Boundedness of the maximal operator for double phase functionals with variable exponents

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Abstract

Our aim in this note is to establish the boundedness of the maximal operator for double phase functionals $\phi(x,t) = t^{p(x)} + \{b(x)t\}^{q(x)}$, where $p(\cdot), q(\cdot)$ are log-Hölder continuous exponents and b is a nonnegative, bounded and Hölder continuous function of order $\theta \in (0,1]$.

1 Introduction

Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [2, 3, 4, 5] studied a double phase functional $\Phi(x,t) = t^p + a(x)t^q$, $x \in \mathbf{R}^N, t \geq 0$, where $1 < p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$. In [4], the minimization problem of the double phase functional was discussed under the assumption $q < (1 + \theta/N)p$. Hästö [9, Theorem 4.7] showed the boundedness of the maximal operator on $L^{\Phi}(G)$ when $\Phi(x,t) = t^p + a(x)t^q$, $1 < p < q$, $G \subset \mathbf{R}^N$ is bounded, $a \in C^{\theta}(\bar{G})$ is non-negative and $q \leq (1 + \theta/N)p$.

In this note, let us consider the double phase functional

$$\phi(x,t) = t^{p(x)} + \{b(x)t\}^{q(x)}, \tag{1.1}$$

where $p(\cdot), q(\cdot)$ are log-Hölder continuous and b is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$. For an open set $G \subset \mathbf{R}^N$, recall that the Lebesgue

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space $L^{p(\cdot)}(G)$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\} < \infty$$

(see e.g. [6], [7], [8], [13]). Further let us consider the Musielak-Orlicz space $\phi(G)$ of all functions f such that

$$\|f\|_{\phi(G)} = \inf\left\{\lambda > 0 : \int_G \phi\left(y, \frac{|f(y)|}{\lambda}\right) dy \le 1\right\} < \infty$$

(see e.g. [9], [11]).

Our main aim in this note is to establish the boundedness of the maximal operator in $\phi(\mathbf{R}^N)$ (see Theorem 2.1 in Section 2). We also extend Theorem 2.1 to the Herz case (Theorem 5.1).

2 Variable exponents

Let $p(\cdot)$ be a real valued measurable function on $\mathbf{R}^N$ such that

(P1) $1 < p^- := \operatorname{ess\,inf}_{x\in\mathbf{R}^N} p(x) \le \operatorname{ess\,sup}_{x\in\mathbf{R}^N} p(x) =: p^+ < \infty$;

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{\log(e + 1/|x-y|)} \quad \text{for all } x,\, y \in \mathbf{R}^N$$

with a constant $C_p \ge 0$.

Moreover we suppose that

(P3) $p(\cdot)$ is log-Hölder continuous at ∞, namely, there exists a constant $p(\infty) > 1$ such that

$$|p(x) - p(\infty)| \le \frac{C_{p,\infty}}{\log(e + |x|)} \quad \text{for all } x \in \mathbf{R}^N$$

with a constant $C_{p,\infty} \ge 0$.

In this case, we write $p(\cdot) \in \mathcal{P}_0$.

Our aim in this note is to give the boundedness of the maximal operator $f \to Mf$ in the Musielak-Orlicz space $\phi(\mathbf{R}^N)$.

Theorem 2.1. *Let $p(\cdot)$ and $q(\cdot)$ be variable exponents in $\mathcal{P}_0$. Suppose*

$$0 \le 1/p(x) - 1/q(x) \le \min\{1/p(\infty), \theta/N\}$$

for $x \in \mathbf{R}^N$. Then there is a constant $C > 0$ such that

$$\|Mf\|_{\phi(\mathbf{R}^N)} \le C\|f\|_{\phi(\mathbf{R}^N)}. \tag{2.1}$$

In what follows, we always assume that $p(\cdot), q(\cdot) \in \mathcal{P}_0$.

3 Maximal functions

Consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r)$ denotes the open ball with center at x and radius r.

For the boundedness of the maximal operator in $L^p(\mathbf{R}^N)$ of constant exponent, we refer to [1], [12] and [15].

Lemma 3.1 (cf. [14, Lemma 3.5]). *There is a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \le C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^{p(y)}\, dy \right)^{1/p(x)} + C(1+|x|)^{-N}$$

for all $x \in \mathbf{R}^N$, $r > 0$ and measurable functions f on $\mathbf{R}^N$ such that $\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} < \infty$.

