[W]^\omega$ if $X \cap Z \neq \emptyset$ for any club $Z \subseteq [W]^\omega$. For $S \subseteq \omega_1$, S is stationary in ω_1 in the usual sense if and only if S is stationary in $[\omega_1]^\omega$ in the above sense.

We will use the following standard facts without any reference. Proofs can be found also in Jech [7].

Fact 2.1 ((1) Kueker [9], (2) Menas [11])**.** *Suppose W is a set $\supseteq \omega_1$ and X is a subset of $[W]^\omega$.*

(1) X is stationary if and only if for any function $F : [W]^{<\omega} \to W$ there is a non-empty $x \in X$ which is closed under F, i.e. $F(a) \in x$ for all $a \in [x]^{<\omega}$.

(2) Suppose $W' \supseteq W$. Then X is stationary in $[W]^\omega$ if and only if the set $\{x' \in [W']^\omega \mid x' \cap W \in X\}$ is stationary in $[W']^\omega$.

Here we slightly simplify DRP at $[\omega_2]^\omega$.

Lemma 2.2. *Assume DRP at $[\omega_2]^\omega$. Then, for any sequence $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ of stationary subsets of $[\omega_2]^\omega$, there is $\delta \in \omega_2 \setminus \omega_1$ such that $X_\alpha \cap [\delta]^\omega$ is stationary in $[\delta]^\omega$ for all $\alpha < \delta$.*

Proof. Suppose $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ is a sequence of stationary subsets of $[\omega_2]^\omega$. We find δ as in the lemma.

For each $\beta < \omega_2$, take a surjection $\pi_\beta : \omega_1 \to \beta$. Let Z be the set of all $x \in [\omega_2]^\omega$ such that $x \cap \omega_1 \in \omega_1$ and x is closed under π_β for all $\beta \in x$. Then, Z is club in $[\omega_2]^\omega$. Moreover, it is easy to see that if $\omega_1 \subseteq R \in [\omega_2]^{\omega_1}$, and $Z \cap [R]^\omega$ is $\subseteq$-cofinal in $[R]^\omega$, then $R \in \omega_2 \setminus \omega_1$.

By shrinking X_0 if necessary, we may assume that $X_0 \subseteq Z$. By DRP at $[\omega_2]^\omega$, take $R \in [\omega_2]^{\omega_1}$ including ω_1 such that $X_\alpha \cap [R]^\omega$ is stationary for all $\alpha \in R$. Then, $R \in \omega_2 \setminus \omega_1$ since $Z \cap [R]^\omega$ is $\subseteq$-cofinal in $[R]^\omega$. So, $\delta := R$ is as desired. $\square$

Next, we present our notation and basic facts about forcing. Suppose $\mathbb{P}$ is a forcing notion and M is a set. We say that $g \subseteq \mathbb{P} \cap M$ is *M-generic* if $g \cap D \neq \emptyset$ for any dense $D \subseteq \mathbb{P}$ with $D \in M$.

We will use the following well-known fact about forcing axioms:

Fact 2.3 (Woodin [15])**.** *Let Γ be a class of forcing notions and μ be a cardinal $\leq \omega_1$, and assume $\mathsf{MA}^{+\mu}(\Gamma)$ holds. Suppose $\mathbb{P} \in \Gamma$ and $\langle \dot{T}_\xi \mid \xi < \mu \rangle$ is a sequence of $\mathbb{P}$-names for stationary subsets of ω_1. Then, for any regular cardinal θ with $\mathbb{P} \in \mathcal{H}_\theta$ and any $A \in [\mathcal{H}_\theta]^{\omega_1}$, there are $M \in [\mathcal{H}_\theta]^{\omega_1}$ and $g \subseteq \mathbb{P} \cap M$ with the following properties.*

(i) $A \subseteq M \prec \langle \mathcal{H}_\theta, \in \rangle$.

(ii) *g is an M-generic filter on $\mathbb{P} \cap M$.*

(iii) *$\dot{T}_\xi^g$ is stationary in ω_1 for any $\xi < \mu$.*

We will also use forcing notions for shooting club sets. For an ordinal $\lambda \geq \omega_1$ and a subset X of $[\lambda]^\omega$, let $\mathbb{R}(X)$ denote the poset of all $\subseteq$-increasing continuous function from some countable successor ordinal to X, which is ordered by reverse inclusions. The following is standard:

Lemma 2.4. *Suppose X is a stationary subset of $[\lambda]^\omega$ for some ordinal $\lambda \geq \omega_1$.*

(1) A forcing extension by $\mathbb{R}(X)$ adds no new countable sequences of ordinals. So it preserves ω_1.

