Geometrical constants of Day-James spaces [1]

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Abstract

We describe some recent results on the von Neumann-Jordan (NJ-) constant $C_{\mathrm{NJ}}(X)$ and the related geometrical constants of concrete Banach spaces X. In particular, we calculate the constants for X being a class of Day-James spaces ℓ_p-ℓ_q by using the Banach-Mazur distance $d(X, H)$ between X and H, where H is a two-dimensional inner product space.

Definition 1 (i) Let X be a Banach space. The NJ-constant $C_{\mathrm{NJ}}(X)$ is the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y \in X$ not both 0 ([2]). An equivalent definition of this constant is

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X,\ y \in B_X \right\},$$

where $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leq 1\}$.

It is well known (cf. [5]) that
(i) $1 \leq C_{\mathrm{NJ}}(X) \leq 2$ for all Banach spaces X
(ii) X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$
(iii) $C_{\mathrm{NJ}}(L_p) = 2^{2/\min\{p,p'\}-1}$, where $1/p + 1/p' = 1, 1 \leq p \leq \infty$
(iv) X is uniformly non-square if and only if $C_{\mathrm{NJ}}(X) < 2$
(v) $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(X^*)$ for all Banach spaces X.

[1] *Keywords.* Day-James space, Banach-Mazur distance, von Neumann-Jordan constant, von Neumann-Jordan type constant,

Definition 2 (cf. [5]) Let $1 \le p, q \le \infty$. The Day-James ℓ_p-ℓ_q space is the space $\mathbb{R}^2$ with the norm $\|\cdot\|_{p,q}$ defined by

$$\|(x,y)\|_{p,q} = \begin{cases} \|(x,y)\|_p, & xy \ge 0, \\ \|(x,y)\|_q, & xy \le 0, \end{cases}$$

where $\|\cdot\|_p$ is the ℓ_p-norm on $\mathbb{R}^2$.

Theorem 1 ([3, 14, 15, 16, 17]) (i) If either $1 \le p \le 2$, or $p > 2$ and $(p-2)2^{2/p-2} < 1$ then

$$C_{\mathrm{NJ}}(\ell_p\text{-}\ell_1) = 1 + 2^{2/p-2}.$$

(ii) If $p > 2$ and $(p-2)2^{2/p-2} \ge 1$, then

$$C_{\mathrm{NJ}}(\ell_p\text{-}\ell_1) = \frac{1}{2} + \frac{1 - t_0^p}{2(t_0 - t_0^{p-1})},$$

where $t_0 \in (0,1)$ is the unique solution to the equation

$$\frac{(t - t^{p-1})(1 + t^p)^{2/p-1}}{1 - t^2} = 1.$$

In particular,

$$C_{\mathrm{NJ}}(\ell_\infty\text{-}\ell_1) = \frac{3 + \sqrt{5}}{4}.$$

We first calculate the NJ-constant for X being a class of Day-James spaces ℓ_p-ℓ_q by using the Banach-Mazur distance.

Definition 3 For isomorphic Banach spaces X and Y, the Banach-Mazur distance between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\| \cdot \|T^{-1}\|$ taken over all bicontinuous linear operators T from X onto Y (cf. [11]).

Lemma 2 ([5]) If X and Y are isomorphic Banach spaces, then

$$\frac{C_{\mathrm{NJ}}(X)}{d(X,Y)^2} \le C_{NJ}(Y) \le C_{\mathrm{NJ}}(X) d(X,Y)^2.$$

In particular, if X and Y are isometric, then $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(Y)$.

Lemma 3 ([5]) Let $X = (X, \|\cdot\|)$ be a non-trivial Banach space and let $X_1 = (X, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $\alpha, \beta > 0$,

$$\alpha\|x\| \leq \|x\|_1 \leq \beta\|x\|, \quad x \in X.$$

Then

$$\frac{\alpha^2}{\beta^2} C_{\mathrm{NJ}}(X) \leq C_{\mathrm{NJ}}(X_1) \leq \frac{\beta^2}{\alpha^2} C_{\mathrm{NJ}}(X).$$

For a norm $\|\cdot\|$ on $\mathbb{R}^2$, we write $C_{\mathrm{NJ}}(\|\cdot\|)$ for $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|))$.