Corollary 3.2. *There is a constant $C > 0$ such that*

$$\|Mf\|_{L^{p(\cdot)}(\mathbf{R}^N)} \le C\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} \quad \text{for all } f \in L^{p(\cdot)}(\mathbf{R}^N).$$

In fact, take $1 < p_0 < p^-$ and apply Lemma 3.1 with $p(\cdot)$ replaced by $p_0(\cdot) = p(\cdot)/p_0$. If $\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} \le 1$, then we find

$$Mf(x) \le C\,(M[f_0](x))^{1/p_0(x)} + C(1+|x|)^{-N},$$

where $f_0(y) = |f(y)|^{p_0(y)}$. Hence

$$\begin{aligned} \int (Mf(x))^{p(x)}\, dx &\le C \int (M[f_0](x))^{p_0}\, dx + C \int (1+|x|)^{-Np(x)} dx \\ &\le C \int (f_0(y))^{p_0}\, dy + C \int (1+|x|)^{-Np^-}\, dx \\ &\le C \int |f(y)|^{p_0(y)} dy + C \le C. \end{aligned}$$

4 Fractional maximal functions

Consider the fractional maximal function

$$M_{\tau(\cdot)}f(x) = \sup_{r>0} \frac{r^{\tau(x)}}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $\tau(\cdot)$ is a measurable function on $\mathbf{R}^N$ satisfying

(τ) $0 \le \tau^- = \inf_{x\in\mathbf{R}^N} \ \tau(x) \le \tau^+ = \sup_{x\in\mathbf{R}^N} \ \tau(x) \le N.$

THEOREM 4.1 (Sobolev type inequality). *Suppose*

$$0 \le 1/p(x) - 1/q(x) = \tau(x)/N \le 1/p(\infty)$$

for all $x \in \mathbf{R}^N$. Then there is a constant $C > 0$ such that

$$\|M_{\tau(\cdot)}f\|_{L^{q(\cdot)}(\mathbf{R}^N)} \le C\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)}$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^N)$.

Proof. Let f be a measurable function on $\mathbf{R}^N$ with $\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} \le 1$. Let $x \in \mathbf{R}^N$ and $r > 0$. First note that for $0 < r \le \delta$ we have

$$\begin{aligned} \frac{r^{\tau(x)}}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy &\le \delta^{\tau(x)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \\ &\le \delta^{\tau(x)} Mf(x) \end{aligned}$$

and for $0 < \delta < r < (1 + |x|)^{p(x)}$ we have by Lemma 3.1

$$\begin{aligned} &\frac{r^{\tau(x)}}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \\ \le\ & Cr^{\tau(x)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^{p(y)}\, dy \right)^{1/p(x)} + Cr^{\tau(x)}(1 + |x|)^{-N} \\ \le\ & Cr^{\tau(x)-N/p(x)} \\ \le\ & C\delta^{\tau(x)-N/p(x)} \end{aligned}$$

since $1/q(x) = 1/p(x) - \tau(x)/N > 0$. If $r \ge (1 + |x|)^{p(x)}$, then, for $z = rx/|x|$, we obtain by Lemma 3.1

$$
\begin{aligned}
& \frac{r^{\tau(x)}}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy \\
\le\ & C\frac{r^{\tau(x)}}{|B(z,3r)|}\int_{B(z,3r)}|f(y)|\,dy \\
\le\ & Cr^{\tau(x)}\left(\frac{1}{|B(z,3r)|}\int_{B(z,3r)}|f(y)|^{p(y)}\,dy\right)^{1/p(z)} + Cr^{\tau(x)}(1+|z|)^{-N} \\
\le\ & Cr^{\tau(x)-N/p(z)} + Cr^{\tau(x)}(1+|z|)^{-N} \\
\le\ & Cr^{\tau(x)-N/p(z)} \\
\le\ & Cr^{\tau(x)-N/p(\infty)} \\
\le\ & C(1+|x|)^{(\tau(x)-N/p(\infty))p(x)} \\
\le\ & C(1+|x|)^{(\tau(x)-N/p(x))p(x)}.
\end{aligned}
$$

Hence

$$M_{\tau(x)}f(x) \le \delta^{\tau(x)}Mf(x) + C\delta^{\tau(x)-N/p(x)} + C(1+|x|)^{(\tau(x)-N/p(x))p(x)}.$$