(2) In $V^{\mathbb{R}(X)}$, X contains a club subset of $[\lambda]^\omega$.

(3) In V, suppose $Y \subseteq X$ and Y is stationary in $[\lambda]^\omega$. Then Y remains stationary in $V^{\mathbb{R}(X)}$.

Proof. Let $\mathbb{R}$ denote $\mathbb{R}(X)$. Before starting, note that the set $\{r \in \mathbb{R} \mid \exists \xi \in \mathrm{dom}(r),\ r(\xi) \supseteq x\}$ is dense in $\mathbb{R}$ for any $x \in [\lambda]^\omega$, since X is $\subseteq$-cofinal in $[\lambda]^\omega$.

First, we prove (1) and (3). We work in V. Suppose $r \in \mathbb{R}$, $\mathcal{D}$ is a countable family of dense open subsets of $\mathbb{R}$ and $\dot{F}$ is an $\mathbb{R}$-name for a function from $[\lambda]^{<\omega}$ to λ. It suffices to find $r^* \leq r$ and $y \in Y$ such that $r^* \in \bigcap \mathcal{D}$ and r^* forces y to be closed under $\dot{F}$.

Take a sufficiently large regular cardinal θ. Since Y is stationary, there is a countable $M \prec \langle \mathcal{H}_\theta, \in \rangle$ such that $\{\lambda, X, r, \dot{F}\} \cup \mathcal{D} \subseteq M$ and $y := M \cap \lambda \in Y$. Then, we can construct a descending sequence $\langle r_n \mid n < \omega \rangle$ in $\mathbb{R} \cap M$ such that $r_0 = r$ and $\{r_n \mid n < \omega\}$ is M-generic. Note that any lower bound of $\{r_n \mid n < \omega\}$ forces y to be closed under $\dot{F}$ by the M-genericity of $\{r_n \mid n < \omega\}$.

Let $r' := \bigcup_{n<\omega} r_n$ and $\zeta := \mathrm{dom}(r')$. Then, using the fact mentioned at the beginning, it is easy to check that ζ is a limit ordinal and $\bigcup_{\xi<\zeta} r'(\xi) = y$. Let r^* be an extension of r' such that $\mathrm{dom}(r^*) = \zeta + 1$ and $r^*(\zeta) = y$. Then $r^* \in \mathbb{R}$, and r^* is a lower bound of $\{r_n \mid n < \omega\}$. So r^* and y are as desired.

Next, we check (2). By (1), the definition of $\mathbb{R}$ and the fact mentioned at the beginning, if G is an $\mathbb{R}$-generic filter over V, then $\mathrm{range}(\bigcup G)$ is a club subset of $[\lambda]^\omega$ consisting of elements of X. So (2) holds. $\square$

3 Proof of Main Theorem

Here we prove the Main Theorem. Throughout this section, assume that $\mathsf{MM}^{+\omega}$ holds in the ground model V.

We construct a forcing notion which preserves $\mathsf{MM}^{+\omega}$ and adds a counter-example $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ of the consequence of Lemma 2.2. Here recall that MM implies wDRP. So we must arrange our forcing notion so that each X_α is not projectively stationary. For some technical reason, we also make $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ pairwise disjoint.

Recall the fact, due to Foreman-Magidor-Shelah [5], that MM implies $2^{\omega_1} = \omega_2$. In V, fix an enumeration $\langle S_\alpha \mid \alpha < \omega_2 \rangle$ of all stationary subsets of ω_1. Let $\mathbb{P}$ be the following forcing notion:

- $\mathbb{P}$ consists of all functions p such that
  - (i) $p : \delta_p \times [\delta_p]^\omega \to 2$ for some $\delta_p < \omega_2$,
  - (ii) for any $\alpha < \delta_p$, $X_{p,\alpha} := \{x \in [\delta_p]^\omega \mid p(\alpha, x) = 1\}$ has size $\leq \omega_1$,
  - (iii) $x \cap \omega_1 \in S_\alpha$ for any $\alpha < \delta_p$ and any $x \in X_{p,\alpha}$,
  - (iv) $X_{p,\alpha} \cap X_{p,\beta} = \emptyset$ for any distinct $\alpha, \beta < \delta_p$,
  - (v) for any $\delta \in \delta_p + 1 \setminus \omega_1$, there is $\alpha < \delta$ with $X_{p,\alpha} \cap [\delta]^\omega$ non-stationary in $[\delta]^\omega$.
- $p \leq p'$ in $\mathbb{P}$ if $p \supseteq p'$.