Definition 4 A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(|x|, |y|)\| = \|(x, y)\|$ for any $x, y \in \mathbb{R}$.

From Lemmas 2 and 3, we have the following.

Theorem 4 ([7], cf. [6]) Let $\|\cdot\|, \|\cdot\|_H$ be absolute norms on $\mathbb{R}^2$. Assume that
(i) $(\mathbb{R}^2, \|\cdot\|_H)$ is an inner product space.
(ii) $\alpha\|(x, y)\|_H \leq \|(x, y)\| \leq \beta\|(x, y)\|_H$ for any $(x, y) \in \mathbb{R}^2$ (α, β are the best constants).
(iii) In (ii) it satisfies either $\alpha\|(1, 0)\|_H = \|(1, 0)\|$ and $\alpha\|(0, 1)\|_H = \|(0, 1)\|$, or $\beta\|(1, 0)\|_H = \|(1, 0)\|$ and $\beta\|(0, 1)\|_H = \|(0, 1)\|$.
Then

$$C_{\mathrm{NJ}}(\|\cdot\|) = \frac{\beta^2}{\alpha^2}.$$

We calculate NJ-constant for X being a class of Day-James spaces, by using Theorem 4. For $1 \leq q < p < \infty$, we define a new norm $\|\cdot\|_X$ on $\mathbb{R}^2$ by

$$\|(x, y)\|_X = \begin{cases} \|T(x, y)\|_p, & |x| \geq |y|, \\ \|T(x, y)\|_q, & |x| \leq |y|, \end{cases}$$

where $T(x, y) = \frac{1}{\sqrt{2}}(x - y, x + y)$. Note that $C_{\mathrm{NJ}}(\ell_p\text{-}\ell_q) = C_{\mathrm{NJ}}(\|\cdot\|_X)$. Also define

$$\|(x, y)\|_H = \sqrt{2^{2/p-1}x^2 + 2^{2/q-1}y^2} \quad (1 \leq q < p < \infty).$$

Note that both norms $\|\cdot\|_X$ and $\|\cdot\|_H$ are absolute and satisfy the conditions in Theorem 4. Applying Theorem 4 we obtain the following.

Theorem 5 ([7]) If $1 \le q \le 2, q \le p < \infty$ and $2^{2/p-2/q}(p-1) \le 1$, then

$$C_{\rm NJ}(\ell_p\text{-}\ell_q) = \frac{2^{2/p}(t_0^2 + 2^{2/q-2/p})}{((1+t_0)^q + (1-t_0)^q)^{2/q}}. \tag{1}$$

where

$$t_0 = \sup\left\{t \in (0,1) : \frac{(2^{2/q-2/p} - t)(1+t)^{q-1}}{(2^{2/q-2/p} + t)(1-t)^{q-1}} \le 1\right\}.$$

In particular, if $1 \le q \le p \le 2$, then (1) holds.

Corollary 6 ([3, 14, 15, 17]) If either $1 \le p \le 2$, or $p > 2$ and $2^{2/p-2}(p-1) \le 1$, then

$$C_{\rm NJ}(\ell_p\text{-}\ell_1) = 1 + 2^{2/p-2}.$$

Remark 1 Let $1 \le q \le 2, q \le p < \infty$ and $2^{2/p-2/q}(p-1) \le 1$. Theorem 7 gives that if H is an inner product space with $\dim H = 2$, then

$$d(\ell_p\text{-}\ell_q, H) = \sqrt{C_{\rm NJ}(\ell_p\text{-}\ell_q)}.$$

We next consider some other geometrical constants for Day-James spaces.