Now, letting $\delta = \{Mf(x)\}^{-p(x)/N}$, we find

$$
\begin{aligned}
M_{\tau(x)}f(x) &\le C\{Mf(x)\}^{1-\tau(x)p(x)/N} + C(1+|x|)^{(\tau(x)-N/p(x))p(x)} \\
&\le C\{Mf(x)\}^{p(x)/q(x)} + C(1+|x|)^{(\tau(x)-N/p(x))p(x)},
\end{aligned}
$$

so that

$$
\begin{aligned}
\int \{M_{\tau(x)}f(x)\}^{q(x)}\,dx &\le C\int \{Mf(x)\}^{p(x)}\,dx + C\int (1+|x|)^{-Np(x)}\,dx \\
&\le C,
\end{aligned}
$$

as required. □

Proof of Theorem 2.1. Let f be a measurable function on $\mathbf{R}^N$ with

$$\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} + \|bf\|_{L^{q(\cdot)}(\mathbf{R}^N)} \le 1.$$

Let $x \in \mathbf{R}^N$ and $r > 0$. Set $\tau(x) = N/p(x) - N/q(x)$. First note that

$$
\begin{aligned}
& b(x)\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy \\
=\ & \frac{1}{|B(x,r)|}\int_{B(x,r)}\{b(x)-b(y)\}|f(y)|\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)}b(y)|f(y)|\,dy \\
\le\ & \frac{1}{|B(x,r)|}\int_{B(x,r)}C|x-y|^{\tau(x)}|f(y)|\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)}b(y)|f(y)|\,dy,
\end{aligned}
$$

so that

$$b(x)Mf(x) \le CM_{\tau(x)}f(x) + M[bf](x).$$

Since $1/q(x) = 1/p(x) - \tau(x)/N$, Theorem 4.1 gives

$$\begin{aligned} \|bMf\|_{L^{q(\cdot)}(\mathbf{R}^N)} &\le C\|M_{\tau(\cdot)}f\|_{L^{q(\cdot)}(\mathbf{R}^N)} + \|M[bf]\|_{L^{q(\cdot)}(\mathbf{R}^N)} \\ &\le C\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} + C\|bf\|_{L^{q(\cdot)}(\mathbf{R}^N)}, \end{aligned}$$

which proves the result. □

EXAMPLE 4.2. Let $1/q(0) < 1/p(0) - \theta/N$. For $0 < \theta \le 1$ and $0 < \beta < N$, consider

$$b(x) = (\min\{\max\{0, x_N\}, 1\})^{\theta} \quad \text{and} \quad f(y) = |y|^{-\beta}\chi_{B_-}(y),$$

where $B_- = \{x = (x_1, ..., x_N) \in B(0,1) : x_N < 0\}$. Then note that

(1) $Mf(x) \ge C|x|^{-\beta}$ on $B(0,1)$;

(2) $\displaystyle\int_{B(0,1)} \phi(y, f(y))\, dy < \infty$ when $-\beta p(0) + N > 0$;

(3) $\displaystyle\int_{B(0,1)} \phi(x, Mf(x))\, dx = \infty$ when $(\theta - \beta)q(0) + N \le 0$.

Now take θ and β such that $0 < \theta \le 1$, $\theta < \beta < N$ and

$$\frac{1}{q(0)} = \frac{\beta - \theta}{N} \quad \text{and} \quad \frac{1}{q(0)} + \frac{\theta}{N} = \frac{\beta}{N} < \frac{1}{p(0)}.$$

Then (2) and (3) hold.

5 Herz spaces

Let $A(r) = B(0, 2r) \setminus B(0, r)$ for $r > 0$. For a real number ν and $0 < q < \infty$, consider the Herz space

$$\|f\|_{\phi,q,\nu} = \|f\|_{\phi(B(2))} + \left(\int_1^{\infty} \left(r^{\nu}\|f\|_{\phi(A(r))}\right)^q \frac{dr}{r} \right)^{1/q} < \infty.$$

Theorem 2.1 is extended in the Herz settings.

Theorem 5.1. *Suppose* $-N/q(\infty) < \nu < N - N/p(\infty)$ *and*

$$0 \le 1/p(x) - 1/q(x) = \tau(x)/N \le \min\{1/p(\infty), \theta/N\}$$

for all $x \in \mathbf{R}^N$. *Then there is a constant* $C > 0$ *such that*

$$\|Mf\|_{\phi,q,\nu} \le C\|f\|_{\phi,q,\nu} \tag{5.1}$$

when $\|f\|_{\phi,q,\nu} < \infty$.