We observe basic properties of $\mathbb{P}$. Note that a forcing extension by $\mathbb{P}$ preserves all cardinals by (1) and (3) of the following lemma.

Lemma 3.1. *(1)* $|\mathbb{P}| = \omega_2$.

(2) $\mathbb{P}$ *is* σ*-closed.*

(3) A forcing extension by $\mathbb{P}$ *adds no new sequences of ordinals of length* ω_1*.*

(4) For any $p \in \mathbb{P}$ *and any* $\delta < \omega_2$*, there is* $p' \leq p$ *with* $\delta \leq \delta_{p'}$*.*

Proof. (1) This is clear from the definition of $\mathbb{P}$, especially the property (ii) of its conditions, and the fact that $2^{\omega_1} = \omega_2$ in V.

(4) Suppose $p \in \mathbb{P}$ and $\delta < \omega_2$. We may assume $\delta_p \leq \delta$. Let $p' : \delta \times [\delta]^\omega \to 2$ be an extension of p' such that $p'(\alpha, x) = 0$ for all $\langle \alpha, x \rangle \notin \delta_p \times [\delta_p]^\omega$. It suffices to prove that $p' \in \mathbb{P}$. We only check that p' satisfies the property (v) of conditions of $\mathbb{P}$. The other properties are easily checked.

Take an arbitrary $\gamma \in \delta + 1 \setminus \omega_1$. We find $\alpha < \delta$ with $X_{p',\alpha} \cap [\gamma]^\omega$ is non-stationary. If $\gamma \leq \delta_p$, then we can find such α since $p \in \mathbb{P}$ and $p \subseteq p'$. Suppose $\gamma > \delta_p$. Then $Z := [\gamma]^\omega \setminus [\delta_p]^\omega$ is club in $[\gamma]^\omega$, and $X_{p',\alpha} \cap Z = \emptyset$ for any $\alpha < \gamma$. So any $\alpha < \gamma$ is as desired in this case.

(2) Suppose $\langle p_n \mid n < \omega \rangle$ is a descending sequence in $\mathbb{P}$. We find a lower bound p^* of $\{p_n \mid n < \omega\}$ in $\mathbb{P}$. We may assume that $\langle p_n \mid n < \omega \rangle$ is not eventually constant.

Let $\delta_n := \delta_{p_n}$ for each $n < \omega$. Let $\delta^* := \bigcup_{n<\omega} \delta_n$, and let $p^* : \delta^* \times [\delta^*]^\omega \to 2$ be an extension of $\bigcup_{n\in\omega} p_n$ such that $p^*(\alpha, x) = 0$ for all $\alpha < \delta^*$ and all $x \in [\delta^*]^\omega \setminus \bigcup_{n\in\omega} [\delta_n]^\omega$. Note that $X_{p^*,\alpha}$ is non-stationary in $[\delta^*]^\omega$ for any $\alpha < \delta^*$ since $Z := [\delta^*]^\omega \setminus \bigcup_{n<\omega} [\delta_n]^\omega$ is club in $[\delta^*]^\omega$ and $X_{p^*,\alpha} \cap Z = \emptyset$. Then it is easy to see that p^* is as desired.

(3) Suppose $p \in \mathbb{P}$ and $\langle D_\xi \mid \xi < \omega_1 \rangle$ is a sequence of dense open subsets of $\mathbb{P}$. It suffices to find $p^* \leq p$ with $p^* \in \bigcap_{\xi < \omega_1} D_\xi$.