Definition 5 ([9]) Let X be a Banach space. The James type constant of X is

$$J_{X,t}(\tau) = \begin{cases} \sup\left\{\left(\dfrac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2}\right)^{1/t} : x, y \in S_X\right\} & \text{if } t \ne -\infty, \\ \sup\left\{\min(\|x+\tau y\|, \|x-\tau y\|) : x, y \in S_X\right\} & \text{if } t = -\infty \end{cases}$$

for $\tau \ge 0$ and $-\infty \le t < \infty$.

In [9], $\rho_X(\tau) = J_{X,1}(\tau) - 1$ and $J(X) = J_{X,-\infty}(1)$, where

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} : x, y \in S_X\right\}$$

is the modulus of smoothness of X and

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

is James constant of X ([4]).

Definition 6 ([9]) (i) Let X be a Banach space. The von Neumann-Jordan type constant of X is

$$C_t(X) = \sup\{J_{X,t}(\tau)^2/(1+\tau^2) : 0 \le \tau \le 1\}$$

for $-\infty \le t < \infty$.

(ii) Let X be a Banach space. The constant $C'_{\rm NJ}(X)$ is

$$C'_{\rm NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x, y \in S_X\right\}.$$

Note that $C_2(X) = C_{\rm NJ}(X)$, $C_0(X) = C_{\rm Z}(X)$ and $C'_{\rm NJ}(X) = J_{X,2}(1)^2/2$, where

$$C_{\rm Z}(X) = \sup\left\{\frac{\|x+y\|\|x-y\|}{\|x\|^2+\|y\|^2} : x, y \in X, \text{ not both zero}\right\}.$$

is the Zbăganu constant of X ([20]).

Some properties of $C_t(X)$ are found in [9]. For example,

$$1 \le J(X)^2/2 \le C_{-\infty}(X) \le C_Z(X) \le C_1(X) \le C_{\rm NJ}(X) \le 2.$$

for any Banach space X. If X is an L_p-space, then

$$J(X)^2/2 = C_{-\infty}(X) = C_Z(X) = C_1(X) = C_{\rm NJ}(X).$$

If X is a Hilbert space, then all these values are equal to 1, and if X is not uniformly non-square, then all these values are equal to 2. If X is ℓ_2-ℓ_1, then

$$C_Z(X) = \sqrt{2} < C_1(X) = \frac{3+2\sqrt{2}}{4} < \frac{3}{2} = C_{\rm NJ}(X) = C'_{\rm NJ}(X).$$

Note that the dual space X^* of X is ℓ_2-ℓ_∞. Then

$$C_t(X^*) = \frac{3}{2} \quad (-\infty \le t \le 2).$$

In particular,

$$C_Z(X^*) = C_{\rm NJ}(X^*) = \frac{3}{2}.$$

Also,

$$C'_{\rm NJ}(X^*) = \frac{3+2\sqrt{2}}{4} < \frac{3}{2} = C_{\rm NJ}(X^*).$$

We give these constants for X being a class of ℓ_p-ℓ_q spaces, as an improvement of Theorem 5.

Theorem 7 Let $1 \le q \le 2, q \le p < \infty$ with $2^{2/p-2/q}(p-1) \le 1$. Let $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Let t_0 be as in Theorem 5. For all t with $-\infty \le t \le 2$,

$$C'_{\rm NJ}(\ell_p\text{-}\ell_q) = C_t(\ell_{p'}\text{-}\ell_{q'}) = \frac{2^{2/p}(t_0^2 + 2^{2/q-2/p})}{((1+t_0)^q + (1-t_0)^q)^{2/q}} \ (= C_{\rm NJ}(\ell_p\text{-}\ell_q)). \tag{2}$$

In particular, if $1 \le q \le p \le 2$, then (2) holds.

Corollary 8 ([9, 14]) Let $1 \le p \le 2$ and $1/p + 1/p' = 1$. For all t with $-\infty \le t \le 2$,

$$C'_{\rm NJ}(\ell_p\text{-}\ell_1) = C_t(\ell_{p'}\text{-}\ell_\infty) = 1 + 2^{2/p-2} \ (= C_{\rm NJ}(\ell_p\text{-}\ell_1)).$$

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