For a proof of Theorem 5.1, we use Theorem 2.1 and the following lemmas.

Lemma 5.2. *There are constants* $C_1, C_2 > 0$ *such that*

$$C_1 r^{N/p(\infty)} \le \|\chi_{B(x,r)}\|_{L^{p(\cdot)}} \le C_2 r^{N/p(\infty)}$$

for all $x \in \mathbf{R}^N$ *and* $r > 1$.

Lemma 5.3 ([10, Lemma 2.5]). *There is a constant* $C > 0$ *such that*

$$\frac{1}{|A(r)|}\int_{A(r)} |f(y)|\,dy \le Cr^{-N/p(\infty)}\|f\|_{L^{p(\cdot)}(A(r))}$$

for all $x \in \mathbf{R}^N$, $r > 1$ *and measurable functions* f *on* $\mathbf{R}^N$ *such that* $\|f\|_{L^{p(\cdot)}(A(r))} < \infty$.

Lemma 5.4. *If* $\varepsilon + \nu - N + N/p(\infty) < 0$ *and* $\varepsilon > 0$, *then*

$$\int_{B(0,r)\setminus B(0,1)} |f(y)|\,dy \le Cr^{-\varepsilon+N-N/p(\infty)-\nu}\left(\int_{1/2}^{r} \left(t^{\varepsilon+\nu}\|f\|_{L^{p(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right)^{1/q}$$

for all $r \ge 1$ *and* $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$.

Proof. We may assume that $f(x) = 0$ for $x \in B(0,1)$. Let j_0 be the smallest integer such that $2^{j_0} \ge r$. By Lemma 5.3, we have

$$\begin{aligned}\int_{B(0,r)\setminus B(0,1)} |f(y)|\,dy &\le C\sum_{j=1}^{j_0}(2^{-j}r)^N \frac{1}{|A(2^{-j}r)|}\int_{A(2^{-j}r)} |f(y)|\,dy \\ &\le C\sum_{j=1}^{j_0}(2^{-j}r)^{N-N/p(\infty)}\|f\|_{L^{p(\cdot)}(A(2^{-j}r))}.\end{aligned}$$

In case $q > 1$, by Hölder's inequality, we have

$$\sum_{j=1}^{j_0} (2^{-j}r)^{N-N/p(\infty)} \|f\|_{L^{p(\cdot)}(A(2^{-j}r))}$$
$$\le \left(\sum_{j=1}^{j_0} \left((2^{-j}r)^{-\varepsilon+N-N/p(\infty)-\nu} \right)^{q'} \right)^{1/q'} \left(\sum_{j=1}^{j_0} \left((2^{-j}r)^{\varepsilon+\nu} \|f\|_{L^{p(\cdot)}(A(2^{-j}r))} \right)^{q} \right)^{1/q}$$
$$\le Cr^{-\varepsilon+N-N/p(\infty)-\nu} \left(\int_{1/2}^{r} \left(t^{\varepsilon+\nu} \|f\|_{L^{p(\cdot)}(A(t))} \right)^{q} \frac{dt}{t} \right)^{1/q}.$$

Therefore, we obtain the required result in this case.

For the case $0 < q \le 1$, by the fact that $(a+b)^q \le a^q + b^q$ for all $a, b \ge 0$ instead of Hölder's inequality, we also obtain the required result. $\square$

In the same manner we have the following result.

Lemma 5.5. *Let $\beta \in \mathbf{R}$. If $\varepsilon + \beta - N/p(\infty) - \nu < 0$ and $\varepsilon > 0$, then*

$$\int_{\mathbf{R}^N \setminus B(0,r)} |y|^{\beta-N} |f(y)|\, dy \le Cr^{\varepsilon+\beta-N/p(\infty)-\nu} \left(\int_{r/2}^{\infty} \left(t^{-\varepsilon+\nu} \|f\|_{L^{p(\cdot)}(A(t))} \right)^{q} \frac{dt}{t} \right)^{1/q}$$

for all $r \ge 1$ and $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$.

Proof of Theorem 5.1. Suppose $\|f\|_{\phi,q,\nu} \le 1$. Set $\tau(x) = N/p(x) - N/q(x)$ and $\tau(\infty) = N/p(\infty) - N/q(\infty)$. We show that

$$\int_1^{\infty} \left(r^{\nu} \|bMf\|_{L^{q(\cdot)}(A(r))} \right)^{q} \frac{dr}{r} \le C$$

when $1 < q < \infty$.