We recursively construct a strictly descending sequence $\langle p_\xi \mid \xi < \omega_1 \rangle$ in $\mathbb{P}$ as follows. For each $\xi < \omega_1$, we let δ_ξ denote δ_{p_ξ}. First, let $p_0 := p$. If p_ξ has been taken, then take $p_{\xi+1} < p_\xi$ with $p_{\xi+1} \in D_\xi$. Suppose ξ is a limit ordinal $< \omega_1$ and $\langle p_\eta \mid \eta < \xi \rangle$ has been constructed. Then define p_ξ as in the proof of (2). That is, let $\delta_\xi := \bigcup_{\eta < \xi} \delta_\eta$, and let $p_\xi : \delta_\xi \times [\delta_\xi]^\omega \to 2$ be an extension of $\bigcup_{\eta < \xi} p_\eta$ such that $p_\xi(\alpha, x) = 0$ for all $\alpha < \delta_\xi$ and all $x \in [\delta_\xi]^\omega \setminus \bigcup_{\eta < \xi} [\delta_\eta]^\omega$. Then p_ξ is a lower bound of $\{p_\eta \mid \eta < \xi\}$ in $\mathbb{P}$.

We have constructed $\langle p_\xi \mid \xi < \omega_1 \rangle$. Let $\delta^* := \sup_{\xi < \omega_1} \delta_\xi$ and $p^* := \bigcup_{\xi < \omega_1} p_\xi$. Here note that $[\delta^*]^\omega = \bigcup_{\xi < \omega_1} [\delta_\xi]^\omega$. So $p^* : \delta^* \times [\delta^*]^\omega \to 2$. Note also that $X_{p^*,0}$ is non-stationary in $[\delta^*]^\omega$ since

$$Z := \{x \in [\delta^*]^\omega \mid x \subseteq \delta_\xi = \sup(x) \text{ for some limit } \xi < \omega_1\}$$

is club in $[\delta^*]^\omega$ and that $X_{p^*,0} \cap Z = \emptyset$ by the construction of p_ξ for a limit $\xi < \omega_1$. Then, it is easy to check that p^* is as desired. □

Let $\dot{G}$ be the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter. For $\alpha < \omega_2$, let $\dot{X}_\alpha$ be the $\mathbb{P}$-name for the set

$$\{x \in [\omega_2]^\omega \mid \exists p \in \dot{G},\ p(\alpha, x) = 1\}.$$

Lemma 3.2. *For each $\alpha < \omega_2$, $\dot{X}_\alpha$ is stationary in $[\omega_2]^\omega$ in $V^{\mathbb{P}}$.*

Proof. We work in V. Take an arbitrary $\alpha < \omega_2$. Suppose $p \in \mathbb{P}$ and $\dot{F}$ is a $\mathbb{P}$-name for a function from $[\omega_2]^{<\omega}$ to ω_2. It suffices to find $p^* \leq p$ and $x \in [\omega_2]^{<\omega}$ such that $p^* \Vdash_{\mathbb{P}}$ "$x \in \dot{X}_\alpha \wedge x$ is closed under $\dot{F}$".

Take a sufficiently large regular cardinal θ and a countable $M \prec \langle \mathcal{H}_\theta, \in \rangle$ such that $\alpha, \mathbb{P}, p, \dot{F} \in M$ and $M \cap \omega_1 = \alpha$. Let $x := M \cap \omega_2$. We can take a descending sequence $\langle p_n \mid n < \omega \rangle$ in $\mathbb{P} \cap M$ such that $p_0 = p$ and $\{p_n \mid n < \omega\}$ is M-generic. Note that any lower bound of $\{p_n \mid n < \omega\}$ forces x to be closed under $\dot{F}$ by the M-genericity. For each $n < \omega$, let $\delta_n := \delta_{p_n}$. Note that $\delta_n \in M \cap \omega_2$ for each $n < \omega$ and that $\delta^* := \sup_{n<\omega} \delta_n = \sup(M \cap \omega_2)$ by Lemma 3.1 (4).

Let $p^* : \delta^* \times [\delta^*]^\omega \to 2$ be an extension of $\bigcup_{n<\omega} p_n$ such that $p^*(\alpha, x) = 1$ and $p^*(\beta, y) = 0$ for any $\beta < \delta^*$ and any $y \in [\delta^*]^\omega \setminus \bigcup_{n<\omega} [\delta_n]^\omega$ with $\langle \beta, y \rangle \neq \langle \alpha, x \rangle$. Then, it is easy to check that p^* and x are as desired. □

The following is immediate from Lemma 2.2, 3.1, 3.2 and the property (v) of conditions of $\mathbb{P}$:

Corollary 3.3. DRP *at ω_2 fails in $V^{\mathbb{P}}$.*

We must show that $\mathbb{P}$ preserves $\mathsf{MM}^{+\omega}$. The following lemma is a key:

Lemma 3.4. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for an ω_1-stationary preserving forcing notion and $\langle \dot{T}_n \mid n < \omega \rangle$ be a sequence of $\mathbb{P} * \dot{\mathbb{Q}}$-names for stationary subsets of ω_1. Then there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot{\gamma}$ of an ordinal $< \omega_2^V$ such that if we let*

$$\mathbb{S} := \mathbb{P} * \dot{\mathbb{Q}} * \mathbb{R}([\omega_2^V]^\omega \setminus \dot{X}_{\dot{\gamma}}),$$

then all elements of $\{S_\alpha \mid \alpha < \omega_2^V\} \cup \{\dot{T}_n \mid n < \omega\}$ remain stationary in $V^{\mathbb{S}}$.

Proof. Let $\lambda := \omega_2^V$. Suppose $G * H$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter over V. In $V[G * H]$, let $X_\alpha := \dot{X}_\alpha^G$ for $\alpha < \lambda$ and $T_n := \dot{T}_n^{G*H}$ for $n < \omega$. Moreover, let $\mathbb{R}_\alpha$ denote $\mathbb{R}([\lambda]^\omega \setminus X_\alpha)$ for $\alpha < \lambda$. In $V[G * H]$, we find $\gamma < \lambda$ such that $\mathbb{R}_\gamma$ forces all elements of $\{S_\alpha \mid \alpha < \lambda\} \cup \{T_n \mid n < \omega\}$ stationary. Here note that all S_α and

T_n are stationary in $V[G * H]$ by the fact that $\mathbb{P} * \dot{\mathbb{Q}}$ is ω_1-stationary preserving and the assumption on $\langle \dot{T}_n \mid n < \omega \rangle$.

We work in $V[G * H]$. For $S \subseteq \omega_1$, let $\bar{S} := \{x \in [\lambda]^\omega \mid x \cap \omega_1 \in S\}$. For $X, Y \subseteq [\lambda]^\omega$, we write $X \subseteq^* Y$ if $X \setminus Y$ is non-stationary in $[\lambda]^\omega$. By Lemma 2.4, for $S \subseteq \omega_1$ and $\alpha < \lambda$, $\mathbb{R}_\alpha$ does not force $S \subseteq \omega_1$ stationary if and only if $\bar{S} \subseteq^* X_\alpha$.

Since $\langle X_\alpha \mid \alpha < \lambda \rangle$ is pairwise disjoint, for each $n < \omega$ there is at most one $\alpha < \lambda$ with $\bar{T}_n \subseteq^* X_\alpha$. Since $|\lambda| \geq \omega_1$, we can take $\beta < \lambda$ such that $\bar{T}_n \not\subseteq^* X_\beta$ for any $n < \omega$. Then $\mathbb{R}_\beta$ forces T_n stationary for all $n < \omega$. Thus, if $\mathbb{R}_\beta$ also forces S_α stationary for all $\alpha < \lambda$, then $\gamma := \beta$ is as desired.

Assume there is $\alpha < \lambda$ such that $\mathbb{R}_\beta$ does not force S_α stationary. By replacing α with α' such that $S_{\alpha'} \subseteq S_\alpha$ if necessary, we may assume that $\alpha \neq \beta$. Here note that $X_\alpha \subseteq \bar{S}_\alpha$ by the property (iii) of conditions of $\mathbb{P}$. Then, $X_\alpha \subseteq \bar{S}_\alpha \subseteq^* X_\beta$ and $X_\alpha \cap X_\beta = \emptyset$. Hence X_α is non-stationary in $[\lambda]^\omega$. Thus $\mathbb{R}_\alpha$ is ω_1-stationary preserving, and so $\gamma := \alpha$ is as desired. $\square$

Now, we can prove that $\mathbb{P}$ preserves $\mathsf{MM}^{+\omega}$ by a similar argument as Beaudoin [1]:

Lemma 3.5. $\mathsf{MM}^{+\omega}$ *holds in* $V^{\mathbb{P}}$.