For $r > 1$, write

$$\begin{aligned} f &= f\chi_{B(0,r/2)} + f\chi_{B(0,4r)\setminus B(0,r/2)} + f\chi_{\mathbf{R}^N \setminus B(0,4r)} \\ &= f_{1,r} + f_{2,r} + f_{3,r}. \end{aligned}$$

By Theorem 2.1 we have

$$\int_1^{\infty} \left(r^{\nu} \|bMf_{2,r}\|_{L^{q(\cdot)}(A(r))} \right)^{q} \frac{dr}{r} \le C.$$

Note that

$$
\begin{aligned}
& b(x)\frac{1}{|B(x,t)|}\int_{B(x,t)} |f_{1,r}(y)|\,dy \\
\le\ & \frac{1}{|B(x,t)|}\int_{B(x,t)} C|x-y|^{\tau(x)}|f_{1,r}(y)|\,dy + \frac{1}{|B(x,t)|}\int_{B(x,t)} b(y)|f_{1,r}(y)|\,dy \\
\le\ & Cr^{\tau(\infty)-N}\int_{B(0,r)} |f(y)|\,dy + Cr^{-N}\int_{B(0,r)} b(y)|f(y)|\,dy
\end{aligned}
$$

and

$$
\begin{aligned}
& b(x)\frac{1}{|B(x,t)|}\int_{B(x,t)} |f_{3,r}(y)|\,dy \\
\le\ & \frac{1}{|B(x,t)|}\int_{B(x,t)} C|x-y|^{\tau(y)}|f_{3,r}(y)|\,dy + \frac{1}{|B(x,t)|}\int_{B(x,t)} b(y)|f_{3,r}(y)|\,dy \\
\le\ & C\int_{\mathbf{R}^N\setminus B(0,2r)} |y|^{\tau(\infty)-N}|f(y)|\,dy + C\int_{\mathbf{R}^N\setminus B(0,2r)} |y|^{-N}b(y)|f(y)|\,dy
\end{aligned}
$$

for all $x \in A(r)$ and $t > 0$. By Lemmas 5.2 and 5.4 we have for $0 < \varepsilon_1 < N - N/p(\infty) - \nu$ and $0 < \varepsilon_2 < N - N/q(\infty) - \nu$

$$
\begin{aligned}
& \int_1^\infty \left(r^\nu \|bMf_{1,r}\|_{L^{q(\cdot)}(A(r))}\right)^q \frac{dr}{r} \\
\le\ & C\int_1^\infty \left(r^{\nu+N/q(\infty)+\tau(\infty)}\left(r^{-\varepsilon_1-N/p(\infty)-\nu}\left(\int_{1/2}^r \left(t^{\varepsilon_1+\nu}\|f\|_{L^{p(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right)^{1/q}\right)\right)^q \frac{dr}{r} \\
& + C\int_1^\infty \left(r^{\nu+N/q(\infty)}\left(r^{-\varepsilon_2-N/q(\infty)-\nu}\left(\int_{1/2}^r \left(t^{\varepsilon_2+\nu}\|bf\|_{L^{q(\cdot)}(A(t))}\right)^q \frac{dt}{t}\right)^{1/q}\right)\right)^q \frac{dr}{r} \\
\le\ & C\int_{1/2}^\infty \left(t^{\varepsilon_1+\nu}\|f\|_{L^{p(\cdot)}(A(t))}\right)^q \left(\int_t^\infty r^{-\varepsilon_1 q}\frac{dr}{r}\right)\frac{dt}{t} \\
& + C\int_{1/2}^\infty \left(t^{\varepsilon_2+\nu}\|bf\|_{L^{q(\cdot)}(A(t))}\right)^q \left(\int_t^\infty r^{-\varepsilon_2 q}\frac{dr}{r}\right)\frac{dt}{t} \\
\le\ & C\int_{1/2}^\infty \left(t^{\nu}\|f\|_{L^{p(\cdot)}(A(t))}\right)^q \frac{dt}{t} + C\int_{1/2}^\infty \left(t^{\nu}\|bf\|_{L^{q(\cdot)}(A(t))}\right)^q \frac{dt}{t} \le C.
\end{aligned}
$$

In the same way, by Lemmas 5.2 and 5.5, we obtain

$$
\int_1^\infty \left(r^\nu \|bMf_{3,r}\|_{L^{q(\cdot)}(A(r))}\right)^q \frac{dr}{r} \le C,
$$

which completes the proof. □

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