Proof. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for an ω_1-stationary preserving foricng notion. For each $\xi < \omega_1$, let $\dot{D}_\xi$ be a $\mathbb{P}$-name for a dense subset of $\dot{\mathbb{Q}}$, and for each $n < \omega$, let $\ddot{T}_n$ be a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name for a stationary subset of ω_1. Take an arbitrary $p_0 \in \mathbb{P}$. It suffices to find $p^* \leq p_0$ in $\mathbb{P}$ such that if G is a $\mathbb{P}$-generic filter over V with $p^* \in G$, then in $V[G]$ there is a filter $h \subseteq \mathbb{Q}$ with the following properties:

(i) $h \cap D_\xi \neq \emptyset$ for any $\xi < \omega_1$.

(ii) $\dot{T}_n^h$ is stationary in ω_1 for all $n < \omega$.

Here $\mathbb{Q}$, D_ξ and $\dot{T}_n$ denote $\dot{\mathbb{Q}}^G$, $\dot{D}_\xi^G$ and $\ddot{T}_n^G$, respectively.

First, we find p^* as above. We work in V. We identify each $\ddot{T}_n$ with a $\mathbb{P} * \dot{\mathbb{Q}}$-name. Let $\dot{\gamma}$ and $\mathbb{S}$ be as in Lemma 3.4. Note that $\mathbb{S}$ is ω_1-stationary preserving and each $\ddot{T}_n$ is stationary in ω_1 in $V^{\mathbb{S}}$. Let $\dot{\mathbb{R}}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for $\mathbb{R}([\omega_2^V]^\omega \setminus \dot{X}_{\dot{\gamma}})$.

Take a sufficiently large regular cardinal θ. By Fact 2.3, there are $M \in [\mathcal{H}_\theta]^{\omega_1}$ and $k \subseteq \mathbb{S} \cap M$ such that

(iii) $\omega_1 \cup \{p_0, \dot{\mathbb{Q}}, \dot{\gamma}, \mathbb{S}\} \cup \{\dot{D}_\xi \mid \xi < \omega_1\} \cup \{\ddot{T}_n \mid n < \omega\} \subseteq M \prec \langle \mathcal{H}_\theta, \in \rangle$,

(iv) k is an M-generic filter on $\mathbb{S} \cap M$ with $p_0 * 1_{\dot{\mathbb{Q}}} * 1_{\dot{\mathbb{R}}} \in k$,

(v) $\ddot{T}_n^k$ is stationary in ω_1 for any $\xi < \mu$.

Let $\delta^* := M \cap \omega_2 \in \omega_2$, and let

$$g := \{p \in \mathbb{P} \cap M \mid \exists \dot{q}\, \exists \dot{r},\ p * \dot{q} * \dot{r} \in k\}.$$

Then, g is an M-generic filter on $\mathbb{P} \cap M$.

Note that $\sup_{p \in g} \delta_p = \delta^*$ by Lemma 3.1 (4) and the M-genericity of g_0. Let $p^* : \delta^* \times [\delta^*]^\omega \to 2$ be an extension of $\bigcup g$ such that $p^*(\alpha, x) = 0$ for all $\langle \alpha, x \rangle \notin \mathrm{dom}(\bigcup g)$. We claim that p^* is as desired. For this, we use the transitive collapse of M. First, we make some preliminaries on it.

Let $\pi : M \to M'$ be the transitive collapse of M, and let $\mathbb{P}'$, $\dot{\mathbb{Q}}'$, $\dot{\mathbb{R}}'$, $\mathbb{S}'$, k' and g' be $\pi(\mathbb{P})$, $\pi(\dot{\mathbb{Q}})$, $\pi(\dot{\mathbb{R}})$, $\pi(\mathbb{S})$, $\pi[k]$ and $\pi[g]$, respectively. Note that $\mathbb{S}' = \mathbb{P}' * \dot{\mathbb{Q}}' * \dot{\mathbb{R}}'$ in M'. Moreover, k' is an $\mathbb{S}'$-generic filter over M', and g' is the $\mathbb{P}'$-generic filter over M' naturally obtained from k'. Let h' be the $(\dot{\mathbb{Q}}')^{g'}$-generic filter over $M'[g']$ naturally obtained from k', and let i' be the $(\dot{\mathbb{R}}')^{g'*h'}$-generic filter over $M'[g'*h']$ naturally obtained from k'.

Now, we start to prove that p^* is as desired. First, we prove that $p^* \in \mathbb{P}$. We only check that $X_{p^*,\gamma}$ is non-stationary in $[\delta^*]^\omega$ for some $\gamma < \delta^*$. The other properties are easily checked.

First of all, note that $\pi \restriction (\mathcal{H}_{\omega_2} \cap M)$ is the identity map since $\mathcal{H}_{\omega_2} \cap M$ is transitive and that $\pi(\omega_2) = \delta^*$. Let $\gamma := \pi(\dot{\gamma})^{g'*h'} < \pi(\omega_2) = \delta^*$. Then range$(\bigcup i')$ is a club subset of $[\delta^*]^\omega$ which does not intersect $\bigcup_{p' \in g'} X_{p',\gamma} = \bigcup_{p \in g} \pi(X_{p,\gamma})$. Here note that $X_{p,\gamma} \in \mathcal{H}_{\omega_2} \cap M$ for all $p \in g$ by the property (ii) of conditions in $\mathbb{P}$. So $\bigcup_{p \in g} \pi(X_{p,\gamma}) = \bigcup_{p \in g} X_{p,\gamma} = X_{p^*,\gamma}$. Hence $X_{p^*,\gamma}$ is non-stationary in $[\delta^*]^\omega$

We have shown that $p^* \in \mathbb{P}$. Note that p^* is a lower bound of g. Then $p^* \le p$ since $p \in g$ by (iv). Suppose G is a $\mathbb{P}$-generic filter over V with $p^* \in G$. Working in $V[G]$, we find a filter $h \subseteq \mathbb{Q}$ satisfying (i) and (ii).

Let $M[G]$ denote the collection of $\dot{a}^G$ for all $\mathbb{P}$-names $\dot{a} \in M$, and define $\hat{\pi} : M[G] \to M'[g']$ by $\hat{\pi}(\dot{a}^G) := \pi(\dot{a})^{g'}$. It is easy to see that $\hat{\pi}$ coincides with the transitive collapse of $M[G]$ and that $\hat{\pi}$ extends π. Let h be the filter on $\mathbb{Q}$ generated by $\hat{\pi}^{-1}[h']$. Then h satisfies (i) since $D_\xi \in M[G]$ and $h' \cap \hat{\pi}(D_\xi) \neq \emptyset$ for all $\xi < \omega_1$. As for (ii), it is easy to see that $\dot{T}_n^h = \ddot{T}_n^k$ for each $n < \omega$. Then, h satisfies (ii) by (v). $\square$

References

[1] R. Beaudoin, *The proper forcing axiom and stationary set reflection*, Pacific J. Math. **149** (1991), no.1, 13–24.

[2] S. D. Cox, *The diagonal reflection principle*, Proc. Amer. Math. Soc. **140** (2012), no.8, 2893–2902.

[3] S. D. Cox, *Forcing axioms, approachability at ω_2 and stationary set reflection*, preprint.

[4] Q. Feng and M. Magidor, *On reflection of stationary sets*, Fund. Math. **140** (1992), no.2, 175–181.

[5] M. Foreman, M. Magidor and S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters I*, Ann. of Math. **127** (1988), no.1, 1–47.

[6] S. Fuchino, A. R. Ottenbreit and H. Sakai, *Strong downward Löwenheim-Skolem theorems for stationary logic, I*, preprint.

[7] T. Jech, *Set Theory, 3rd Edition*, Springer-Verlag, Berlin, 2002.

[8] B. König and Y. Yoshinobu, *Fragments of Martin's Maximum in generic extensions*, Math. Logic Quart. **50** (2004), no.3, 296–302.

[9] D. W. Kueker, *Countable approximations and Löwenheim-Skolem theorems*, Ann. Math. Logic **11** (1977), no.1, 57–103.

[10] P. B. Larson, *Separating stationary reflection principles*, J. Symbolic Logic **65** (2000), no.1, 247–258.

[11] T. K. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic **7** (1974/75), no.1, 33–57.

[12] S. Shelah, *Reflection implies the SCH*, Fund. Math. **198** (2008), no.2, 95–111.

[13] Y. Yoshinobu, *Operations, climbability and the Proper Forcing Axiom*, Ann. of Pure and Appl. Logic **164** (2013), no.7–8, 749–762.

[14] Y. Yoshinobu, *The *-variation of Banach-Mazur game and forcing axioms*, Ann. of Pure and Appl. Logic **168** (2017), no.6, 1335–1359.

[15] W. H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Non-stationary Ideal*, De Gruyter Series in Logic and its Applications; 1, Walter de Gruyter, Berlin, New York, 1999.